Department of Mathematics, University of Wisconsin-Madison

Math 431 — Midterm Exam 1 — Fall 2024

NAME : (as it appears on Canvas)

EMAIL: @wisc.edu

PROFESSOR: David Clancy or Mikhail Ivanov

Time: **90 minutes**

- This exam contains 6 questions some with multiple parts, 10 pages (including the cover) for the total of 93 marks. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Justify your steps.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	Total
Points:	16	15	16	26	10	10	93
Score:							

1. (16 points) The following questions are **Answers Only**.

   Every morning, Albert flips five fair (5) coins. He calls a day good if at least four (4) coins are heads.

   a) What is the probability that today (October 9th) was a good day? Call this number p.

   b) October has 31 days. Let X be the number of good days Albert has in October. What is the distribution of X? Either name it or write down explicitly the probability mass function.

   c) September 30th was a Monday and a good day for Albert. What is the probability the first good day after September 30th was Monday October 7th?

   d) What is the probability that the first good day *after* September 30th is also on a Monday? You can leave your answer in terms of the number p you computed in part (a).

2. You have one fair coin, and one biased coin which lands Heads with probability 3/4.

   (a) (5 points) You pick one of the coins uniformly at random and flip it **once**. what is the probability that the coin you picked lands Head?

   (b) (10 points) You pick one of the coins at random and flip it **three times**. It lands Heads all three times. Given this information, what is the probability that the coin you picked is the fair one?

3. The following questions are **Answers Only**.

(a) (4 points) Let X be a random variable with cumulative distribution function

$$F(x) = \begin{cases} \frac{1}{2}e^x & : x \leq 0 \\ \frac{1}{2} + \frac{1}{4}x & : 0 \leq x \leq 2 \\ 1 & : x \geq 2. \end{cases}$$

What is $P(-1 < X \leq 1)$?

(b) (4 points) A and B are independent events such that $P(A) = 0.44$ and $P(B) = \frac{1}{4} = 0.25$. What is $P(A \cup B)$?

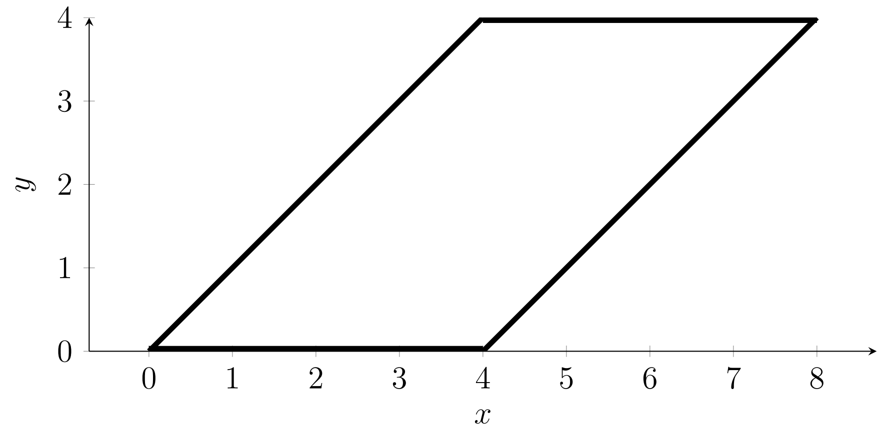
(c) (4 points) An urn contains 17 red balls, 13 green balls, and 4 yellow balls. Sample 15 **without replacement**. Let X be the number of green balls drawn. What is the distribution of X? Either name it (with parameters) or write down its probability mass function.

(d) (4 points) Let X be a random variable with probability mass function

$$P(X = 0) = \frac{1}{3} \qquad \text{and} \qquad P(X = n) = c\left(\frac{1}{4}\right)^n \text{ for } n = 1, 2, \ldots.$$

What is the value of c?

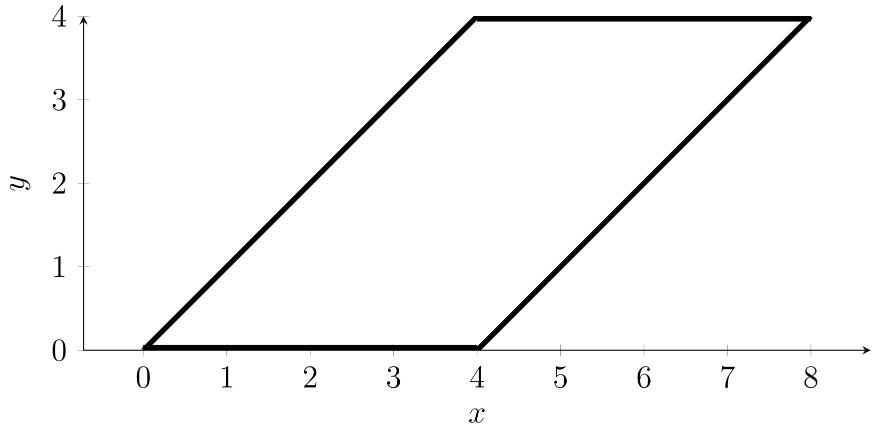
4. Sample a point (X, Y) uniformly at random inside the parallelogram with vertices $(0, 0)$, $(4, 0)$, $(8, 4)$, and $(4, 4)$. This is depicted in figure below (and copied on the second page of this problem).



(a) (4 points) What is $P(Y \leq 3)$?

(b) (4 points) What is $P(X \leq 4, Y \leq 3)$?

(c) (4 points) Are X and Y independent? Explain your answer.



(d) (7 points) What is the cumulative distribution function of X?

(e) (7 points) What is the probability density function of X?

5. (10 points) Suppose that in the population of college applicants, being good at baseball is independent of having a good math score on a certain standardized test. A certain college has a simple admissions procedure: admit an applicant if and only if the applicant is good at baseball or has a good math score on the test. Let's show why it makes sense that among students that the college admits, having a good math score decreases their likelihood of being good at baseball, i.e., conditioning on having a good math score decreases the chance of being good at baseball.

   Mathematically speaking show that if A and B are independent with $P(A) = 3/4$ and $P(B) = 1/3$ and $C = A \cup B$, then A and B are conditionally dependent given C, with

$$P(A|B, C) < P(A|C).$$

   where $P(A|B, C) = P(A|BC) = P(A|B \cap C)$.

6. (10 points) Bob is playing a video game that has 7 levels. He starts at level 1, and has probability 0.2 of reaching level 2. In general, given that he reaches level j, he has probability 0.2 of reaching level $j+1$, for $1 \leq j \leq 6$. Let X be the highest level that he reaches. Find the PMF of X.

First Name: ______________________ Last Name: ______________________

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