Department of Mathematics, University of Wisconsin-Madison

Math 431 — Final Exam — Spring 2025

NAME : (as it appears on Canvas)

EMAIL: @wisc.edu

PROFESSOR: Mikhail Feldman or Mikhail Ivanov

Time: **120 minutes**

- This exam contains 7 questions some with multiple parts, 12 pages (including the cover) for the total of 91 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Justify your steps.
- For binomial random variable approximations, you will NOT get full credit for your answer if you do not include some explanation of why that approximation is valid.
- You can use the attached table of the values of the $\Phi(x)$ function.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	7	Total
Points:	16	16	11	12	8	20	8	91
Score:								

1. For this problem, you need to provide **answers only**.

(a) (4 points) Find $\text{Corr}(-2X+1, X)$.

(b) (4 points) Suppose A, B, C are events on one probability space. Write down explicit conditions for them to be mutually independent.

(c) (4 points) Let X and Y be random variables such that $X \sim N(0,1)$ and conditional on $X = x$, Y is $N(x,1)$. Find the PDF of Y as an integral.

(d) (4 points) X_1 and X_2 are iid random variables with the cumulative distribution function $F(t)$. Find the cumulative distribution function of $Y = \max(X_1, X_2)$.

2. Suppose that X is a non-negative random variable.

   (a) (5 points) We know that the mean of X is 80. How can you estimate $P(X \geq 120)$ using this (and only this) information?

   (b) (5 points) Suppose that we also learn (in addition to its mean) that the variance of X is 400. How can you improve on your upper bound on $P(X \geq 120)$?

   (c) (6 points) Suppose that we also learn (in addition to its mean and variance) that X is actually the sum of a 200 i.i.d. random variables. How would you estimate the probability $P(X \geq 120)$ now?

3. (a) (5 points) A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll, given that the first roll is 3.

   (b) (6 points) A fair 6-sided die is rolled once. Find the expected number of additional rolls needed to obtain a value at least as large as that of the first roll.

4. We are given a biased coin and we are told that because of manufacturing defects, the probability of heads, denoted by Y, is itself random, with a distribution over the interval $[0, 0.5]$ given by density $1 + 4y$. We toss the coin a fixed number n of times, and we let X be the number of heads obtained.

   (a) (6 points) Find $E[X|Y]$.

   (b) (6 points) Find $E[X]$.

5. (8 points) Suppose $X \sim \mathrm{Exp}(\lambda)$, that is, X is an exponential random variable with parameter $\lambda > 0$. Find the probability density function of X^2.

6. An urn contains six balls, one is red, three are green, two are yellow. Randomly draw three balls from the urn **without** replacement, and let R and G denote the number of red and green balls (within the three balls), respectively.

   (a) (6 points) Compute $P(R = 1, G = 1)$.

   (b) (6 points) Compute the expectations $E[R]$ and $E[G]$.

(c) (8 points) Compute the covariance $\mathrm{Cov}(R, G)$.

7. (8 points) Let X and Y be independent random variables with distributions $\text{Unif}(0, 2)$ and $\text{Unif}(1, 3)$. Let $W = X + Y$. Find the probability density function of W.

Table of Named Distributions

Name	p.m.f. or p.d.f.	$E[X]$	$Var(X)$
Ber(p)	$p_X(0) = 1 - p$, $p_X(1) = p$	p	$p(1-p)$
Bin(n, p)	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \leq k \leq n$	np	$np(1-p)$
Geom(p)	$p_X(k) = p(1-p)^{k-1}$, $k \geq 1$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(k, p)	$p_X(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$, $n \geq k$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
HyperGeom(N, N_A, n)	$p_X(k) = \frac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}$	$\frac{nN_A}{N}$	$\frac{nN_A(N-N_A)(N-n)}{N^2(N-1)}$
Poisson(λ)	$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k \geq 0$	λ	λ
Unif$[a, b]$	$f_X(x) = \frac{1}{b-a}$, $x \in [a, b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
N(μ, σ^2)	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Exp(λ)	$f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(n, λ)	$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}$, $x \geq 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$

Table of values for $\Phi(x)$, the CDF of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED