Department of Mathematics, University of Wisconsin-Madison

Math 431 — Midterm Exam 2 — Solutions — Fall 2024

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Time: **90 minutes**

- This exam contains 5 questions some with multiple parts, 11 pages (including the cover) for the total of 93 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Justify your steps.
- For binomial random variable approximations, you will NOT get full credit for your answer if you do not include some explanation of why that approximation is valid.
- You can use the attached table of the values of the $\Phi(x)$ function.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	Total
Points:	16	29	8	30	10	93
Score:						

1. The following questions are **Answers Only**.

(a) (4 points) A random variable Y has moment generating function

$$M_Y(t) = \frac{1}{15} + \frac{1}{3}e^{2t} + \frac{1}{5}e^{3t} + \frac{1}{3}e^{4t} + \frac{1}{15}e^{7t}.$$

What is the probability that Y is even?

Solution:

$$P(Y \text{ is even}) = P(Y=0) + P(Y=2) + P(Y=4) = \frac{1}{15} + \frac{1}{3} + \frac{1}{3} = \frac{11}{15}.$$

(b) (4 points) Let $X \sim \text{Exp}(1/3)$. What is $P(X < 5 | X \geq 3)$?

Solution:

$$P(X < 5|X \geq 3) = 1 - P(X \geq 3 + 2|X \geq 3) = 1 - P(X \geq 2) = 1 - e^{-\frac{2}{3}}.$$

Or

$$P(X < 5|X \geq 3) = \frac{P(3 \leq X < 5)}{P(X \geq 3)} = \frac{e^{-1} - e^{-5/3}}{e^{-1}} = 1 - e^{-2/3}.$$

(c) (4 points) The joint probability mass function of the random variables (X, Y) is given below:

		Y			
		0	1	2	3
X	0	1/10	1/5	3/10	0
	1	1/5	0	1/10	1/10

What is the marginal probability mass function of Y?

Solution:

Y			
0	1	2	3
3/10	1/5	2/5	1/10

(d) (4 points) Find $\text{Var}(X)$ where $X \sim \text{Unif}\{1, 2, 3, 4\}$.

Solution:

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{4}(1+4+9+16) - \left(\frac{1+2+3+4}{4}\right)^2 = 7.5 - 6.25 = 1.25.$$

2. Let X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} x+y, & \text{for } 0<x<1 \text{ and } 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Check that this is a valid joint pdf.

Solution: $f(x,y) = f_{X,Y}(x,y) \geq 0$ for all x, y and

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy &= \int_{y=0}^{1}\int_{x=0}^{1}(x+y)\,dx\,dy \\ &= \int_{y=0}^{1}\left(\frac{1}{2}x^2+xy\right)\Bigg|_{x=0}^{x=1} dy = \int_0^1 \frac{1}{2}+y\,dy \\ &= \frac{1}{2}(y+y^2)\big|_0^1 = \frac{1}{2}+\frac{1}{2} = 1\end{aligned}$$

(b) (5 points) Find the marginal pdfs of X and Y.

Solution: The marginals

$$f_X(x) = \int_{y=0}^{1}(x+y)\,dy = xy+\frac{1}{2}y^2\Bigg|_{y=0}^{y=1} = x+\frac{1}{2} \text{ for } x\in(0,1)$$

otherwise $f_X(x) = 0$. We computed in part (a)

$$f_Y(y) = \int_{x=0}^{1}(x+y)\,dx = \frac{1}{2}+y \qquad \text{for } y \in (0,1).$$

Thus

$$f_X(x) = \begin{cases} \frac{1}{2}+x & : 0<x<1 \\ 0 & : \text{else} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2}+y & : 0<y<1 \\ 0 & : \text{else} \end{cases}.$$

(c) (5 points) Are X and Y independent?

Solution: No. For $x, y \in (0,1)$ we have

$$f_X(x)f_Y(y) = \frac{1}{4}+\frac{1}{2}(x+y)+xy \neq x+y = f_{X,Y}(x,y).$$

(problem 2 continues)

Let X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} x+y, & \text{for } 0<x<1 \text{ and } 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

(d) (7 points) Find $P(X<Y)$.

Solution:

$$P(X<Y) = \iint_{x<y} f_{X,Y}(x,y)\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{y} (x+y)\,dx\,dy$$
$$= \int_{y=0}^{1} \frac{1}{2}x^2 + xy\Big|_{x=0}^{x=y} dy = \int_0^1 \frac{3}{2}y^2\,dy = \frac{1}{2}y^3\Big|_{y=0}^{1} = \frac{1}{2}.$$

(e) (7 points) Find $E[XY]$.

Solution:

$$E[XY] = \int_{y=0}^{1}\int_{x=0}^{1} (xy)(x+y)\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1} (x^2y+xy^2)\,dx\,dy = \int_0^1 \frac{1}{3}x^3y + \frac{1}{2}x^2y^2\Big|_{x=0}^{x=1} dy$$
$$= \int_0^1 \frac{1}{3}y + \frac{1}{2}y^2\,dy = \frac{1}{6}y^2 + \frac{1}{6}y^3\Big|_{y=0}^{1} = \frac{1}{3}.$$

3. (8 points) Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 8x^{-3} & \text{for } x \geq 2 \\ 0 & \text{else.} \end{cases}$$

Let $Y = X^3$. What is the probability density function of Y?

Solution: The CDF of Y is

$$P(Y \leq t) = P(X^3 \leq t) = P(X \leq t^{1/3}).$$

For $t^{1/3} < 2$, i.e. $t < 8$ $P(X \leq t) = 0$. For $t \geq 8$

$$P(Y \leq t) = \int_2^{t^{1/3}} 8x^{-3}\, dx = -4x^{-2}\Big|_{x=2}^{x=t^{1/3}} = 1 - 4t^{-2/3}.$$

So

$$\frac{d}{dt}P(Y \leq t) = \begin{cases} 0 & : t < 8 \\ \text{undefined} & : t = 8 \\ \frac{8}{3}t^{-5/3} & : t > 8. \end{cases} \quad \text{and} \quad f_Y(t) = \begin{cases} 0 & : t < 8 \\ \frac{8}{3}t^{-5/3} & : t \geq 8. \end{cases}$$

4. Each day, John plays 7 games of chess. He wins, draws, or loses each game independently with probabilities 1/2, 1/6, 1/3.

(a) (6 points) Compute the exact probability that John wins 7 times, draws 5 times and loses 2 times this weekend.

Solution: This is a multinomial $\text{Mult}(14, 3, (1/2, 1/6, 1/3))$ so

$$P(7 \text{ wins}, 5 \text{ draws}, \ 2 \text{ losses}) = \binom{14}{7,5,2} \left(\frac{1}{2}\right)^7 \left(\frac{1}{6}\right)^5 \left(\frac{1}{3}\right)^2$$

$$= \frac{14!}{7!\,5!\,2!} \left(\frac{1}{2}\right)^7 \left(\frac{1}{6}\right)^5 \left(\frac{1}{3}\right)^2.$$

(b) (8 points) John calls a day "great" if he wins all 7 of the games he plays that day. Approximate that probability that over the next $256(= 2^8)$ days John has 3 great days.

Solution: The probability of a great day is $(1/2)^7 = 2^{-7}$. So

$$P(3 \text{ great days over } 256 \text{ days}) = P(\text{Bin}(2^8, 2^{-7}) = 3).$$

We can use Poisson approximation because $np^2 = 2^8 \times 2^{-14} = 2^{-6} = \frac{1}{64} \leq \frac{1}{10}$. So

$$P(\text{Bin}(2^8, 2^{-7}) = 3) \approx P(\text{Poisson}(2 = np) = 3) = \frac{e^{-2}2^3}{3!}.$$

(c) (8 points) John calls a day "good" if he wins at least half of his games (i.e. he wins 4, 5, 6, 7). The probability that a day is good is $\frac{1}{2}$ (you do **NOT** need to check it). Approximate the probability that over the next 256 days the number of good days is between 116 and 132 (inclusive).

Solution: Let S_n be the number of good days over the next 256 days. Then $S_n \sim \text{Bin}(2^8, 1/2)$. So $np(1-p) = 2^6 = 64 > 10$ and therefore the normal approximation to the binomial is good.
We compute $np = 2^7 = 128$ and $np(1-p) = 2^6$ so $\sqrt{np(1-p)} = 2^3 = 8$.

$$\begin{aligned} P(116 \le S_n \le 132) &= P\left(\frac{116-128}{8} \le \frac{S_n - np}{\sqrt{np(1-p)}} \le \frac{132-128}{8}\right) \\ &\approx P\left(\frac{-12}{8} \le Z \le \frac{4}{8}\right) = P(-1.5 \le Z \le .5) = \Phi(.5) - \Phi(-1.5) \\ &= \Phi(.5) - (1 - \Phi(1.5)) \approx 0.6915 + 0.9332 - 1 = 0.6247. \end{aligned}$$

With continuity correction you get

$$P \approx 0.6529.$$

(d) (8 points) John's friend Mark is skeptical that John actually only loses 1/3 of his games. Mark notices that over the last 4 weeks (=28 days) John lost 80 of the $196(= 2^2 \times 7^2)$ games he played. Does $p = 1/3 \approx 0.333$ land in the 95% interval of $\widehat{p} = 80/196 \approx 0.408$?

Solution: The 95% confidence interval is $(\widehat{p} - \varepsilon, \widehat{p} + \varepsilon)$

$$\begin{aligned} 2\Phi(2\varepsilon\sqrt{n}) - 1 &= 0.95 \\ \Phi(2\varepsilon\sqrt{196}) &= 0.975 \\ 28\varepsilon &= 1.96 \\ \varepsilon &= \frac{1.96}{2^2 \times 7} = \frac{196/100}{2^2 \times 7} = \frac{7}{100} = 0.07. \end{aligned}$$

Hence

$$(\widehat{p} - \varepsilon, \widehat{p} + \varepsilon) = (0.338, 0.478).$$

So $\widehat{p} - \varepsilon \approx 0.338 > 1/3$. So p is **not** in the confidence interval.

5. Suppose that X is a random variable with moment generating function

$$M_X(t) = \begin{cases} \dfrac{25}{(5-t)^2} & : t < 5 \\ \infty & : \text{else} \end{cases}.$$

(a) (5 points) Compute the mean of X.

Solution: Write $M = M_X$. The mean is

$$E[X] = M'(0) = \left.\frac{50}{(5-t)^3}\right|_{t=0} = \frac{50}{5^3} = \frac{2}{5}.$$

(b) (5 points) Compute the variance of X.

Solution: The second moment is

$$E[X^2] = M''(0) = \left.\frac{150}{(5-t)^4}\right|_{t=0} = \frac{150}{5^4} = \frac{6}{25}.$$

So the variance is

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{6}{25} - \frac{4}{25} = \frac{2}{25}.$$

Solution:

$$M_X(t) = \frac{5}{5-t} \cdot \frac{5}{5-t},$$

and we can recognize $\frac{5}{5-t}$ near $t = 0$ as a moment generation function of Exp(5) distribution. So, given independent $X_1, X_2 \sim \text{Exp}(\lambda)$, we have $X \sim X_1 + X_2$, so

$$E(X) = E(X_1 + X_2) = E(X_1) + E(X_2) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$
$$\text{Var}(X) = \text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \frac{1}{25} + \frac{1}{25} = \frac{2}{25}$$

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED

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Table of values for $\Phi(x)$, the CDF of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998