Department of Mathematics, University of Wisconsin-Madison

Math 240 — Midterm Exam II — Spring 2024

NAME : (as it appears on Canvas)

EMAIL: @wisc.edu

PROFESSOR: MIKHAIL IVANOV or LUKE JEFFREYS

Choose your discussion section

LEC-001				LEC-002			
302	Alexandra Bonat	Tu	8:50am	322	Yewei Xu	Mo	8:50am
304	Owen Goff	Tu	11:00am	323	Josiah Jacobsen-Grocott	Mo	9:55am
305	Owen Goff	Tu	12:05am	324	Diego Rojas La Luz	Mo	11:00am
306	Chenghuang Chen	Tu	3:30am	325	Diego Rojas La Luz	Mo	12:05pm
307	Alexandra Bonat	Th	7:45am	327	Yewei Xu	We	7:45am
308	Alexandra Bonat	Th	8:50am	328	Yewei Xu	We	8:50am
309	Chenghuang Chen	Th	9:55am	329	Josiah Jacobsen-Grocott	We	9:55am
310	Robert Argus	Th	1:20pm	330	Josiah Jacobsen-Grocott	We	1:20pm
311	Robert Argus	Th	12:05pm	331	Aviva Englander	We	2:25pm
312	Chenghuang Chen	Th	3:30pm	332	Aviva Englander	We	3:30pm

INSTRUCTIONS:

Time: **90 minutes**

Please write your name on every page.

No Calculators, No Notecards, No Notes

With the exception of the True/False questions, Multiple Choice questions, and Short Answer questions you must justify your claims and use complete sentences in proofs.

You must use correct notation to receive full credit.

For multiple choice questions with answers listed by ◯, choose one answer and completely fill the circle.

For multiple choice questions with answers listed by ☐, choose all of the answers that you believe are correct and completely fill each square.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
Points:	3	3	4	3	4	4	4	3	4	4	3	4	3	6	6	8	66

You may only assume the following results in written proofs:

- The sum and product of two integers is an integer.
- Every integer is either even or odd.
- For nonnegative real numbers a, b, c, and d: if $a < b$, then $ac < bc$. Likewise, if $a < b$ and $c < d$, then $ac < bd$.
- If x is an integer, then there is no integer between x and $x + 1$.
- The product of two integers m and n is odd if and only if both m and n are odd.
- For an integer a, if a^2 is even, then a is even.
- The square of any real number is greater than or equal to 0.
- $\sqrt{2}$ is irrational.

The Master Theorem

Suppose $T(n)$ satisfies the recurrence relation

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d)$$

for $a > 0, b > 1$ and $d \geq 0$.

- If $a < b^d$, then $T(n) = \Theta(n^d)$.
- If $a = b^d$, then $T(n) = \Theta(n \log(n))$.
- If $a > b^d$, then $T(n) = \Theta(n^{\log_b a})$.

Sums of arithmetic series

For $a, d \in \mathbb{R}$ and $n \in \mathbb{Z}^+$,

$$\sum_{i=0}^{n-1}(a + id) = an + \frac{d(n-1)n}{2}.$$

Sums of geometric series

For $a, r \in \mathbb{R}$, $r \neq 1$, and $n \in \mathbb{Z}^+$,

$$\sum_{i=0}^{n-1} ar^i = \frac{a(r^n - 1)}{r - 1}.$$

Any other results you wish to use must be proven.

First Name: ____________________ Last Name: ____________________

1. (3 points) What is smallest positive integer k such that the function

$$f(n) = (n \log(n) + 1)^2$$

is $O(n^k)$?

- ◯ $k = 1$
- ◯ $k = 2$
- ◯ $k = 3$
- ◯ $k = 4$

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2. (3 points) Consider the functions

$$f(n) = 5 \log \log(n), \quad g(n) = n! - 5, \quad h(n) = 6^n + 2n, \quad k(n) = n^3 + 5.$$

Which of the following is a correct ordering of these functions by **increasing** growth rate?

- ◯ $f(n), h(n), g(n), k(n)$
- ◯ $g(n), k(n), f(n), h(n)$
- ◯ $h(n), k(n), f(n), g(n)$
- ◯ $f(n), k(n), h(n), g(n)$

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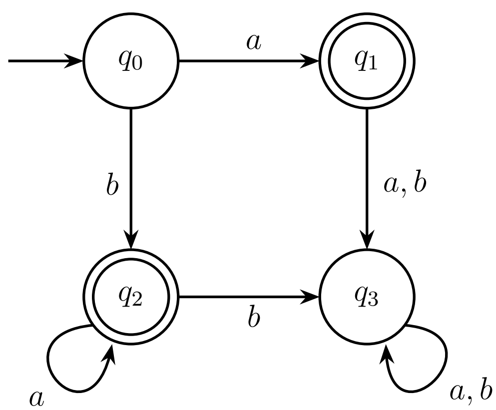
3. (4 points) Consider the following finite state machine:



Select from the list below the statements that are true:

- ☐ the string a is accepted by the finite state machine
- ☐ the string ab is accepted by the finite state machine
- ☐ the machine accepts all strings that start with a b
- ☐ the machine accepts all strings of the form b followed by any number of as

4. (3 points) The inductive step of a proof by induction shows that for a positive integer k, if $11^k - 6$ is divisible by 5, then $11^{k+1} - 6$ is also divisible by 5. In which step of the proof is the **inductive hypothesis** used?

$$\begin{aligned} 11^{k+1} - 6 &= 11(11^k) - 6 && \text{(Step 1)} \\ &= 11(5m+6) - 6 && \text{(Step 2)} \\ &= 55m + 66 - 6 && \text{(Step 3)} \\ &= 5(11m+12) && \text{(Step 4)} \end{aligned}$$

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- ◯ Step 1
- ◯ Step 2
- ◯ Step 3
- ◯ Step 4

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5. (4 points) Recursively define the set $S \subseteq \mathbb{Z}$ by

Basis: $2 \in S$
Recursive rule: If $n \in S$, then $2n - 1 \in S$.

Select from the list below the elements that are in S:

- ☐ 4
- ☐ 5
- ☐ 7
- ☐ 9

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6. (4 points) Consider the following program:

Algorithm **mystic**(n) where $n \in \mathbb{Z}^+$

(1) If $n = 0$ return 0
(2) else return $((n \bmod 2) + \textbf{mystic}(\lfloor \frac{n}{2} \rfloor))$.

Select from the list below the statements that are true:

- ☐ This program is recursive
- ☐ **mystic**(2)=1
- ☐ $n > 0$ is a loop invariant
- ☐ This program never terminates

7. (4 points) Consider the algorithm below which takes as input two 2-dimensional arrays $A[i, j]$ and $B[i, j]$, where each array has n rows and n columns.

Algorithm **sum**(A, B) where input A, B are square arrays with elements from $\mathbb{Z}$.

(1) for $i = 1$ to n
(2)   for $j = 1$ to n
(3) $C[i, j] \leftarrow 0$
(4) for $i = 1$ to n
(5)   for $j = 1$ to n
(6) $C[i, j] \leftarrow A[i, j] + B[i, j]$
(7) return C

The notation $C[i, j] \leftarrow 0$ means the same thing as $C[i, j] := 0$.

Let $f(n)$ be the number of arithmetic operations (additions and assignments) performed while executing the program on two arrays with n rows and n columns. Find the asymptotic growth rate of $f(n)$ in big theta notation. Fill in your answer below.

$$f(n) = \Theta(\quad).$$

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8. (3 points) Select the values of x and y that make the following equation true:

$$\sum_{i=0}^{4}(i + 2) = \sum_{j=x}^{y}(j - 5)$$

- ◯ $x = 2, y = 5$
- ◯ $x = 7, y = 11$
- ◯ $x = 5, y = 9$
- ◯ $x = 7, y = 12$

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9. (4 points) Complete the table below by filling in the values of f_2, f_3, f_4 and f_5 for the recurrence relation:

$$f_0 = 1, \quad f_1 = 1, \quad f_n = (f_{n-1} - 2) \cdot f_{n-2}, \quad n \geq 2.$$

You can use $f_6 = -55$ to check your working.

n	0	1	2	3	4	5	6
f_n	1	1					-55

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10. (4 points) From the recurrence relations below, select those that are linear and homogeneous:

- ☐ $a_n = 2a_{n-1} + 5n$
- ☐ $b_n = \sqrt{2}b_{n-3} - 4b_{n-10}$
- ☐ $c_n = 2(c_{n-1} - c_{n-2})$
- ☐ $d_n = d_{n-1} + d_{n-2} \cdot d_{n-3}$

11. (3 points) Consider the recurrence relation $a_n = 2a_{n-2} - a_{n-1}$. Choose the correct expression for the general solution for a_n, where c_1 and c_2 are constants.

   - ◯ $c_1(-2)^n + c_2(-1)^n$.
   - ◯ $c_1 1^n + c_2 2^n$.
   - ◯ $c_1(-1)^n + c_2 2^n$.
   - ◯ $c_1(-2)^n + c_2 1^n$.

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12. (4 points) Recall that the set of finite length binary strings $B = \{0,1\}^*$ is recursively defined by

   Basis: $\lambda \in B$
   Recursive rule: If $w \in B$, then $w0 \in B$ and $w1 \in B$.

   Define the function $g : B \to B$ by

   Basis: $g(\lambda) = \lambda$
   Recursive rule: If $w \in B$, then $g(w0) = 1g(w)$ and $g(w1) = 0g(w)$.

   Select all of the statements below that are true.

   - ☐ $g(0) = 1$
   - ☐ $g(011) = 011$
   - ☐ $g(01) = 01$
   - ☐ $g(00) = 10$

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13. (3 points) It is determined that the time complexity $T(n)$ of a divide and conquer algorithm satisfies the recurrence relation:

$$T(n) = 2T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + T\left(\left\lceil \frac{n}{3} \right\rceil\right) + 6n - 5.$$

   After simplifying this recurrence relation, which of the following statements is true according to the Master Theorem?

   - ◯ $T(n) = \Theta(n \log(n))$
   - ◯ $T(n) = \Theta(n)$
   - ◯ $T(n) = \Theta(\log_3 2)$
   - ◯ $T(n) = \Theta(\log(n))$

14. (6 points) Prove by induction that for all integers $n \geq 1$ we have $2n^2 + n \leq 5^n$.

15. (6 points) Consider the function $f(n) = n^3 + 5n - 6$.

(a) (3 points) Prove directly from the definition (i.e., by specifying witnesses c and n_0) that $f(n) = O(n^3)$.

(b) (3 points) Prove directly from the definition (i.e., by specifying witnesses c and n_0) that $f(n) = \Omega(n^3)$.

16. (8 points) Consider the algorithm **double_power**, computing x^{2^n}, where $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Algorithm **double_power**(x, n) where $x \in \mathbb{R}$, $n \in \mathbb{N}$

(1) $i \leftarrow 0$
(2) $p \leftarrow x$
(3) while $(i < n)$
(4) $\quad i \leftarrow i + 1$
(5) $\quad p \leftarrow p^2$
(6) return p

$\{$output is $x^{2^n}\}$

The notation $i \leftarrow 0$ means the same thing as $i := 0$.

Use the loop invariants $i \leq n$ and $p = x^{2^i}$ to prove program correctness. You do not need to prove that these are loop invariants. You must argue both partial correctness and termination.

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM
SCRATCH WORK WILL NOT BE GRADED

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