Department of Mathematics, University of Wisconsin-Madison

Math 431 — Midterm Exam 1 — Solutions — Fall 2024

NAME : (as it appears on Canvas)

EMAIL: @wisc.edu

PROFESSOR: David Clancy or Mikhail Ivanov

Time: **90 minutes**

- This exam contains 6 questions some with multiple parts, 12 pages (including the cover) for the total of 93 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Justify your steps.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	Total
Points:	16	15	16	26	10	10	93
Score:							

1. (16 points) The following questions are **Answers Only**.

Every morning, Albert flips five fair (5) coins. He calls a day good if at least four (4) coins are heads.

a) What is the probability that today (October 9th) was a good day? Call this number p.

> **Solution:** The number of heads on any day is $\text{Bin}(n = 5, p = 1/2)$ so
>
> $$P(\geq 4 \text{ heads}) = \binom{5}{4}\left(\frac{1}{2}\right)^4 \frac{1}{2} + \binom{5}{5}\left(\frac{1}{2}\right)^5 = \frac{5+1}{2^5} = \frac{3}{16}.$$

b) October has 31 days. Let X be the number of good days Albert has in October. What is the distribution of X? Either name it or write down explicitly the probability mass function.

> **Solution:** Each day is good independently with probability $p = \frac{3}{16}$ (from part (a)). There are $n = 31$ days in October. So $X \sim \text{Bin}(n = 31, p = \frac{3}{16})$.
> Its probability mass function is
>
> $$p_X(k) = \binom{31}{k} p^k (1-p)^{31-k} \qquad \text{for } k = 0, 1, \cdots, 31.$$

c) September 30th was a Monday and a good day for Albert. What is the probability the first good day after September 30th was Monday October 7th?

> **Solution:** Let N be the number of days after September 30th for Albert to have another good day. Then $N \sim \text{Geom}(p)$ where p is as in part (a). Then
>
> $$P(N = 7) = (1-p)^{7-1} p = (1-p)^6 p.$$

d) What is the probability that the first good day *after* September 30th is also on a Monday? You can leave your answer in terms of the number p you computed in part (a).

Solution: Let N be as above. We want to compute

$$\begin{aligned}
P(N = 7, 14, 21, \cdots) &= \sum_{n=1}^{\infty} P(N = 7n) \\
&= \sum_{n=1}^{\infty} (1-p)^{7n-1} p \\
&= \frac{p}{1-p} \sum_{n=1}^{\infty} (1-p)^{7n} \\
&= \frac{p}{1-p} \sum_{n=1}^{\infty} r^n \qquad \text{where } r = (1-p)^7 \\
&= \frac{p}{1-p} \cdot \frac{r}{1-r} = \frac{p}{1-p} \cdot \frac{(1-p)^7}{1-(1-p)^7} = \frac{p(1-p)^6}{1-(1-p)^7}.
\end{aligned}$$

2. You have one fair coin, and one biased coin which lands Heads with probability 3/4.

(a) (5 points) You pick one of the coins uniformly at random and flip it **once**. what is the probability that the coin you picked lands Head?

Solution: Let H be the event you flip heads, F the event you flip the fair coin and B the biased coin. Then, by law of total probability

$$P(H) = P(H|F)P(F) + P(H|B)P(B) = \frac{1}{2} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{5}{8}.$$

(b) (10 points) You pick one of the coins at random and flip it **three times**. It lands Heads all three times. Given this information, what is the probability that the coin you picked is the fair one?

Solution: Let H_3 be the event that we flip heads 3 times. Let B and F be the same as before. We want

$$P(F|H_3) = \frac{P(H_3|F)P(F)}{P(H_3)}.$$

We compute

$$P(H_3 \cap F) = P(H_3|F)P(F) = \left(\frac{1}{2}\right)^3 \frac{1}{2} = \frac{1}{16} = \frac{8}{128}$$

and

$$P(H_3 \cap B) = P(H_3|B)P(B) = \left(\frac{3}{4}\right)^3 \frac{1}{2} = \frac{27}{128}.$$

So $P(H_3) = P(H_3 \cap F) + P(H_3 \cap B) = \frac{27}{128} + \frac{8}{128} = \frac{35}{128}$. Now

$$P(F|H_3) = \frac{P(H_3|F)P(F)}{P(H_3)} = \frac{8/128}{35/128} = \frac{8}{35}.$$

3. The following questions are **Answers Only**.

(a) (4 points) Let X be a random variable with cumulative distribution function

$$F(x) = \begin{cases} \frac{1}{2}e^x & : x \le 0 \\ \frac{1}{2} + \frac{1}{4}x & : 0 \le x \le 2 \\ 1 & : x \ge 2. \end{cases}$$

What is $P(-1 < X \le 1)$?

> **Solution:** $F(1) - F(-1) = \frac{3}{4} - \frac{1}{2}e^{-1}$.

(b) (4 points) A and B are independent events such that $P(A) = 0.44$ and $P(B) = \frac{1}{4} = 0.25$. What is $P(A \cup B)$?

> **Solution:** $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = .44 + .25 - .11 = 0.58$.

(c) (4 points) An urn contains 17 red balls, 13 green balls, and 4 yellow balls. Sample 15 **without replacement**. Let X be the number of green balls drawn. What is the distribution of X? Either name it (with parameters) or write down its probability mass function.

> **Solution:** X is a hypergeometric random variable with parameters $N = 17 + 13 + 4 = 34$ (total balls), $N_A = 13$ (green balls), and $n = 15$ (drawn balls). So $X \sim \text{HyperGeom}(34, 13, 15)$.
>
> Alternatively,
>
> $$P(X = k) = \frac{\binom{13}{k}\binom{21}{15-k}}{\binom{34}{15}} \qquad \text{for } k = 0, 1, 2, \ldots, 13, \quad \text{and 0 otherwise.}$$

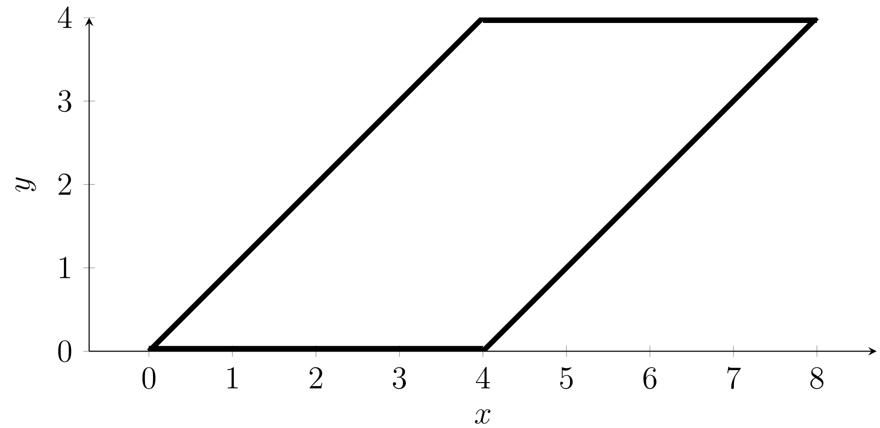
(d) (4 points) Let X be a random variable with probability mass function

$$P(X = 0) = \frac{1}{3} \qquad \text{and} \qquad P(X = n) = c\left(\frac{1}{4}\right)^n \text{ for } n = 1, 2, \ldots.$$

What is the value of c?

> **Solution:** We need
>
> $$1 = \sum_{n=0}^{\infty} P(X = n) = \frac{1}{3} + \sum_{n=1}^{\infty} c\frac{1}{4^n} = \frac{1}{3} + c\frac{1/4}{1 - 1/4} = \frac{1}{3} + \frac{c}{3}$$
>
> so $c = 2$.

4. Sample a point (X, Y) uniformly at random inside the parallelogram with vertices $(0, 0)$, $(4, 0)$, $(8, 4)$, and $(4, 4)$. This is depicted in figure below (and copied on the second page of this problem).



(a) (4 points) What is $P(Y \leq 3)$?

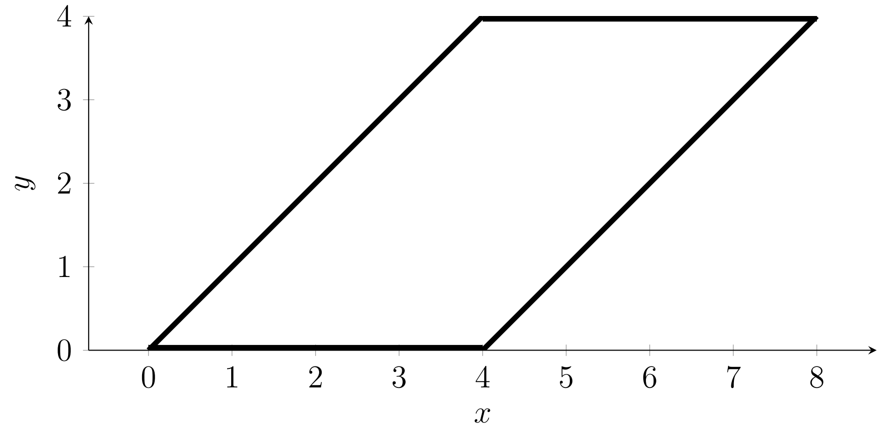
> **Solution:** For $Y \leq 3$, the point (X, Y) needs to be within the parallelogram with vertices $(0, 0), (4, 0), (7, 3), (3, 3)$. This has area $12 = 3 \times 4$. The total area of Ω is $16 = 4 \times 4$. Hence $P(Y \leq 3) = \frac{12}{16} = \frac{3}{4}$.

(b) (4 points) What is $P(X \leq 4, Y \leq 3)$?

> **Solution:** We now wan the trapezoid with vertices $(0, 0), (4, 0), (4, 3), (3, 3)$ which has area $\frac{1+4}{2} \times 3 = 15/2$. Hence $P(X \leq 4, Y \leq 3) = (15/2)/16 = 15/32$.

(c) (4 points) Are X and Y independent? Explain your answer.

> **Solution:** No. The probability $P(X \leq 4) = \frac{1}{2}$ and
>
> $$P(X \leq 4)P(Y \leq 3) = \frac{1}{2}\frac{3}{4} = \frac{3}{8} \neq \frac{15}{32} = P(X \leq 4, Y \leq 3).$$

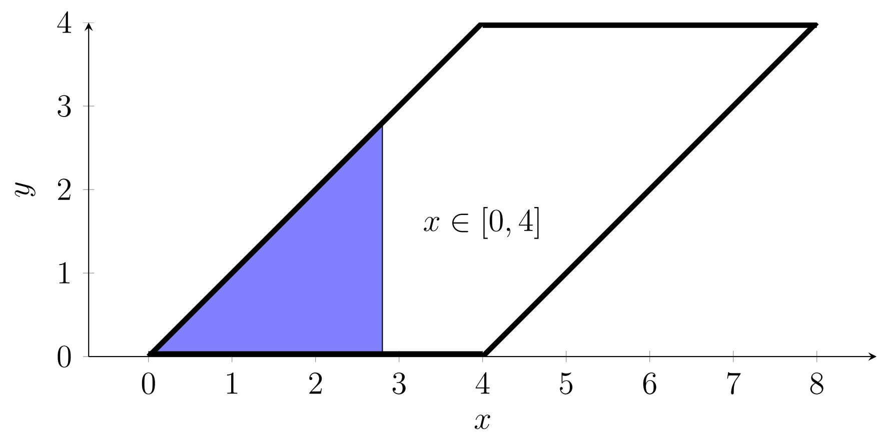


(d) (7 points) What is the cumulative distribution function of X?

Solution: The area of the parallelogram is 16. For $x \leq 0$, $F_X(x) = 0$.
For $x \in [0, 4]$ we can compute

$$F_X(x) = \frac{\text{area of triangle with height } x \text{ and base } x}{\text{area of parallelogram}} = \frac{x^2}{32}.$$

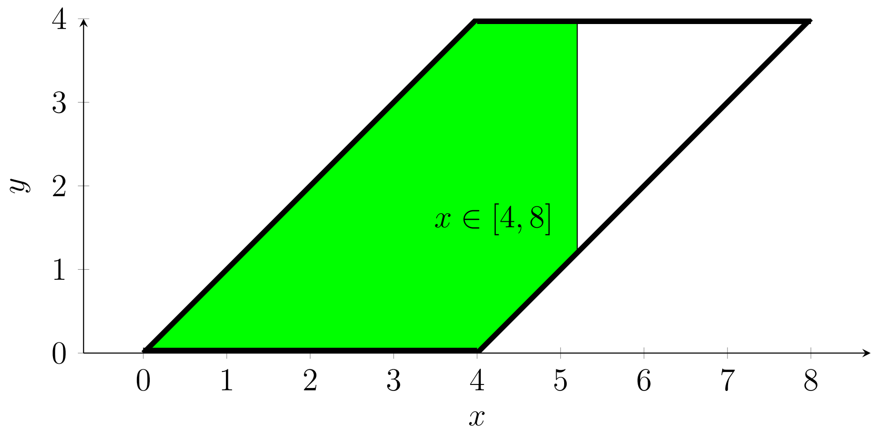
This is in the figure below.



For $x \in [4, 8]$ it is easier to compute the complement probability.

$$F_X(x) = 1 - \frac{\text{area of triangle with base } 8 - x \text{ and height } (8 - x)}{16} = 1 - \frac{(8 - x)^2}{32}.$$

This is the picture below.



Lastly, $F_X(x) = 1$ for all $x \geq 4$. Combining everything:

$$F_X(x) = \begin{cases} 0 & : x \leq 0 \\ \frac{x^2}{32} & : 0 \leq x \leq 4 \\ 1 - \frac{(8-x)^2}{32} = \frac{-x^2+16x-32}{32} & : 4 \leq x \leq 8 \\ 1 & : x \geq 4 \end{cases}.$$

(e) (7 points) What is the probability density function of X?

Solution: We differentiate:

$$F_X'(x) = \begin{cases} 0 & : x \leq 0 \\ \frac{x}{16} & : 0 \leq x \leq 4 \\ -\frac{x}{16} + \frac{1}{2} & : 4 \leq x \leq 8 \\ 0 & : x \geq 4 \end{cases}.$$

5. (10 points) Suppose that in the population of college applicants, being good at baseball is independent of having a good math score on a certain standardized test. A certain college has a simple admissions procedure: admit an applicant if and only if the applicant is good at baseball or has a good math score on the test. Let's show why it makes sense that among students that the college admits, having a good math score decreases their likelihood of being good at baseball, i.e., conditioning on having a good math score decreases the chance of being good at baseball.

   Mathematically speaking show that if A and B are independent with $P(A) = 3/4$ and $P(B) = 1/3$ and $C = A \cup B$, then A and B are conditionally dependent given C, with

$$P(A|B, C) < P(A|C).$$

   where $P(A|B, C) = P(A|BC) = P(A|B \cap C)$.

Solution: First, $B \cap C = B$ so

$$P(A|B, C) = P(A|B \cap C) = \frac{P(A \cap (B \cap C))}{P(B \cap C)} = \frac{P(A \cap B)}{P(B)}$$

and since A, B are independent

$$P(A \cap B) = P(A)P(B).$$

Hence $P(A|B, C) = P(A) = \frac{3}{4}$.

Second,

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)}{P(C)} = \frac{P(A)}{P(A \cup B)}.$$

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = \frac{3}{4} + \frac{1}{3} - \frac{1}{4} = \frac{5}{6}$. So

$$P(A|C) = \frac{3/4}{5/6} = \frac{9}{10}.$$

We can also do this generically. We want to show

$$P(A) = P(A|B, C) < P(A|C) = \frac{P(A)}{P(A \cup B)}$$

so (rearranging the outer-most terms)

$$P(A) < \frac{P(A)}{P(A \cup B)} \qquad \text{which is just} \qquad P(A \cup B) < 1.$$

The last is true because $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = \frac{3}{4} + \frac{1}{3} - \frac{3}{4} \cdot \frac{1}{3} < 1$.

6. (10 points) Bob is playing a video game that has 7 levels. He starts at level 1, and has probability 0.2 of reaching level 2. In general, given that he reaches level j, he has probability 0.2 of reaching level $j+1$, for $1 \leq j \leq 6$. Let X be the highest level that he reaches. Find the PMF of X.

Solution:

Solution: This is similar to geometric random variable. Each time Bob makes it to a level, he gets stuck with probability 4/5 (or he finishes the game and is at level 7).

$$\begin{aligned}
&P(X=1) = P(\text{fails at level 1}) = 4/5, \\
&P(X=2) = P(\text{pass at 1, fails at 2}) = P(\text{pass 1})P(\text{fails at 2}|\text{passes level 1}) = \frac{1}{5}\frac{4}{5} = \frac{4}{25} \\
&P(X=3) = P(\text{pass 1, 2, fail at 3}) = P(\text{pass 1})P(\text{pass 2}|\text{pass 1})P(\text{fail 3}|\text{pass 2}) \\
&\qquad\qquad \frac{1}{5}\frac{1}{5}\frac{4}{5} = \frac{4}{5^3}.
\end{aligned}$$

This continues:

$$P(X=k) = \frac{4}{5^k} \qquad \text{for } k = 1,2,3,4,5,6$$

but

$$P(X=7) = \frac{1}{5^6}.$$

because you need to pass the first 6 levels.

First Name: ____________________ Last Name: ____________________

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM.
SCRATCH WORK WILL NOT BE GRADED

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM
SCRATCH WORK WILL NOT BE GRADED