Midterm 1 — Solutions

1. You have a 6-sided, a 10-sided, and a 12-sided die. You choose one at random and roll it.

   (a) What is the probability that your roll is greater than or equal to 6?

   (b) Given that your roll is greater than or equal to 6, what is the probability you rolled the the 10-sided?

   Solution. Let S, T, W be events that we choose 6-sided, a 10-sided, and a 12-sided die resp. $P(S) = P(T) = P(W) = \frac{1}{3}$. Now

   (a) $P(\geq 6) = P(\geq 6|S) \cdot P(S) + P(\geq 6|T) \cdot P(T) + P(\geq 6|W) \cdot P(W) = \frac{1}{6} \cdot \frac{1}{3} + \frac{5}{10} \cdot \frac{1}{3} + \frac{7}{12} \cdot \frac{1}{3} = \frac{5}{12}$.

   (b)
   $$P(T| \geq 6) = \frac{P(\geq 6|T)P(T)}{P(\geq 6)} = \frac{\frac{5}{10} \cdot \frac{1}{3}}{\frac{5}{12}} = \frac{2}{5}.$$

2. A forest has N elk. Today, $M \geq 2$ of the elk are captured, tagged, and released into the wild. At a later date, again M elk are recaptured at random. Assume that the recaptured elk are equally likely to be any set of M of the elk, e.g., an elk that has been captured does not learn how to avoid being captured again. What is the probability that exactly 2 tagged elks were captured?

   Solution. By the construction of the Hypergeometric distribution, the number of tagged elk in the recaptured sample has distribution $HyperGeom(N, M; M)$. The M tagged elk in this story correspond to the "good" balls and the $N - M$ untagged elk correspond to the "bad" balls. Instead of sampling M balls from the urn, we recapture M elk from the forest.

   So, but the formula for the Hypergeometric distribution:
   $$p = \frac{\binom{M}{2} \cdot \binom{N-M}{M-2}}{\binom{N}{M}}.$$

3. A certain candy company put golden tickets in the wrappers of 1% of their candy bars. If you find a golden ticket you win a trip to their candy factory. Assume that every candy bar is equally likely to have a golden ticket.

   (a) Suppose you bought 10 candy bars. What is the probability you find exactly 1 ticket? exactly 2 tickets?

   (b) Suppose you bought candy bars until you find your first ticket. What is the probability that you must bought more than 100?

   Solution: Number N of tickets has Binomial distribution with parameters $n = 10$ and $p = 0.01$, so

   (a) $P(N = 1) = \binom{10}{1}(0.01)^1(0.99)^9$, $P(N = 2) = \binom{10}{1}(0.01)^2(0.99)^8$.

(b) $P(\text{first success will be after 100 candy}) = P(\text{no success in the first 100 candys}) = (1-0.01)^{100} = (0.99)^{100}$.

4. All the buildings in a certain city have between 1 and 10 floors. Strangely the city has k buildings with k floors for each $k = 1, \dots, 10$. (That is, there is 1 building with a single story, 2 buildings two stories tall, etc.)

   (a) Suppose pick a building uniformly at random to enter. Let X be the number of floors the building has. Write down the probability mass function for X.

   (b) What is the probability that the building has more than 5 floors?

   (c) Suppose you start climbing the stairs and reach the second floor before you get tired and stop. Now what is the probability that the building has more than 5 floors?

   (You might find the identity $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ useful.)

   Solution: First of all, we have $1+2+3+4+5+6+7+8+9+10 = 55$. Now

   (a) Possible values for our random variables X is $\{1,2,3,4,5,6,7,8,9,10\}$ with p.m.f. $P(X=k) = \frac{k}{55}$.

   (b)
   $$P(X > 5) = 1 - P(X \le 5) = \frac{55-1-2-3-4-5}{55} = \frac{40}{55} = \frac{8}{11}.$$

   (c) We know that $X \ge 2$, so
   $$P(X > 5 | X \ge 2) = \frac{P(X > 5, X \ge 2)}{X \ge 2} = \frac{P(X > 5)}{X \ge 2} = \frac{\frac{40}{55}}{\frac{54}{55}} = \frac{40}{54} = \frac{20}{27}.$$

5. Pick a uniformly chosen random point inside the triangle with vertices $(0,0)$, $(10,0)$ and $(10,10)$. (Draw a picture!) Let X be the x-coordinate of the chosen point.

   (a) What is the probability $P(5 < X < 8)$?

   (b) What is the c.d.f. of X?

   (c) What is the p.d.f. of X?

   (d) What is the $\mathbb{E}[X]$?

   (e) Now we pick two uniformly chosen random points inside the triangle. Let X_1 and X_2 be the x-coordinates of the chosen points. What is the probability $P(\min(X_1, X_2) \ge 4)$?

   Solution: First of all we need to find $P(0 \ge X \ge t)$. We can find it using area, or by integrating
   $$P(0 \le X \le t) = \frac{\int_0^t x dx}{\int_0^{10} x dx} = \frac{\frac{t^2}{2}}{\frac{10^2}{2}} = \frac{t^2}{100}.$$

   Now

   (a) $P(5 < X < 8) = P(X < 8) - P(X < 5) = \frac{64-25}{100} = \frac{39}{100}$.

(b) $F_X(t) = P(X \le t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{100} & 0 \le t \le 10 \\ 1 & t > 10. \end{cases}$

(c) $\rho_X(t) = F'(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{50} & 0 \le t \le 10 \\ 0 & t > 10. \end{cases}$

(d) $\mathbb{E}[X] = \int_0^{10} t \cdot \frac{t}{50} dt = \frac{t^3}{150}\Big|_0^{10} = \frac{1000}{150} = 6\frac{2}{3}$.

(e) $P(\min(X_1, X_2) \ge 4) = P(X_1 \ge 4, X_2 \ge 4) = P(X_1 \ge 4) \cdot P(X_2 \ge 4) = (1 - P(X < 4))^2 = (1 - \frac{16}{100}) = \frac{84^2}{100^2}$.

6. Students in a school have a choice of three foreign language courses (F)rench, (G)erman and (S)panish (students can attend more than one course or none of them). Let F, G, S denote the event that a randomly selected student plans to attend respectively French, German and Spanish courses respectively. From past experience suppose the school knows that the probabilities of these events are

$$P(F) = 0.6, \qquad P(G) = 0.6, \qquad P(S) = 0.5.$$

and further

$$P(F \cap G) = 0.4, \qquad P(F \cap S) = 0.3, \qquad P(G \cap S) = 0.2$$

and finally

$$P(F \cap G \cap S) = 0.1.$$

(a) Find the probability that a randomly selected student plans to attend at least one of the three language courses.

(b) Find the probability that a randomly selected student plans to attend **only** Spanish.

Solution:

(a) By the inclusion-exclusion principle we get that

$$\begin{aligned} P(F \cup G \cup S) &= P(F) + P(G) + P(S) - \big(P(FG) + P(FS) + P(GS)\big) + P(FGS) \\ &= 0.6 + 0.6 + 0.5 - 0.4 - 0.3 - 0.2 + 0.1 \\ &= 0.9 \end{aligned}$$

(b) $P(\text{only } S) = P(S) - P(FS) - P(GS) + P(FGS) = 0.5 - 0.3 - 0.2 + 0.1 = 0.1$

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