Final Exam

Name:

Student ID number:

- This is exam for section 002, instructor Mikhail Ivanov.
- **There are 8 problems on the exam, some of them have multiple parts.**
- Read the problems carefully and budget your time wisely.
- No calculators or other electronic devices, please. **Turn off your phone.**
- You do not have to carry out complicated numerical computations, but you should simplify your answer if it is possible with reasonable effort. In particular, geometric series must be evaluated.
- Please present your solutions in a clear manner. **Justify your steps.** A numerical answer without explanation cannot get credit. Cross out the writing that you do not wish to be graded on.

Problem	Points
1	/10
2	/15
3	/15
4	/15
5	/10
6	/10
7	/10
8	/15
Total	/100

1. A vote on certain matter is to be held. A polling company takes a random sample of 10000 voting adults to measure support for the new legislation. Suppose in fact 20% of the total population of voting adults support the legislation. Let X be the number of individuals in the *sample* who **oppose** the legislation.

   (a) Suppose X has Binomial distribution $Bin(n, p)$. Fill in the blanks: the parameters of the Binomial distribution for X are

   $n =$ $\qquad\qquad\qquad$ $p =$

   (b) The mean and the variance of X are:

   $\mu_X =$ $\qquad\qquad\qquad$ $\sigma_X^2 =$

   (c) Use the normal approximation to the Binomial to find an approximate value for the probability $P(7960 < X < 8060)$.

2. Let (X, Y) be a uniformly chosen point from the triangle given by the points $(0, 0), (1, 0)$ and $(1, 1)$.

   (a) What is the joint density of (X, Y)?

   (b) Find the conditional density of X given $Y = y$.

   (c) Find the conditional expectation $E[X|Y]$.

3. The daily number of customers that visit my store is $N \sim \text{Poisson}(\lambda)$. Each customer buys something with probability p, independently of other customers. Let X be the daily number of customers that buy something.

   (a) Write down the conditional probability mass function $p_{X|N}(k|n) = P(X = k|N = n)$. Be careful about the range of n and k.

   (b) Find the conditional expectations $E(X|N = n)$.

   (c) Deduce $E[X]$ from the conditional expectation from the previous part.

4. Suppose that 8 balls are put into 4 boxes with each ball independently put into box i with probability p_i, where $p_i > 0$ and satisfy $p_1 + p_2 + p_3 + p_4 = 1$. In the followings, your answers should be written in terms of p_1, p_2, p_3, p_4.

   (a) Find the probability that all balls fall into the same box.

   (b) Write down the probability that there are exactly 2 balls in each box.

   (c) Let T be the number of boxes containing exactly one ball. Find the expectation $\mathbb{E}[T]$.
   Hint: Consider the relation between T and the random variables $\{X_i\}_{i=1}^4$, where

$$X_i = \begin{cases} 1, & \text{if the i-th box has exactly one ball} \\ 0, & \text{otherwise} \end{cases}$$

5. You are on a game show. There are 3 bins in front of you labeled 1, 2, 3, with bin 1 having 1 green balls and 1 red ball, bin 2 have 3 green and 1 red ball and bin 3 having 5 green and 1 red ball. Without you seeing, the game show host chooses one of the bins at random (so that bin 1 gets selected with probability 1/3, the same for bin 2 and 3) and then selects a ball at random from the bin selected. He tells you that the color of the ball selected is green.

   What is the conditional probability that bin 3 was selected?

6. Let the joint density function of (X, Y) be

$$f(x,y) = \begin{cases} \frac{x+y}{3} & \text{if } 0 < x < 2, \quad 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the covariance $Cov(X, Y)$.

7. Let X be a non-negative random variable with mean 1.

(a) Use Markov's inequality to find an upper bound for $P(X \geq 3)$

(b) Suppose we further know that the variance of X is $\frac{1}{3}$. Use Chebyshev's inequality to find an upper bound for $P(X \geq 3)$.

8. Suppose that the moment generating function of the random variable Y is given by

$$M_Y(t) = \frac{11}{20} + \frac{1}{4}e^{-4t} + \frac{1}{5}e^{5t}.$$

(a) Find $\mathbb{E}[Y^3]$;

(b) Let X be the random variable independent with Y and with the same distribution. Find pmf or pdf of random variable $Z = X + Y$.

Table of values for $\Phi(x)$, the cumulative distribution function of a standard normal random variable

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998