Department of Mathematics, University of Wisconsin-Madison

Math 431 — Final Exam — Solutions — Fall 2024

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Time: **120 minutes**

- This exam contains 8 questions some with multiple parts, 14 pages (including the cover) for the total of 109 points. Read the problems carefully and budget your time wisely.
- You are allowed a single sheet of hand-written notes.
- **NO CALCULATORS** or other electronic devices are to be used. Turn off your phone so as to not disturb others.
- You **do not** need to simplify binomial coefficients or factorials.
- Please present your solutions in a clear manner. Cross out any writing that you do not wish to be graded.
- Justify your steps.
- For binomial random variable approximations, you will NOT get full credit for your answer if you do not include some explanation of why that approximation is valid.
- You can use the attached table of the values of the $\Phi(x)$ function.
- If you use an additional page for a particular problem, be sure to **CLEARLY** indicate this on the problem's page so I know to look further.

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	15	8	20	11	20	15	8	109
Score:									

1. (12 points) For this problem, you need to provide **answers only**.

   (a) Let X be the random variable with moment generating function

$$M_X(t) = \frac{1}{4} + \frac{1}{8}e^{2t} + \frac{1}{8}e^{3t} + \frac{1}{2}e^{5t}.$$

   What is the p.m.f. of X?

   Solution:

$$P(X=0) = \frac{1}{4}, \qquad P(X=2) = \frac{1}{8}, \qquad P(X=3) = \frac{1}{8}, \qquad P(X=5) = \frac{1}{2}.$$

   The rest are 0.

   (b) A full house in poker consists of 5 cards where three are the same value and the remaining two are the same value (e.g. three Q's and two 4's). If 5 cards are dealt from a shuffled deck without replacement, what is the probability of drawing a full house?

   Solution: There are

   $\binom{13}{1}$ ways of selecting the value for the 3-of-a-kind

   $\binom{4}{3}$ ways of selecting those 3 of the 4 cars

   $\binom{12}{1}$ ways of selecting the value for the pair

   $\binom{4}{2}$ ways of selecting the pair from the 4 cards of the value.

   Hence

$$p = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}.$$

   (c) Let X and Y be i.i.d. Unif$[0,1]$. Compute the covariance of $X+Y$ and $X-Y$.

   Solution:

$$\begin{aligned}\mathrm{Cov}(X+Y, X-Y) &= \mathrm{Cov}(X,X) - \mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) - \mathrm{Cov}(Y,Y) \\ &= \mathrm{Var}(X) - 0 + 0 - \mathrm{Var}(Y) = 0.\end{aligned}$$

   (d) We flip a coin over and over again. Let N_n be the number of flips required to see the nth Heads. What is $\lim_{n\to\infty} P(N_n > 3n)$?

   Solution: $N_n = \sum_{j=1}^{n} X_j$ where X_j are i.i.d. Geom$(1/2)$. So $E[X_j] = 2$ and

$$P(N_n/n > 3) \leq P\left(\left|\frac{N_n}{n} - 2\right| > 1\right) \to 0.$$

2. (15 points) **Please provide explanation for all parts.**

Suppose that X and Y are random variables with

$$E[X] = 2, \qquad E[X^2] = 5, \qquad E[Y] = 4, \qquad \mathrm{Cov}(X, Y) = -4.$$

(a) What is $\mathrm{Var}(X)$.

Solution: $\mathrm{Var}(X) = E[X^2] - E[X]^2 = 5 - 4 = 1.$

(b) What is $E[XY]$?

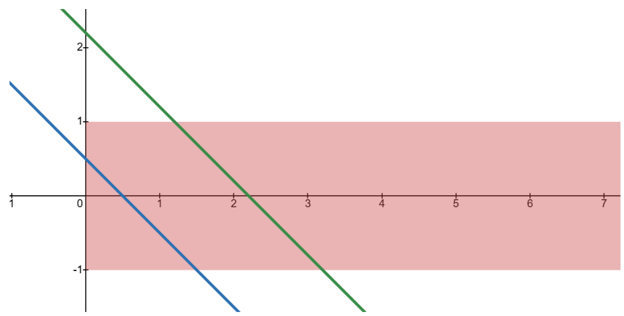
Solution: $E[XY] = \mathrm{Cov}(X, Y) + E[X]E[Y] = -4 + 8 = 4.$

(c) What is $\mathrm{Cov}(2X + Y, X)$?

Solution: $\mathrm{Cov}(2X + Y, X) = 2\,\mathrm{Var}(X) + \mathrm{Cov}(Y, X) = 2 \cdot 1 + (-4) = -2.$

3. (8 points) Let X and Y be two independent random variables where $X \sim \text{Exp}(1)$ and $Y \sim \text{Unif}(-1, 1)$. What is the probability density function of $Z = X + Y$?

Solution: For $z \in (-1, 1)$ we integrate the convolution from $x = 0$ to $x = z$, while for $z \geq 1$ we integrate from $x = z - 1$ to $x = z$. See the picture below.



For $z \in (-1, 1)$:

$$f_Z(z) = \int_0^{z+1} f_X(x) f_Y(z - x)\, dx = \int_0^{z+1} e^{-x} \cdot \frac{1}{2}\, dx = -\frac{1}{2} e^{-x} \bigg|_{x=0}^{z+1} = \frac{1}{2}\left(1 - e^{-z-1}\right).$$

For $z \geq 1$

$$f_Z(z) = \int_{z-1}^{z+1} e^{-x} \cdot \frac{1}{2}\, dx = -\frac{1}{2} e^{-x} \bigg|_{x=z-1}^{z+1} = \frac{1}{2}\left(e^{-(z-1)} - e^{-(z+1)}\right)$$

For $z \leq -1$, $f_Z(z) = 0$. Hence

$$f_Z(z) = \begin{cases} 0 & : z \leq -1 \\ \frac{1}{2}(1 - e^{-z-1}) & : z \in (-1, 1) \\ \frac{1}{2}(e^{-z+1} - e^{-z-1}) & : z \geq 1. \end{cases}$$

4. Let X and Y be jointly continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{2}{y}e^{-2y} & : 0 < x < y \\ 0 & : \text{else} \end{cases}$$

(a) (5 points) Find the marginal density of Y, i.e. $f_Y(y)$. Specify the name of the distribution and its parameters.

Solution: For $y > 0$:

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^y \frac{2}{y}e^{-2y}\,dx = 2e^{-2y}.$$

For $y \leq 0$ the p.d.f. of Y is zero. So $Y \sim \text{Exp}(2)$.

(b) (5 points) Determine the conditional distribution $f_{X|Y}(x|y)$ for each $y > 0$?

Solution: For $x \in (0, y)$:

$$f_{X|Y}(x|y) = \frac{\frac{2}{y}e^{-2y}}{2e^{-2y}} = \frac{1}{y}$$

and for $x \geq y$ the conditional p.d.f. is 0. So, for all $y > 0$:

$$f_{X|Y}(x|y) = \begin{cases} y^{-1} & : x \in (0,y) \\ 0 & : \text{else} \end{cases}$$

and, actually, $X|Y = y \sim \text{Unif}[0, y]$.

Problem 4, cont.

(c) (5 points) What is $E[X|Y=y]$ for each $y>0$?

Solution:

$$E[X|Y=y] = E[\text{Unif}[0,y]] = \frac{y}{2}.$$

(d) (5 points) What is $E[X]$?

Solution: From (c) $E[X|Y] = \frac{1}{2}Y$. So

$$E[X] = E[E[X|Y]] = E[\tfrac{1}{2}Y] = \frac{1}{2}E[\text{Exp}(2)] = \frac{1}{4}.$$

5. There is an urn with 30 red balls, 20 green balls, and 20 yellow balls. You sample 40 balls from the urn without replacement, one after the other.

   (a) (5 points) Given that the last ball drawn is red, what is the probability the first ball drawn is yellow?

Solution: Let D_k be the color of the kth draw and let the colors be $\{\mathtt{R}, \mathtt{G}, \mathtt{Y}\}$ for red, green and yellow. Then D_k are exchangeable and

$$\begin{aligned} P(D_1 = \mathtt{Y} | D_{40} = \mathtt{R}) &= \frac{P(D_1 = \mathtt{Y}, D_{40} = \mathtt{R})}{P(D_{40} = \mathtt{R})} \\ &= \frac{P(D_2 = \mathtt{Y}, D_1 = \mathtt{R})}{P(D_1 = \mathtt{R})} \\ &= P(D_2 = \mathtt{Y} | D_1 = \mathtt{R}) = \frac{20}{69}. \end{aligned}$$

   (b) (6 points) Let X be the number of times that you draw three consecutive balls of the same color. What is $E[X]$?

Solution: Let I_k be the indicator that balls $k-2$, $k-1$, k are the same color, i.e.

$$I_k = \begin{cases} 1 & : D_{k-2} = D_{k-1} = D_k \\ 0 & : \text{else} \end{cases} \qquad k = 3, 4, \dots, 40.$$

There are 38 indicators and, by exchangeability of D_k

$$E[I_k] = E[I_3] = P(D_3 = D_2 = D_1).$$

This probability is

$$\begin{aligned} &P(D_3 = D_2 = D_1 = \mathtt{R}) + P(D_3 = D_2 = D_1 = \mathtt{G}) + P(D_3 = D_2 = D_1 = \mathtt{Y}) \\ &\quad = \frac{30 \cdot 29 \cdot 28}{70 \cdot 69 \cdot 68} + \frac{20 \cdot 19 \cdot 18}{70 \cdot 69 \cdot 68} + \frac{20 \cdot 19 \cdot 18}{70 \cdot 69 \cdot 68} \end{aligned}$$

So

$$E[X] = 38 E[I_3] = 38 \left(\frac{30 \cdot 29 \cdot 28}{70 \cdot 69 \cdot 68} + 2 \cdot \frac{20 \cdot 19 \cdot 18}{70 \cdot 69 \cdot 68} \right)$$

6. Let X be a non-negative random variable. Each part requires an explanation.

(a) (5 points) We know that $E[X] = 200$, what can we say about $P(X > 300)$?

Solution:

$$P(X \geq 300) \leq \frac{E[X]}{300} = \frac{2}{3} = 0.6667.$$

(b) (5 points) Suppose we also know that $\mathrm{Var}(X) = 3600$, what can we say about $P(X > 300)$?

Solution: Chebyshev's inequality says that $P(|X - E[X]| > c) \leq \mathrm{Var}(X)/c^2$ for $c > 0$. Hence

$$\begin{aligned} P(X \geq 300) &= P(X - 200 \geq 300) \leq P(|X - 200| \geq 300) \\ &\leq \mathrm{Var}(X)/100^2 = \frac{3600}{10000} = \frac{36}{400} = 0.36. \end{aligned}$$

(c) (5 points) Suppose we also know that X is the sum of 50 i.i.d. random variables. What is the expectation and variance of the summand? That is if $X = \sum_{j=1}^{50} Y_j$, where Y_j are i.i.d., then what is $E[Y_j]$ and $\mathrm{Var}(Y_j)$?

Solution: Suppose $X = Y_1 + \ldots + Y_{50}$. We have $E[Y_i] = 200/50 = 4$, $\mathrm{Var}(Y_i) = 3600/50 = 72$.

(d) (5 points) Given the information in (c) provide an estimate for the precise value of $P(X > 300)$, assuming it is reasonable.

Solution: We can estimate $P(X > 400)$ by the CLT:

$$P(X > 300) = P\left(\frac{Y_1 + Y_2 + \ldots + Y_{50} - 200}{\sqrt{3600}} > \frac{300 - 200}{60}\right) \approx 1 - \Phi(1.67) \approx 0.0475$$

$1 - \Phi(1.66) \approx 0.0485$

7. A fair 6-sided die is rolled, and then a fair coin is flipped as many times as the die roll says, e.g., if the result of the die roll is a 3, then the coin is flipped 3 times. Let X be the result of the die roll and Y be the number of times the coin lands Heads.

   (a) (5 points) Find the joint pmf of X and Y.

Solution:

$$P(X=k, Y=\ell) = \frac{1}{6}\binom{k}{\ell}\frac{1}{2^k}, \qquad k \in \{1,2,3,4,5,6\}, \quad \ell \in \{0,1,2,\ldots,k\}.$$

   (b) (5 points) Find $P(Y=5)$.

Solution:

$$P(Y=5) = P(Y=5, X=5) + P(Y=5, X=6) = \frac{1}{6}\binom{5}{5}\frac{1}{32} + \frac{1}{6}\binom{6}{5}\frac{1}{64} = \frac{1}{48}.$$

   (c) (5 points) What is $E[X|Y=5]$?

Solution: We use part (c) and Bayes' formula.

$$\begin{aligned} E[X|Y=5] &= 5P(X=5|Y=5) + 6P(X=6|Y=5) \\ &= 5\,\frac{P(X=5,Y=5)}{P(Y=5)} + 6\,\frac{P(X=6,Y=5)}{P(Y=5)} \\ &= 5\,\frac{\frac{1}{6}\binom{5}{5}\frac{1}{32}}{\frac{1}{6}\binom{5}{5}\frac{1}{32} + \frac{1}{6}\binom{6}{5}\frac{1}{64}} + 6\,\frac{\frac{1}{6}\binom{6}{5}\frac{1}{64}}{\frac{1}{6}\binom{5}{5}\frac{1}{32} + \frac{1}{6}\binom{6}{5}\frac{1}{64}} \\ &= 5\frac{1}{4} + 6\frac{3}{4} = 5.75. \end{aligned}$$

8. We have two bins. The first bin has 6 blue marbles and 4 yellow marbles. The second bin has 3 blue marbles and 4 yellow marbles. We flip a coin. If it is heads, we sample from the first bin, otherwise we sample from the second.

   (a) (4 points) If the marble we select is yellow, what is the probability that we chose the first bin?

   Solution: Let A_i be the event that bin i was chosen ($i = 1, 2$) and Y_j the event that draw j ($j = 1, 2$) is yellow.

$$
\begin{aligned}
P(A_1|Y_1) &= \frac{P(Y_1|A_1)P(A_1)}{P(Y_1|A_1)P(A_1) + P(Y_1|A_2)P(A_2)} \\
&= \frac{\frac{4}{10} \cdot \frac{1}{2}}{\frac{4}{10} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{1}{2}} = \frac{14}{34} \approx 0.4118.
\end{aligned}
$$

   (b) (4 points) Now suppose we put the yellow marble from (a) back in the bin it was drawn from and then draw a marble from the same bin. This marble is also yellow. What is the probability we chose the first bin?

   Solution: This question asks for the conditional probability of A_1, given that two draws with replacement from the chosen urn yield yellow. We assume that draws with replacement from the same urn are independent. This translates into conditional independence of Y_1 and Y_2, given A_i.

$$
\begin{aligned}
P(A_1|Y_1Y_2) &= \frac{P(Y_1Y_2|A_1)P(A_1)}{P(Y_1Y_2|A_1)P(A_1) + P(Y_1Y_2|A_2)P(A_2)} \\
&= \frac{P(Y_1|A_1)P(Y_1|A_1)P(A_1)}{P(Y_1|A_1)P(Y_1|A_1)P(A_1) + P(Y_1|A_2)P(Y_1|A_2)P(A_2)} \\
&= \frac{\frac{4}{10} \cdot \frac{4}{10} \cdot \frac{1}{2}}{\frac{4}{10} \cdot \frac{4}{10} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{4}{7} \cdot \frac{1}{2}} = \frac{196}{596} \approx 0.3289.
\end{aligned}
$$

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SCRATCH WORK WILL NOT BE GRADED

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Table of Named Distributions

Name	p.m.f. or p.d.f.	$E[X]$	$Var(X)$
Ber(p)	$p_X(0) = 1-p$, $p_X(1) = p$	p	$p(1-p)$
Bin(n,p)	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \le k \le n$	np	$np(1-p)$
Geom(p)	$p_X(k) = p(1-p)^{k-1}$, $k \ge 1$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(k,p)	$p_X(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$, $n \ge k$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
HyperGeom(N, N_A, n)	$p_X(k) = \frac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}$	$\frac{nN_A}{N}$	$\frac{nN_A(N-N_A)(N-n)}{N^2(N-1)}$
Poisson(λ)	$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k \ge 0$	λ	λ
Unif$[a,b]$	$f_X(x) = \frac{1}{b-a}$, $x \in [a,b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
N(μ, σ^2)	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} 2e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Exp(λ)	$f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(n, λ)	$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda}}{(n-1)!}$, $x \ge 0$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$

Table of values for $\Phi(x)$, the CDF of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998