Department of Mathematics, University of Wisconsin-Madison
Math 431 — Midterm Exam 2 — Fall 2024

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PROFESSOR: David Clancy   or   Mikhail Ivanov

Time: **90 minutes**

- This exam contains 5 questions some with multiple parts, 11 pages (including the cover) for the total of 93 points. Read the problems carefully and budget your time wisely.

- You are allowed a single sheet of hand-written notes.

- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.

- You **do not** need to simplify binomial coefficients or factorials.

- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.

- Justify your steps.

- For binomial random variable approximations, you will NOT get full credit for your answer if you do not include some explanation of why that approximation is valid.

- You can use the attached table of the values of the $\Phi(x)$ function.

- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	Total
Points:	16	29	8	30	10	93
Score:						

1. The following questions are **Answers Only**.

   (a) (4 points) A random variable Y has moment generating function

$$M_Y(t) = \frac{1}{15} + \frac{1}{3}e^{2t} + \frac{1}{5}e^{3t} + \frac{1}{3}e^{4t} + \frac{1}{15}e^{7t}.$$

   What is the probability that Y is even?

   (b) (4 points) Let $X \sim \text{Exp}(1/3)$. What is $P(X < 5 | X \geq 3)$?

   (c) (4 points) The joint probability mass function of the random variables (X, Y) is given below:

		Y			
		0	1	2	3
X	0	1/10	1/5	3/10	0
	1	1/5	0	1/10	1/10

   What is the marginal probability mass function of Y?

   (d) (4 points) Find $\text{Var}(X)$ where $X \sim \text{Unif}\{1, 2, 3, 4\}$.

2. Let X and Y have joint pdf

$$f_{X,Y}(x, y) = \begin{cases} x + y, & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 points) Check that this is a valid joint pdf.

(b) (5 points) Find the marginal pdfs of X and Y.

(c) (5 points) Are X and Y independent?

(problem 2 continues)

Let X and Y have joint pdf

$$f_{X,Y}(x, y) = \begin{cases} x + y, & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(d) (7 points) Find $P(X < Y)$.

(e) (7 points) Find $E[XY]$.

3. (8 points) Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} 8x^{-3} & \text{for } x \geq 2 \\ 0 & \text{else.} \end{cases}$$

Let $Y = X^3$. What is the probability density function of Y?

4. Each day, John plays 7 games of chess. He wins, draws, or loses each game independently with probabilities 1/2, 1/6, 1/3.

   (a) (6 points) Compute the exact probability that John wins 7 times, draws 5 times and loses 2 times this weekend.

   (b) (8 points) John calls a day "great" if he wins all 7 of the games he plays that day. Approximate that probability that over the next $256(= 2^8)$ days John has 3 great days.

(c) (8 points) John calls a day "good" if he wins at least half of his games (i.e. he wins 4, 5, 6, 7). The probability that a day is good is $\frac{1}{2}$ (you do **NOT** need to check it). Approximate the probability that over the next 256 days the number of good days is between 116 and 132 (inclusive).

(d) (8 points) John's friend Mark is skeptical that John actually only loses 1/3 of his games. Mark notices that over the last 4 weeks (=28 days) John lost 80 of the $196(= 2^2 \times 7^2)$ games he played. Does $p = 1/3 \approx 0.333$ land in the 95% interval of $\widehat{p} = 80/196 \approx 0.408$?

5. Suppose that X is a random variable with moment generating function

$$M_X(t) = \begin{cases} \dfrac{25}{(5-t)^2} & : t < 5 \\ \infty & : \text{else} \end{cases}.$$

(a) (5 points) Compute the mean of X.

(b) (5 points) Compute the variance of X.

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED

Table of values for $\Phi(x)$, the CDF of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998