Department of Mathematics, University of Wisconsin-Madison

Math 240 — Final Exam — Solutions — Spring 2024

NAME : (as it appears on Canvas)

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PROFESSOR: MIKHAIL IVANOV or LUKE JEFFREYS

Choose your discussion section

LEC-001				LEC-002			
302	Alexandra Bonat	Tu	8:50am	322	Yewei Xu	Mo	8:50am
304	Owen Goff	Tu	11:00am	323	Josiah Jacobsen-Grocott	Mo	9:55am
305	Owen Goff	Tu	12:05am	324	Diego Rojas La Luz	Mo	11:00am
306	Chenghuang Chen	Tu	3:30am	325	Diego Rojas La Luz	Mo	12:05pm
307	Alexandra Bonat	Th	7:45am	327	Yewei Xu	We	7:45am
308	Alexandra Bonat	Th	8:50am	328	Yewei Xu	We	8:50am
309	Chenghuang Chen	Th	9:55am	329	Josiah Jacobsen-Grocott	We	9:55am
310	Robert Argus	Th	1:20pm	330	Josiah Jacobsen-Grocott	We	1:20pm
311	Robert Argus	Th	12:05pm	331	Aviva Englander	We	2:25pm
312	Chenghuang Chen	Th	3:30pm	332	Aviva Englander	We	3:30pm

INSTRUCTIONS:

Time: **120 minutes**

Please write your name on every page.

No Calculators, No Notecards, No Notes

With the exception of the True/False questions, Multiple Choice questions, and Short Answer questions you must justify your claims and use complete sentences in proofs.

You must use correct notation to receive full credit.

For multiple choice questions with answers listed by ◯, choose one answer and completely fill the circle.

For multiple choice questions with answers listed by □, choose all of the answers that you believe are correct and completely fill each square.

Question:	1	2	3	4	5	6	7	8	9	10	11	12
Points:	4	3	3	4	3	6	4	4	3	3	4	3

Question:	13	14	15	16	17	18	19	20	21	22		Total
Points:	3	4	4	4	4	3	6	6	6	8		92

You may only assume the following results in written proofs:

- The sum and product of two integers is an integer.
- Every integer is either even or odd.
- For nonnegative real numbers a, b, c, and d: if $a < b$, then $ac < bc$. Likewise, if $a < b$ and $c < d$, then $ac < bd$.
- If x is an integer, then there is no integer between x and $x + 1$.
- The product of two integers m and n is odd if and only if both m and n are odd.
- For an integer a, if a^2 is even, then a is even.
- The square of any real number is greater than or equal to 0.
- $\sqrt{2}$ is irrational.

The Master Theorem

Suppose $T(n)$ satisfies the recurrence relation

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^d)$$

for $a > 0, b > 1$ and $d \geq 0$.

- If $a < b^d$, then $T(n) = \Theta(n^d)$.
- If $a = b^d$, then $T(n) = \Theta(n^d \log(n))$.
- If $a > b^d$, then $T(n) = \Theta(n^{\log_b a})$.

Any other results you wish to use must be proven.

First Name: ____________________ Last Name: ____________________

1. (4 points) Consider the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

Select each of the statements below that are true:

- ☐ R is symmetric
- ■ **R is a partial order**
- ■ **R is reflexive**
- ☐ R is not transitive

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2. (3 points) You have 6 people (3 men and 3 women). How many ways are there to order them into a line if the line must start with a woman and end with a man?

- ○ 36
- ○ 144
- ● **216**
- ○ 720

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3. (3 points) What is the coefficient of xy^3 in the expansion of $(3x - 2y)^4$?

- ○ -216
- ● -96
- ○ 81
- ○ 216

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4. (4 points) A coffee shop sells 5 varieties of coffee. What is the minimum number of coffees you must purchase in order to guarantee that you will have had one of the varieties at least 7 times?

Answer: **31**

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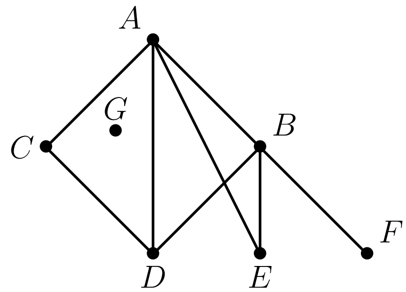
5. (3 points) The dishes on a restaurant's menu contain meat, vegetables, or both. 21 of the dishes contain meat, 19 contain vegetables, and 15 contain both meat and vegetables. How many dishes are on the menu?

- ○ 23
- ● **25**
- ○ 40
- ○ 55

6. (6 points) Consider the following graph:



   (a) (1 point) What is the degree of vertex B. Answer: **4**

   (b) (1 point) Which vertex is isolated? Answer: **G**

   (c) (1 point) Give a cycle of length 4 starting and ending at C. Answer: **C, A, B, D, C**

   (d) (1 point) Does this graph contain C_5 as a subgraph? (Yes/No) Answer: **Yes**

   (e) (1 point) Is F contained in a walk from D to A? (Yes/No) Answer: **Yes**

   (f) (1 point) Is this graph connected? (Yes/No) Answer: **No**

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7. (4 points) A graph G has 5 vertices. Which of the following lists of vertex degrees are **not** possible?

   ☐ 2,2,2,2,2
   ■ **3,3,2,2,1**
   ☐ 2,1,1,1,1
   ■ **5,4,3,2,1**

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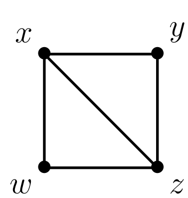
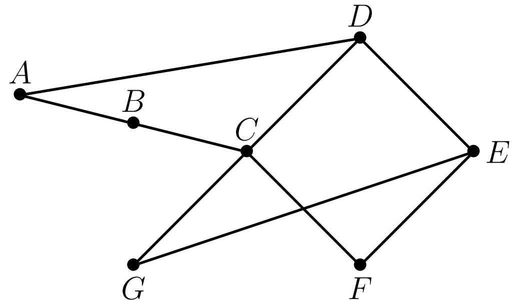
8. (4 points) Consider the following graph G:



   Select the statements below that are true:

   ☐ G has an Euler circuit
   ■ G **has an Euler trail**
   ☐ The path $\langle x, y, z \rangle$ is a Hamiltonian path
   ■ G **has a Hamiltonian cycle**

9. (3 points) Which of the following partitions show that the graph below is bipartite?



- ○ $\{A, B, E\}$ and $\{C, D, F, G\}$
- ○ $\{B, D, F\}$ and $\{A, C, E, G\}$
- ● $\{A, C, E\}$ **and** $\{B, D, F, G\}$
- ○ $\{B, D, G\}$ and $\{A, C, E, F\}$

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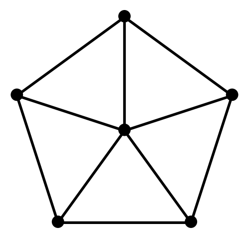
10. (3 points) What is the chromatic number of the graph below?



- ○ 2
- ○ 3
- ● **4**
- ○ 5

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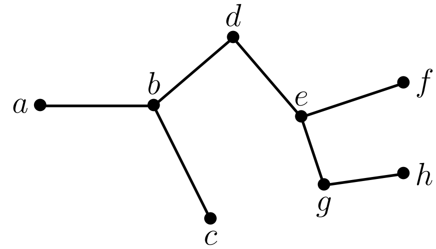
11. (4 points) Consider the following tree:



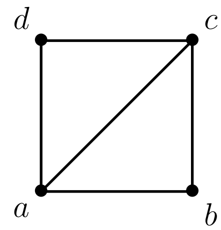
(a) (2 points) If e is chosen to be the root, name all of the vertices that have exactly 1 child.

Answer: d, g

(b) (2 points) If b is chosen to be the root, how many descendants does vertex d have?

Answer: **4**

12. (3 points) How many spanning trees does the graph below have?



- ○ 4
- ○ 6
- ● **8**
- ○ 10

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13. (3 points) For the domain of all students at UW-Madison, define the predicates:

- $C(x)$: "x is taking 15 credits this semester"
- $M(x)$: "x is a Math major"

Choose the logical statement below that is the **negation** of the statement "There exists a Math major who is not taking 15 credits this semester."

- ● $\forall x\,(M(x) \to C(x))$
- ○ $\exists x\,\neg(M(x) \wedge C(x))$
- ○ $\exists x\,(C(x) \wedge \neg M(x))$
- ○ $\forall x\,(M(x) \vee \neg C(x))$

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14. (4 points) Let $A = \{1, 5, 8\}$, $B = \{1, 3, 5\}$, and $C = \{1, 2, 4, 8, 16\}$.

(a) (2 points)

$$A \cup B = \underline{\textbf{\{1,3,5,8\}}}$$

(b) (2 points)

$$C - (A \cup B) = \underline{\textbf{\{2,4,16\}}}$$

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15. (4 points) Consider the function $f : \mathbb{Z} \to \{0, 1, 2, 3, 4\}$ defined by

$$f(x) = (x + 7) \bmod 5.$$

Which of the following are true:

- ■ $f(2) = 4$
- □ f is injective
- ■ f **is surjective**
- □ f has an inverse

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16. (4 points) In a proof that $f(n) = 4n^2 + n + 2$ is $O(n^2)$ which pairs of values below could be used as the witnesses? Select all of the values that could be used.

    ☐ $c = 1$ and $n_0 = 8$

    ☐ $c = 3$ and $n_0 = 1$

    ☐ $c = 5$ and $n_0 = 1$

    ■ $c = 8$ **and** $n_0 = 2$

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17. (4 points) Consider the following program (symbol $:=$ is equivalent to $\leftarrow$):

    mystery(a, b) Input: $a, b \in \mathbb{Z}^+$

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if (b > a) return 0
i := 1
s := a
while (i < b)
      s := s · (a − i)
      i := i + 1
return s
```

    Select the statements below that are true:

    ☐ **mystery**(2,2) $= 1$

    ■ **mystery(3,1) = 3**

    ☐ **mystery**$(a, b) = \binom{a}{b}$

    ■ **mystery**(a, b) **=** $P(a, b)$

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18. (3 points) Let S be the set of graphs recursively defined as follows:

    Basis: The graph consisting of a single vertex with no edges is in S.

    Recursive rule: If a graph G is in S, then the graph obtained by adding a vertex to G and connecting this new vertex to every vertex originally in G is also in S.

    Select which family of graphs are being constructed as the set S:

    ○ Trees

    ○ Cycle graphs

    ○ Complete bipartite graphs

    ● **Complete graphs**

19. (6 points) Prove by induction that for all integers $n \geq 1$ we have

$$\sum_{i=1}^{n}(2i+1) = n(n+2).$$

Solution:

By induction on n.

Base case: For $n = 1$, the LHS is $(2 \cdot 1 + 1) = 3$ and the RHS is $1(1+2) = 3 =$ LHS. So the equality holds for $n = 1$.

Inductive step: Assume that the equation is true for some $k \geq 1$. So we have

$$\sum_{i=1}^{k}(2i+1) = k(k+2).$$

Now,

$$\begin{aligned}\sum_{i=1}^{k+1}(2i+1) &= \sum_{i=1}^{k}(2i+1) + 2(k+1) + 1 \\ &= k(k+2) + 2(k+1) + 1 \\ &= k^2 + 4k + 3 \\ &= (k+1)(k+3) \\ &= (k+1)((k+1)+2),\end{aligned}$$

and so the equality also holds for $k + 1$.

Hence, by induction, the equality holds for all $n \geq 1$. $\square$

20. (6 points) Let $B = \{0, 1\}^*$ be the set of finite length binary strings.

For $u, w \in B$, we say that u is the **reversal** of w if writing w in reverse order gives us u.

For example, if $u = 01$ and $w = 10$, then u is the reversal of w.

Define a relation R on B by

$$uRw \Leftrightarrow u = w \text{ or } u \text{ is the reversal of } w.$$

Prove that R is an equivalence relation.

Solution:

Reflexivity: Since for all $x \in B$, $x = x$, we have that xRx is true for all $x \in B$. Hence, R is reflexive.

Symmetry: Suppose that we have xRy. Then either $x = y$ so that $y = x$ and we also have yRx, or x is the reversal of y in which case y is also the reversal of x and again we have yRx. Hence, R is symmetric.

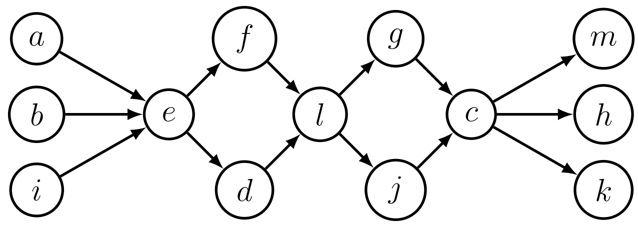
Transitivity: Suppose that we have xRy and yRz. We have 4 cases:

1. $x = y$ and $y = z$. In this case, we also have $x = z$ and so xRz.
2. $x = y$ and y is the reversal of z. In this case, x is also the reversal of z and so we have xRz.
3. x is the reversal of y and $y = z$. In this case, x is also the reversal of z and so we have xRz.
4. x is the reversal of y and y is the reversal of z. In this case, $x = z$ and so we have xRz.

So in all cases we have xRz and R is therefore transitive.

Since R is reflexive, symmetric and transitive it is an equivalence relation. $\square$

21. (6 points) Consider the following digraph:



(a) (2 points) Write down a total order/topological sort for this digraph.

Solution: $a, b, i, e, f, d, l, g, j, c, m, h, k$

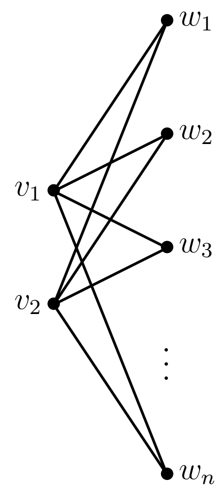
(b) (4 points) Determine how many topological orders/sorts this digraph has. Explain your answer fully.

Solution: The first 3 vertices in the order must be a, b, and c in some order. There are 3! ways to order these. Next, we are forced to have e. Following this, we will have the vertices f and d in some order, and there are 2! ways to order these. Next, we are forced to have l. Following this, we will have the vertices g and j in some order, there are 2! ways to order these. Next we are forced to have c. Finally, we finish with some ordering of the vertices m, h and k. There are 3! ways to order these. Hence, the digraph has

$$3! \cdot 2! \cdot 2! \cdot 3! = 6 \cdot 2 \cdot 2 \cdot 6 = 144$$

topological orders.

22. (8 points) Consider the complete bipartite graph $K_{2,n}$ for $n \geq 1$. It has bipartition $A = \{v_1, v_2\}$ and $B = \{w_1, w_2, \ldots, w_n\}$.



(a) (4 points) Prove that $K_{2,n}$ does not contain any circuits of odd length.

Solution: Each edge in a circuit takes you from a vertex in $A = \{v_1, v_2\}$ to a vertex in $B = \{w_1, w_2, \ldots, w_n\}$ or vice versa. So if a walk starts in A (resp. B), after an odd number of edges we will be at a vertex in B (resp. A) and so cannot be a closed walk and therefore cannot be a circuit. Hence, there are no circuits of odd length. □

(b) (4 points) Every spanning tree for $K_{2,n}$ can be constructed as follows. (You don't need to prove this fact)

Step 1. Choose a vertex w_i for some $1 \leq i \leq n$. This is the first vertex in the spanning tree T.

Step 2. Add the edges $\{v_1, w_i\}$ and $\{v_2, w_i\}$ to T.

Step 3. For each of the remaining vertices w_j, add either the edge $\{v_1, w_j\}$ or $\{v_2, w_j\}$ to T but not both.

By considering the number of choices in each step, explain why the number of spanning trees of $K_{2,n}$ is $n \cdot 2^{n-1}$. You may assume that different choices lead to distinct spanning trees.

Solution: In Step 1, we choose a vertex w_i to start the tree T. We have n choices for doing this.

In Step 2, there are no choices to be made.

In Step 3, for each of the remaining w_j, of which there are $n-1$, we have 2 choices. We add the edge $\{v_1, w_j\}$ or we add the edge $\{v_2, w_j\}$. So we have 2 choices for each of the $n-1$ remaining w_j and so 2^{n-1} choices in Step 3.

So altogether, we have $n \cdot 2^{n-1}$ choices, and so there are $n \cdot 2^{n-1}$ spanning trees of $K_{2,n}$.

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