Department of Mathematics, University of Wisconsin-Madison

Math 240 — Midterm Exam I — Spring 2024

NAME : (as it appears on Canvas)

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PROFESSOR: MIKHAIL IVANOV or LUKE JEFFREYS

Choose your discussion section

LEC-001				LEC-002			
302	Alexandra Bonat	Tu	8:50am	322	Yewei Xu	Mo	8:50pm
304	Owen Goff	Tu	11:00am	323	Josiah Jacobsen-Grocott	Mo	9:55pm
305	Owen Goff	Tu	12:05am	324	Diego Rojas La Luz	Mo	11:00am
306	Chenghuang Chen	Tu	3:30am	325	Diego Rojas La Luz	Mo	12:05pm
307	Alexandra Bonat	Th	7:45am	327	Yewei Xu	We	7:45pm
308	Alexandra Bonat	Th	8:50am	328	Yewei Xu	We	8:50pm
309	Chenghuang Chen	Th	9:55am	329	Josiah Jacobsen-Grocott	We	9:55pm
310	Robert Argus	Th	1:20pm	330	Josiah Jacobsen-Grocott	We	1:20pm
311	Robert Argus	Th	12:05pm	331	Aviva Englander	We	2:25pm
312	Chenghuang Chen	Th	3:30pm	332	Aviva Englander	We	3:30pm

INSTRUCTIONS:

Time: **90 minutes**

Please write your name on every page.

No Calculators, No Notecards, No Notes

With the exception of the True/False questions, Multiple Choice questions, and Short Answer questions you must justify your claims and use complete sentences in proofs.

You must use correct notation to receive full credit.

For multiple choice questions with answers listed by ◯, choose one answer and completely fill the circle.

For multiple choice questions with answers listed by □, choose all of the answers that you believe are correct and completely fill each square.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total
Points:	4	4	4	4	4	4	4	4	4	4	4	4	6	6	6	66

You may only assume the following results in written proofs:

- The sum and product of two integers is an integer.
- Every integer is either even or odd.
- For nonnegative real numbers a, b, c, and d: if $a < b$, then $ac < bc$. Likewise, if $a < b$ and $c < d$, then $ac < bd$.
- If x is an integer, then there is no integer between x and $x + 1$.
- The product of two integers m and n is odd if and only if both m and n are odd.
- For an integer a, if a^2 is even, then a is even.
- The square of any real number is greater than or equal to 0.
- $\sqrt{2}$ is irrational.

Any other results you wish to use must be proven.

The rest of this page is intentionally left blank. You may put scratch work here, but it will not be graded.

First Name: ____________________ Last Name: ____________________

1. (4 points) Complete the final column of the following truth table, using T for "true" and F for "false". (You may add additional columns during your working but only the final column will be graded).

p	q		$(q \to p) \lor (p \land q)$
T	T		T
T	F		T
F	T		F
F	F		T

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2. (4 points) Select the sentence below that is the **converse** of the statement "If it is raining, then the field is flooded."

   ○ "If it is raining, then the field is not flooded."
   ○ "If the field is not flooded, then it is not raining."
   ● **"If the field is flooded, then it is raining."**
   ○ "If it is not raining, then the field is not flooded."

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3. (4 points) For the domain of all animals, define the predicates:

   - $S(x)$: "x has a flipper"
   - $F(x)$: "x is a fish"

   Choose the logical statement below that is the **negation** of the statement "Every animal that has a flipper is a fish."

   ○ $\forall x\,(S(x) \to F(x))$
   ○ $\exists x\,\neg(S(x) \land F(x))$
   ● $\boldsymbol{\exists x\,(S(x) \land \neg F(x))}$
   ○ $\forall x\,(S(x) \lor \neg F(x))$

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4. (4 points) Given the propositions:

   - p: "It is the end of the lecture."
   - q: "I am not happy."

   Which of the following sentences is a valid translation of the logical statement $\neg p \to q$?

   ○ "It is the end of the lecture and I am not happy."
   ● **"If it is not the end of the lecture, then I am not happy."**
   ○ "It is not the end of the lecture and I am not happy."
   ○ "If it is the end of the lecture, then I am happy."

5. (4 points) Consider the predicate:

$$P(x, y) : \text{“}y = x - 1\text{”}$$

with the domain $\mathbb{N}$ of natural numbers for both variables. Select all of the statements below that are true.

- ■ $\boldsymbol{\forall y \, \exists x \, P(x, y)}$
- □ $\exists y \, P(0, y)$
- □ $\exists x \, \forall y \, P(x, y)$
- ■ $\boldsymbol{\exists x \, P(x, 0)}$

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6. (4 points) Select the mistake that is made in the proof given below:

Theorem. *For any two integers, x and y, if $x^2 \mid y^2$, then $x \mid y$.*

Proof. Let $x = 3$ and $y = 6$. Therefore, x and y are both integers. $x^2 = 3^2 = 9$, $y^2 = 6^2 = 36$ and $9 \mid 36$. Since $3 \mid 6$, then $x \mid y$. Therefore, if $x^2 \mid y^2$, then $x \mid y$. □

- ● **Generalizing from examples.**
- ○ Misuse of existential instantiation.
- ○ Failure to properly introduce a variable.
- ○ Assuming facts that have not yet been proven.

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7. (4 points) Consider the set A defined by

$$A = \{x \in \mathbb{Z} : x \text{ is odd and } 5 < x < 17\}.$$

Select the statement below that is true.

- ○ $5 \in A$,
- ○ $16 \in A$,
- ○ $|A| = 6$,
- ● $\boldsymbol{\exists x \in A \, (\exists k \in \mathbb{Z} \, (x = k^2))}$.

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8. (4 points) Let $A = \{1, 5, 8\}$, $B = \{1, 3, 5\}$, and $C = \{1, 2\}$. Compute $A \times (B \cup C)$.

Solution:

$$A \times (B \cup C) = \{1, 5, 8\} \times \{1, 2, 3, 5\}$$

$$= \{(1,1), (1,2), (1,3), (1,5), (5,1), (5,2), (5,3), (5,5), (8,1), (8,2), (8,3), (8,5)\}$$

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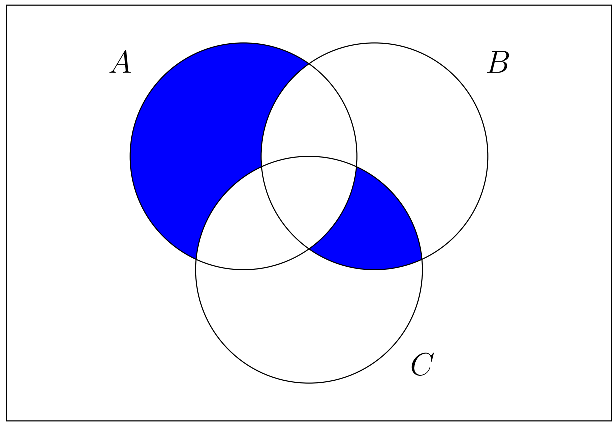
9. (4 points) On the Venn diagram below, shade in the portion representing the set

$$(A - (B \cup C)) \cup ((B \cap C) - A).$$



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10. (4 points) Select all of the statements below that are true. Be mindful of the notation being used.

- ■ **$\forall x \in \mathbb{R}, \lfloor x+1 \rfloor = \lfloor x \rfloor + 1$**
- ☐ $\lceil -1.1 \rceil = -2$
- ■ **$\lfloor 3.9 \rfloor = 3$**
- ☐ $\forall x \in \mathbb{R}, \lceil 2x \rceil = 2\lceil x \rceil$

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11. (4 points) Consider the functions $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{R}$ defined by $f(x) = x + 2$ and $g(x) = x^2$. Select the function below that is the correct definition of $h = g \circ f$.

- ○ $h : \mathbb{R} \to \mathbb{N}$ with $h(x) = x^2 + 2$
- ○ $h : \mathbb{R} \to \mathbb{R}$ with $h(x) = (x + 2)^2$
- ● **$h : \mathbb{N} \to \mathbb{R}$ with $h(x) = (x + 2)^2$**
- ○ $h : \mathbb{N} \to \mathbb{R}$ with $h(x) = x^2 + 2$

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12. (4 points) Select the function below that is the inverse of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x - 1)^3 + 3$.

- ○ $f^{-1}(x) = (x - 1)^3 + 3$
- ○ $f^{-1}(x) = \sqrt[3]{x - 1} + 3$
- ● **$f^{-1}(x) = \sqrt[3]{x - 3} + 1$**
- ○ $f^{-1}(x) = (x - 3)^3 - 1$

13. (6 points) Let x and y be integers. Prove that if xy is not divisible by 3, then both x and y are not divisible by 3.

Solution:

Proof. We will prove the contrapositive. That is, we will show that if one of x or y is divisible by 3 then so is xy.

Without loss of generality, suppose that $3 \mid x$. Then we have that $x = 3k$ for some integer k. So $xy = 3ky = 3(ky)$. Since k and y are both integers, so is ky. Therefore $3 \mid xy$ and we have proven the contrapositive.

Hence, if xy is not divisible by 3, then both x and y are not divisible by 3. □

14. (6 points) Prove that for any sets A, B, C,

$$(A - B) - (B - C) = A - B.$$

Solution:

Proof. We will show that both sets are subsets of each other.

If $x \in (A-B)-(B-C)$, then by definition $x \in (A-B)$ and $x \notin (B-C)$. So, taking the first condition alone, $x \in (A-B)$. Therefore, $(A-B)-(B-C) \subseteq (A-B)$. [We can also use the fact that $D - E \subseteq D$ for any sets D and E.]

Now let $x \in (A-B)$. Then $x \in A$ and $x \notin B$. Since $x \notin B$ we also have that $x \notin (B-C)$ (this is because if $x \in (B-C)$ then it must be true that $x \in B$). So we have that $x \in (A-B)$ and $x \notin (B-C)$. Therefore, $x \in (A-B)-(B-C)$ and we have that $(A-B) \subseteq (A-B)-(B-C)$.

Since both sets are subsets of each other, they must be equal. □

OR

Proof. We have

$$\begin{aligned}
x \in (A-B)-(B-C) &\Leftrightarrow (x \in (A-B)) \wedge (x \notin (B-C)) \\
&\Leftrightarrow (x \in A \wedge x \notin B) \wedge \neg(x \in B \wedge x \notin C) \\
&\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \notin B \vee x \in C) \quad \text{[De Morgan's Law]} \\
&\Leftrightarrow x \in A \wedge (x \notin B \wedge (x \notin B \vee x \in C)) \quad \text{[Associative Law]} \\
&\Leftrightarrow x \in A \wedge (x \notin B) \quad \text{[Absorption Law]} \\
&\Leftrightarrow x \in (A-B)
\end{aligned}$$

□

15. (6 points) Is it true that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = \lfloor \frac{x}{2} + 1 \rfloor$ has an inverse? If yes, find the inverse function (with explanation). If not, provide explanation as to why the inverse function does not exist.

Solution: The function f does not have an inverse. This is because f is not a bijection. Indeed, we can check that

$$f(0) = \left\lfloor \frac{0}{2} + 1 \right\rfloor = \lfloor 1 \rfloor = 1$$

and

$$f(1) = \left\lfloor \frac{1}{2} + 1 \right\rfloor = \left\lfloor \frac{3}{2} \right\rfloor = 1$$

but $0 \neq 1$. Hence f is not injective, so not bijective, and therefore does not have an inverse.

SCRATCH PAPER - DO NOT REMOVE FROM YOUR EXAM
SCRATCH WORK WILL NOT BE GRADED

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