

Department of Mathematics, University of Wisconsin-Madison  
 Math 240 — Midterm Exam I — Spring 2024

NAME : (as it appears on Canvas)

EMAIL: @wisc.edu

PROFESSOR: MIKHAIL IVANOV or LUKE JEFFREYS

Choose your discussion section

LEC-001				LEC-002			
302	Alexandra Bonat	Tu	8:50am	322	Yewei Xu	Mo	8:50pm
304	Owen Goff	Tu	11:00am	323	Josiah Jacobsen-Grocott	Mo	9:55pm
305	Owen Goff	Tu	12:05am	324	Diego Rojas La Luz	Mo	11:00am
306	Chenghuang Chen	Tu	3:30am	325	Diego Rojas La Luz	Mo	12:05pm
307	Alexandra Bonat	Th	7:45am	327	Yewei Xu	We	7:45pm
308	Alexandra Bonat	Th	8:50am	328	Yewei Xu	We	8:50pm
309	Chenghuang Chen	Th	9:55am	329	Josiah Jacobsen-Grocott	We	9:55pm
310	Robert Argus	Th	1:20pm	330	Josiah Jacobsen-Grocott	We	1:20pm
311	Robert Argus	Th	12:05pm	331	Aviva Englander	We	2:25pm
312	Chenghuang Chen	Th	3:30pm	332	Aviva Englander	We	3:30pm

**INSTRUCTIONS:**

Time: **90 minutes**

Please write your name on every page.

No Calculators, No Notecards, No Notes

With the exception of the True/False questions, Multiple Choice questions, and Short Answer questions you must justify your claims and use complete sentences in proofs.

You must use correct notation to receive full credit.

For multiple choice questions with answers listed by  $\bigcirc$ , choose one answer and completely fill the circle.

For multiple choice questions with answers listed by  $\square$ , choose all of the answers that you believe are correct and completely fill each square.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total
Points:	4	4	4	4	4	4	4	4	4	4	4	4	6	6	6	66

You may only assume the following results in written proofs:

- The sum and product of two integers is an integer.
- Every integer is either even or odd.
- For nonnegative real numbers  $a, b, c$ , and  $d$ : if  $a < b$ , then  $ac < bc$ . Likewise, if  $a < b$  and  $c < d$ , then  $ac < bd$ .
- If  $x$  is an integer, then there is no integer between  $x$  and  $x + 1$ .
- The product of two integers  $m$  and  $n$  is odd if and only if both  $m$  and  $n$  are odd.
- For an integer  $a$ , if  $a^2$  is even, then  $a$  is even.
- The square of any real number is greater than or equal to 0.
- $\sqrt{2}$  is irrational.

Any other results you wish to use must be proven.

The rest of this page is intentionally left blank. You may put scratch work here, but it will not be graded.

1. (4 points) Complete the final column of the following truth table, using T for “true” and F for “false”. (You may add additional columns during your working but only the final column will be graded).

$p$	$q$	$(q \rightarrow p) \vee (p \wedge q)$
T	T	
T	F	
F	T	
F	F	

2. (4 points) Select the sentence below that is the **converse** of the statement “If it is raining, then the field is flooded.”
- “If it is raining, then the field is not flooded.”
  - “If the field is not flooded, then it is not raining.”
  - “If the field is flooded, then it is raining.”
  - “If it is not raining, then the field is not flooded.”

3. (4 points) For the domain of all animals, define the predicates:

- $S(x)$  : “ $x$  has a flipper”
- $F(x)$  : “ $x$  is a fish”

Choose the logical statement below that is the **negation** of the statement “Every animal that has a flipper is a fish.”

- $\forall x (S(x) \rightarrow F(x))$
- $\exists x \neg (S(x) \wedge F(x))$
- $\exists x (S(x) \wedge \neg F(x))$
- $\forall x (S(x) \vee \neg F(x))$

4. (4 points) Given the propositions:

- $p$ : “It is the end of the lecture.”
- $q$ : “I am not happy.”

Which of the following sentences is a valid translation of the logical statement  $\neg p \rightarrow q$ ?

- “It is the end of the lecture and I am not happy.”
- “If it is not the end of the lecture, then I am not happy.”
- “It is not the end of the lecture and I am not happy.”
- “If it is the end of the lecture, then I am happy.”

5. (4 points) Consider the predicate:

$$P(x, y) : "y = x - 1"$$

with the domain  $\mathbb{N}$  of natural numbers for both variables. Select all of the statements below that are true.

$\forall y \exists x P(x, y)$

$\exists y P(0, y)$

$\exists x \forall y P(x, y)$

$\exists x P(x, 0)$

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6. (4 points) Select the mistake that is made in the proof given below:

**Theorem.** For any two integers,  $x$  and  $y$ , if  $x^2 \mid y^2$ , then  $x \mid y$ .

*Proof.* Let  $x = 3$  and  $y = 6$ . Therefore,  $x$  and  $y$  are both integers.  $x^2 = 3^2 = 9$ ,  $y^2 = 6^2 = 36$  and  $9 \mid 36$ . Since  $3 \mid 6$ , then  $x \mid y$ . Therefore, if  $x^2 \mid y^2$ , then  $x \mid y$ .  $\square$

- Generalizing from examples.
  - Misuse of existential instantiation.
  - Failure to properly introduce a variable.
  - Assuming facts that have not yet been proven.
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7. (4 points) Consider the set  $A$  defined by

$$A = \{x \in \mathbb{Z} : x \text{ is odd and } 5 < x < 17\}.$$

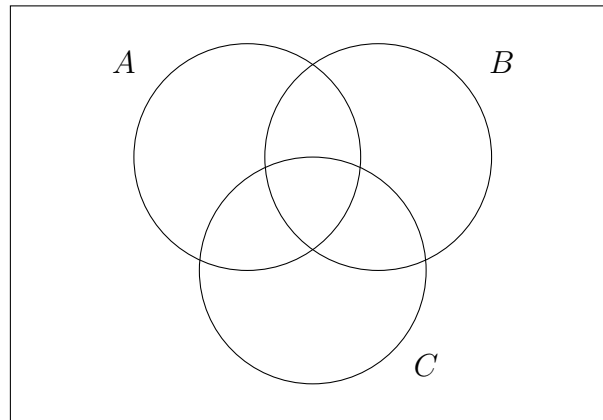
Select the statement below that is true.

- $5 \in A$ ,
  - $16 \in A$ ,
  - $|A| = 6$ ,
  - $\exists x \in A (\exists k \in \mathbb{Z} (x = k^2))$ .
- 

8. (4 points) Let  $A = \{1, 5, 8\}$ ,  $B = \{1, 3, 5\}$ , and  $C = \{1, 2\}$ . Compute  $A \times (B \cup C)$ .

9. (4 points) On the Venn diagram below, shade in the portion representing the set

$$(A - (B \cup C)) \cup ((B \cap C) - A).$$



10. (4 points) Select all of the statements below that are true. Be mindful of the notation being used.

- $\forall x \in \mathbb{R}, \lfloor x + 1 \rfloor = \lfloor x \rfloor + 1$
- $\lceil -1.1 \rceil = -2$
- $\lfloor 3.9 \rfloor = 3$
- $\forall x \in \mathbb{R}, \lceil 2x \rceil = 2\lceil x \rceil$

11. (4 points) Consider the functions  $f : \mathbb{N} \rightarrow \mathbb{Z}$  and  $g : \mathbb{Z} \rightarrow \mathbb{R}$  defined by  $f(x) = x + 2$  and  $g(x) = x^2$ . Select the function below that is the correct definition of  $h = g \circ f$ .

- $h : \mathbb{R} \rightarrow \mathbb{N}$  with  $h(x) = x^2 + 2$
- $h : \mathbb{R} \rightarrow \mathbb{R}$  with  $h(x) = (x + 2)^2$
- $h : \mathbb{N} \rightarrow \mathbb{R}$  with  $h(x) = (x + 2)^2$
- $h : \mathbb{N} \rightarrow \mathbb{R}$  with  $h(x) = x^2 + 2$

12. (4 points) Select the function below that is the inverse of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = (x - 1)^3 + 3$ .

- $f^{-1}(x) = (x - 1)^3 + 3$
- $f^{-1}(x) = \sqrt[3]{x - 1} + 3$
- $f^{-1}(x) = \sqrt[3]{x - 3} + 1$
- $f^{-1}(x) = (x - 3)^3 - 1$

13. (6 points) Let  $x$  and  $y$  be integers. Prove that if  $xy$  is not divisible by 3, then both  $x$  and  $y$  are not divisible by 3.

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14. (6 points) Prove that for any sets  $A, B, C$ ,

$$(A - B) - (B - C) = A - B.$$

15. (6 points) Is it true that the function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $f(x) = \lfloor \frac{x}{2} + 1 \rfloor$  has an inverse? If yes, find the inverse function (with explanation). If not, provide explanation as to why the inverse function does not exist.



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