Department of Mathematics, University of Wisconsin-Madison

Math 431 — Final Exam — Solutions — Spring 2025

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Time: **120 minutes**

- This exam contains 7 questions some with multiple parts, 12 pages (including the cover) for the total of 91 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Justify your steps.
- For binomial random variable approximations, you will NOT get full credit for your answer if you do not include some explanation of why that approximation is valid.
- You can use the attached table of the values of the $\Phi(x)$ function.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	7	Total
Points:	16	16	11	12	8	20	8	91
Score:								

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1. For this problem, you need to provide **answers only**.

(a) (4 points) Find $\text{Corr}(-2X+1, X)$.

Solution: $\text{Corr}(-2X+1, X) = \text{Corr}(-2X, X) + \text{Corr}(1, X) = -\text{Corr}(X, X) = \boxed{-1}$.

(b) (4 points) Suppose A, B, C are events on one probability space. Write down explicit conditions for them to be mutually independent.

Solution:

$$\begin{aligned} P(AB) &= P(A)P(B) \\ P(AC) &= P(A)P(C) \\ P(BC) &= P(B)P(C) \\ P(ABC) &= P(A)P(B)P(C) \end{aligned}$$

(c) (4 points) Let X and Y be random variables such that $X \sim N(0, 1)$ and conditional on $X = x$, Y is $N(x, 1)$. Find the PDF of Y as an integral.

Solution:

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x) F_X(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-x)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

(d) (4 points) X_1 and X_2 are iid random variables with the cumulative distribution function $F(t)$. Find the cumulative distribution function of $Y = \max(X_1, X_2)$.

Solution: $F_Y(t) = P(Y \le y) = P(\max(X_1, X_2) \le t) = P(X_1 \le t, X_2 \le t) = P(X_1 \le t)P(X_2 \le t) = \boxed{F(t)^2}$.

2. Suppose that X is a non-negative random variable.

(a) (5 points) We know that the mean of X is 80. How can you estimate $P(X \geq 120)$ using this (and only this) information?

Solution: Since X is non-negative, according to Markov's inequality, we have

$$P(X \geq 120) \leq \frac{E[X]}{120} = \frac{80}{120} = \frac{2}{3}.$$

(b) (5 points) Suppose that we also learn (in addition to its mean) that the variance of X is 400. How can you improve on your upper bound on $P(X \geq 120)$?

Solution: According to Chebyshev's inequality, we get

$$P(X \geq 120) = P(X - 80 \geq 40) \leq P(|X - 80| \geq 40) \leq \frac{\mathrm{Var}(X)}{40^2} = \frac{400}{40 \times 40} = \frac{1}{4}.$$

(c) (6 points) Suppose that we also learn (in addition to its mean and variance) that X is actually the sum of a 200 i.i.d. random variables. How would you estimate the probability $P(X \geq 120)$ now?

Solution: We can apply CLT here. Since $\sqrt{\mathrm{Var}(X)} = 20$, we get

$$P(X \geq 120) = P\Big(\frac{X - 80}{20} \geq \frac{120 - 80}{20}\Big) \approx P(Z \geq 2) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228.$$

Here Z is the standard normal random variable.

Side note: if we denote the 200 i.i.d. random variables by $Y_1, \ldots, Y_{200}$ then $E[Y_1] = \frac{E[X]}{200} = \frac{2}{5}$ and $\mathrm{Var}(Y_1) = \frac{\mathrm{Var}(X)}{200} = 2$, but this is actually not required for the solution.

3. (a) (5 points) A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll, given that the first roll is 3.

Solution: Given that the first roll is 3, we will roll a die until we have 3, 4, 5 or 6. Probability of it is $\frac{2}{3}$, so the distribution of number of rolls is Geom($\frac{2}{3}$) and expectation is $\frac{3}{2}$.

(b) (6 points) A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll.

Solution: Let N the number of additional rolls and X_1 be the number on the first roll. Now

$$\begin{aligned} E[N] &= P(X_1 = 1)E[N|X_1 = 1] + P(X_1 = 2)E[N|X_1 = 2] + P(X_1 = 3)E[N|X_1 = 3] \\ &\quad + P(X_1 = 4)E[N|X_1 = 4] + P(X_1 = 5)E[N|X_1 = 5] + P(X_1 = 6)E[N|X_1 = 6]. \end{aligned}$$

$P(X_1 = i) = \frac{1}{6}$, and, similarly to part (a), we have $N|X_1 = i \sim \text{Geom}(\frac{6-i+1}{6})$, and $E[N|X_1 = i] = \frac{6}{6-i+1}$. We can write $E[N|X_1] = \frac{6}{7-X_1}$ or do it explicitly in each case:

$$\begin{aligned} N|X_1 = 1 &\sim \text{Geom}(\tfrac{6}{6}), & E[N|X_1 = 1] &= \tfrac{6}{6}, \\ N|X_1 = 2 &\sim \text{Geom}(\tfrac{5}{6}), & E[N|X_1 = 2] &= \tfrac{6}{5}, \\ N|X_1 = 3 &\sim \text{Geom}(\tfrac{4}{6}), & E[N|X_1 = 3] &= \tfrac{6}{4}, \\ N|X_1 = 4 &\sim \text{Geom}(\tfrac{3}{6}), & E[N|X_1 = 4] &= \tfrac{6}{3}, \\ N|X_1 = 5 &\sim \text{Geom}(\tfrac{2}{6}), & E[N|X_1 = 5] &= \tfrac{6}{2}, \\ N|X_1 = 6 &\sim \text{Geom}(\tfrac{1}{6}), & E[N|X_1 = 6] &= \tfrac{6}{1}. \end{aligned}$$

Combining everything in the formula:

$$E[N] = \frac{1}{6} \cdot \frac{6}{6} + \frac{1}{6} \cdot \frac{6}{5} + \frac{1}{6} \cdot \frac{6}{4} + \frac{1}{6} \cdot \frac{6}{3} + \frac{1}{6} \cdot \frac{6}{2} + \frac{1}{6} \cdot \frac{6}{1} = \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{147}{60} = 2\,\frac{9}{20}.$$

4. We are given a biased coin and we are told that because of manufacturing defects, the probability of heads, denoted by Y, is itself random, with a distribution over the interval $[0, 0.5]$ given by density $1 + 4y$. We toss the coin a fixed number n of times, and we let X be the number of heads obtained.

   (a) (6 points) Find $E[X|Y]$.

   > **Solution:** Distribution of X given $Y = y$ is $\text{Bin}(n, y)$, so $E[X|Y = y] = ny$ and $E[X|Y] = nY$.

   (b) (6 points) Find $E[X]$.

   > **Solution:**
   >
   > $$E[X] = E[E[X|Y]] = E[nY] = nE[Y] = n\int_0^{0.5} y(1+4y)dy = n\left(\frac{y^2}{2} + \frac{4y^3}{3}\right)\Bigg|_0^{0.5} = \boxed{\tfrac{7}{24}n}.$$

5. (8 points) Suppose $X \sim \text{Exp}(\lambda)$, that is, X is an exponential random variable with parameter $\lambda > 0$. Find the probability density function of X^2.

Solution: The p.d.f. of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = X^2$ and note that Y is a non-negative random variable. Hence for any $y \geq 0$, the c.d.f. of Y is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) \\ &= P(X \leq \sqrt{y}) \qquad X \text{ is non-negative, and so } X \geq -\sqrt{y} \text{ is automatic.} \\ &= F_X(\sqrt{y}) \\ &= \int_{-\infty}^{\sqrt{y}} f_X(x)\,dx = \int_0^{\sqrt{y}} \lambda e^{-\lambda x} \\ &= 1 - e^{-\lambda\sqrt{y}}. \end{aligned}$$

Taking derivative of $F_Y(y)$, we obtain for $y \geq 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}}.$$

Therefore, recalling that Y is non-negative, the p.d.f. of Y is

$$f_Y(y) = \begin{cases} \frac{\lambda e^{-\lambda\sqrt{y}}}{2\sqrt{y}} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

6. An urn contains six balls, one is red, three are green, two are yellow. Randomly draw three balls from the urn **without** replacement, and let R and G denote the number of red and green balls (within the three balls), respectively.

(a) (6 points) Compute $P(R = 1, G = 1)$.

Solution: Let X_1, X_2, X_3 denote the colors of the first three balls. Then X_1, X_2, X_3 are exchangeable. Let r,g,y denote the colors red, green and yellow, respectively. Let Y denote the number of yellow balls. Then

$$\begin{aligned} P(R = 1, G = 1) &= P(R = 1, G = 1, Y = 1) \\ &= P(\{X_1, X_2, X_3\} = \{r, g, y\}) \\ &= 3!P(X_1 = r, X_2 = g, X_3 = y) \qquad \text{Here we used exchangeability} \\ &= 6 \times \frac{1}{6} \times \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}. \end{aligned}$$

(b) (6 points) Compute the expectations $E[R]$ and $E[G]$.

Solution: For $i = 1, 2, 3$, we let

$$I_i = \begin{cases} 1 & \text{if the i-th ball is red} \\ 0 & \text{otherwise,} \end{cases} \qquad J_i = \begin{cases} 1 & \text{if the i-th ball is green} \\ 0 & \text{otherwise.} \end{cases}$$

Then $R = I_1 + I_2 + I_3$ and $G = J_1 + J_2 + J_3$. Note that I_i's are exchangeable, and so are the J_i's. Moreover,

$$E[I_1] = P(\text{the first ball is red}) = \frac{1}{6}, \quad E[J_1] = P(\text{the first ball is green}) = \frac{3}{6} = \frac{1}{2}.$$

Therefore, by the linearity of expectations,

$$E[R] = E[I_1]+E[I_2]+E[I_3] = 3E[I_1] = \frac{1}{2}, \quad E[G] = E[J_1]+E[J_2]+E[J_3] = 3E[J_1] = \frac{3}{2}$$

(c) (8 points) Compute the covariance $\text{Cov}(R, G)$.

Solution:

Method 1.

Since $\text{Cov}(R, G) = E[RG] - E[R]E[G]$, it remains to compute $E[RG]$ (given that $E[R], E[G]$ are obtained in (b)). Note that

$$\begin{aligned} E[RG] &= 0 + 1 \times 1 \times P(R = 1, G = 1) + 1 \times 2P(R = 1, G = 2) \\ &\overset{\text{by (a)}}{=} \frac{3}{10} + 2P(R = 1, G = 2). \end{aligned}$$

Moreover, using the knowledge of counting,

$$P(R = 1, G = 2) = \frac{1 \times \binom{3}{2}}{\binom{6}{3}} = \frac{3}{20}.$$

Hence $E[RG] = \frac{3}{10} + 2 \times \frac{3}{20} = \frac{3}{5}$ and so

$$\text{Cov}(R, G) = E[RG] - E[R]E[G] = \frac{3}{5} - \frac{1}{2} \times \frac{3}{2} = -\frac{3}{20}.$$

Method 2

$$\begin{aligned} \text{Cov}(R, G) &= \sum_{i,j=1}^{3} \text{Cov}(I_i, J_j) \\ &= \sum_{i=1}^{3} \text{Cov}(I_i, J_i) + \sum_{i \neq j} \text{Cov}(I_i, J_j) \\ &= 3\text{Cov}(I_1, J_1) + 6\text{Cov}(I_1, J_2), \end{aligned}$$

where we used exchangeability in the last step. Since

$$\text{Cov}(I_1, J_1) = E[I_1 J_1] - E[I_1]E[J_1] = 0 - \frac{1}{6} \cdot \frac{1}{2} = -\frac{1}{12}$$

and

$$\text{Cov}(I_1, J_2) = E[I_1 J_2] - E[I_1]E[J_2] = \frac{1}{6} \cdot \frac{3}{5} - \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{60},$$

we conclude that

$$\text{Cov}(R, G) = 3 \cdot (-\frac{1}{12}) + 6 \cdot \frac{1}{60} = -\frac{3}{20}.$$

7. (8 points) Let X and Y be independent random variables with distributions $\mathrm{Unif}(0,2)$ and $\mathrm{Unif}(1,3)$. Let $W = X + Y$. Find the probability density function of W.

Solution: Probability densities of X and Y are

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } 1 < y < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Density of $W = X + Y$ is the convolution

$$f_w(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx.$$

The product $f_X(x)f_Y(w-x)$ is nonzero and equal to $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ if and only if $0 \le x \le 2$ and $1 \le w - x \le 3$. The second inequality is equivalent to $w - 3 \le x \le w - 1$. To simultaneously satisfy $0 \le x \le 2$ and $w - 3 \le x \le w - 1$, we take the larger of the lower bounds and the smaller of the upper bounds on x. Thus $f_X(x)f_Y(w-x) = \frac{1}{4}$ if and only if

$$\max(0, w-3) \le x \le \min(2, w-1). \tag{1}$$

If $w - 1 < 0$ or $w - 3 > 2$, then $\min(2, w-1) < \max(0, w-3)$, so there is no x satisfying (1). Thus $f_W(w) = 0$ if $w < 0$ or $w > 5$. For $1 \le w \le 5$ we have

$$f_W(w) = \int_{\max(0,w-3)}^{\min(2,w-1)} \frac{1}{4} dx = \frac{1}{4}(\min(2, w-1) - \max(0, w-3)).$$

Note that $\min(2, w-1) = 2$ and $\max(0, w-3) = w - 3$ if $w \ge 3$, and that $\min(2, w-1) = w - 1$ and $\max(0, w-3) = 0$ if $w \le 3$. Thus we have

$$f_W(w) = \begin{cases} \frac{1}{4}(w-1) & \text{if } 1 \le w \le 3 \\ \frac{1}{4}(5-w) & \text{if } 3 < w \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Table of Named Distributions

Name	p.m.f. or p.d.f.	$E[X]$	$Var(X)$
Ber(p)	$p_X(0) = 1-p$, $p_X(1) = p$	p	$p(1-p)$
Bin(n, p)	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \le k \le n$	np	$np(1-p)$
Geom(p)	$p_X(k) = p(1-p)^{k-1}$, $k \ge 1$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(k, p)	$p_X(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$, $n \ge k$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
HyperGeom(N, N_A, n)	$p_X(k) = \frac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}$	$\frac{nN_A}{N}$	$\frac{nN_A(N-N_A)(N-n)}{N^2(N-1)}$
Poisson(λ)	$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k \ge 0$	λ	λ
Unif$[a, b]$	$f_X(x) = \frac{1}{b-a}$, $x \in [a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
N(μ, σ^2)	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Exp(λ)	$f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(n, λ)	$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}$, $x \ge 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$

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Table of values for $\Phi(x)$, the CDF of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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