To appear in
Topology Proceedings

ON SQUARES OF SPACES AND F_σ-SETS

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Abstract. We show that the continuum hypothesis implies there exists a Lindelöf space X such that X^2 is the union of two metrizable subspaces but X is not metrizable. This gives a consistent solution to a problem of Balogh, Gruenhage, and Tkachuk. The main lemma is that assuming the continuum hypothesis there exist disjoint sets of reals X and Y such that X is Borel concentrated on Y, i.e., for any Borel set B if $Y \subseteq B$ then $X \setminus B$ is countable, but $X^2 \setminus \Delta$ is relatively F_σ in $X^2 \cup Y^2$.

In Balogh, Gruenhage, and Tkachuk [1] the following question is asked:

Question 4.1. Let X be a regular paracompact space X such that $X \times X$ is the union of two metrizable subspaces. Must X be metrizable? What if X is Lindelöf?

Theorem 1. *Assume the continuum hypothesis. Then there exists a nonmetrizable regular Lindelöf space X such that X^2 is the union of two metrizable subspaces.*

We first prove the following Lemma.

Lemma 2. *(CH) There are uncountable disjoint sets $X, Y \subseteq 2^\omega$ such that*

2000 *Mathematics Subject Classification.* 03E35 54B10 54E35.
Thanks to the Fields Institute for Research in Mathematical Sciences at the University of Toronto for their support during the time these results were obtained, and to Juris Steprans who directed the special program in set theory and analysis. Also we thank Gary Gruenhage for his translation of the topology problem into properties of sets of reals.

(1) *X is Borel concentrated on Y, i.e. every Borel set in 2^ω containing Y contains all but countably many elements of X,*
(2) *$Y^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$, and*
(3) *$X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$.*

Here $\Delta = \{(x, x) : x \in 2^\omega\}$.

Proof.
We identify the Cantor space 2^ω with the power set $P(\omega)$ of ω. We use $[\omega]^\omega$ to stand for the infinite subsets of ω. Define for $y \in [\omega]^\omega$

$$[y]^{*\omega} = \{x \in [\omega]^\omega : x \subseteq^* y\}$$

where $\subseteq^*$ stands for inclusion mod finite. Let $\langle B_\alpha : \alpha < \omega_1\rangle$ be all Borel subsets of $[\omega]^\omega$. We construct y_α for $\alpha < \omega_1$ so that

(1) $\alpha < \beta$ implies $y_\beta \subseteq^* y_\alpha$ and $y_\beta \neq^* y_\alpha$ and
(2) either $y_\alpha \notin B_\alpha$ or $[y_\alpha]^{*\omega} \subseteq B_\alpha$.

These conditions are easy to get. Given y_β for $\beta < \alpha$ and B_α let $y \in [\omega]^\omega$ be arbitrary with $y \subseteq^* y_\beta$ but $y_\beta \neq^* y$ for each $\beta < \alpha$. If $[y]^{*\omega}$ is not a subset of B_α, then simply take $y_\alpha \in [y]^{*\omega} \setminus B_\alpha$, otherwise take $y_\alpha = y$.

Let

$$X = \{y_\alpha \setminus y_{\alpha+1} : \alpha < \omega_1\} \text{ and } Y = \{y_\alpha : \alpha < \omega_1\}$$

Iff B is any Borel set containing Y, then choose α so that $B = B_\alpha$. At stage α of the construction it must have been that $[y_\alpha]^{*\omega} \subseteq B_\alpha$. But this means that $x_\beta \in B_\alpha$ for all $\beta \geq \alpha$. So X is Borel concentrated on Y.

If we define

$$F = \{(u, v) \in P(\omega) \times P(\omega) : (u \subseteq^* v \text{ or } v \subseteq^* u) \text{ and } u \neq v\}$$

then F is an F_σ set and

$$F \cap (X^2 \cup Y^2) = (Y^2 \setminus \Delta)$$

Also if we define

$$H = \{(u, v) \in P(\omega) \times P(\omega) : u \cap v =^* \emptyset\}$$

then H is an F_σ set and

$$H \cap (X^2 \cup Y^2) = (X^2 \setminus \Delta)$$

This finishes the proof of the Lemma.

QED

Now define the following Michael-line like topology. Suppose that M is a topological space and $X \subseteq M$. Then $M(X)$ is the topological space on the same set but with the following topology. For $x \in X$ we make x an isolated point, i.e., add $\{x\}$ to the topology of $M(X)$. For any point $y \in M \setminus X$ neighborhoods in M form a neighborhood basis for y in $M(X)$. It is easy to see that $M(X)$ is regular for any regular space M and subset $X \subseteq M$.

The following is Exercise 5.5.2 from Engelking [2]:

Proposition 3. *Suppose M is a metric space and $X \subseteq M$. Then $M(X)$ is metrizable iff X is an F_σ set in M.*

Our example is $M(X)$ where X and Y are from the Lemma and $M = X \cup Y$ has its usual (separable metric) topology as a subspace of 2^ω. It follows from the Proposition that $M(X)$ is not metrizable.

Claim 1. $M(X)$ is a Lindelöf space.

Take any open cover $\mathcal{U}$ of $M(X)$. Open sets in $M(X)$ have the form $U \cup Z$ where U is open in M and $Z \subseteq X$ is arbitrary. Then since Y has its standard topology, countably many elements of $\mathcal{U}$ will cover Y, say

$$\{(U_n \cup X_n : n < \omega\} \subseteq \mathcal{U}$$

where each U_n open in M and $X_n \subseteq X$. But since X is Borel concentrated on Y we have that $X \setminus \cup\{U_n : n < \omega\}$ is countable, so we need only add countably many more elements of $\mathcal{U}$ to cover all of $M(X)$.

Claim 2. $M(X)^2$ is the union of two metrizable subspaces.

Let

$M_1 = (X^2 \setminus \Delta) \cup Y^2$ and

$M_2 = (X \times Y) \cup (Y \times X) \cup (X^2 \cap \Delta)$.

Note that M_1 is $N(X^2 \setminus \Delta)$ where $N = (X^2 \setminus \Delta) \cup Y^2$ in its separable metric topology as a subspace of $2^\omega \times 2^\omega$. By the Lemma we have that $X^2 \setminus \Delta$ is relatively F_σ in N and so by Proposition 3 M_1 is metrizable.

To see that M_2 is metrizable use the Bing Metrization Theorem:

> A topological space is metrizable iff it is regular and has a σ-discrete basis.

A family $\mathcal{B}$ of subsets of X is discrete iff every point of X has a neighborhood meeting at most one element of $\mathcal{B}$. σ-discrete means the countable union of discrete families.

Note that for each $x \in X$ the sets $\{x\} \times Y$ and $Y \times \{x\}$ are open in M_2. Let $\mathcal{B}$ be a countable open basis for Y. Then

$$\mathcal{C} = \{U \times \{x\}, \ \ \{x\} \times U, \ \ \{(x,x)\} \ : \ \ x \in X, U \in \mathcal{B}\}$$

is an open basis for M_2. It is σ-discrete. The family $\{\{(x,x)\} : x \in X\}$ is discrete in M_2 since $X^2 \cap \Delta$ is closed in M_2. And for each fixed $U \in \mathcal{B}$ the family $\{U \times \{x\} \ : \ x \in X\}$ is discrete in M_2. (For $(x,x) \in X$ use the neighborhood $\{x\} \times \{x\}$. For (y,x) with $y \in Y$ and $x \in X$ use the neighborhood $Y \times \{x\}$ and for (x,y) use the neighborhood $\{x\} \times Y$.) Similarly, for each $U \in \mathcal{B}$ the family $\{\{x\} \times U \ : \ x \in X\}$ is discrete in M_2. Since $\mathcal{B}$ is countable, M_2 has a σ-discrete basis and is therefor metrizable.

This proves Theorem 1.

QED

The next Theorem is an easy generalization of Theorem 1 using the tower cardinal $\mathfrak{t}$ which is defined as follows. $\mathfrak{t}$ is the minimum cardinality of a set $T \subseteq [\omega]^\omega$ which is linearly ordered by $\subseteq^*$ but there does not exist $z \in [\omega]^\omega$ with $z \subseteq^* y$ for every $y \in T$. Martin's axiom implies that $\mathfrak{t} = \mathfrak{c}$.

Theorem 4. *Suppose $\mathfrak{t} = \mathfrak{c}$. Then there exists a nonmetrizable regular paracompact space X such that X^2 is the union of two metrizable subspaces.*

Proof.

The main Lemma changes to:

Lemma 5. *($\mathfrak{t} = \mathfrak{c}$) There are disjoint sets $X, Y \subseteq 2^\omega$ of cardinality $\mathfrak{c}$ such that*

1. *X is Borel $\mathfrak{c}$-concentrated on Y, i.e., for every Borel set B in 2^ω, if $Y \subseteq B$ then $|X \setminus B| < \mathfrak{c}$,*
2. *$Y^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$, and*
3. *$X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$.*

The proof is similar. The space $M = X \cup Y$ is the same. Since X is not relatively Borel in M we have by Proposition 3 that $M(X)$ is not metrizable. But $M(X)$ is regular and paracompact for any $X \subseteq M$ and metric M, see example 5.1.22 Engelking [2].

QED

Remark. The Michael line is the topological space $M(X)$ where M is the unit interval, $[0,1]$, and X the irrationals in $[0,1]$. Michael Granado in unpublished work has shown that the square of the Michael line is not the union of two metrizable subspaces.

Question 6. *(Using just ZFC) Do there exist disjoint sets of reals X and Y such that X is not F_σ in $X \cup Y$ but $X^2 \setminus \Delta$ is F_σ in $X^2 \cup Y^2$?*

References

1. Balogh, Zoltan; Gruenhage, Gary; Tkachuk, Vladimir; *Additivity of metrizability and related properties*, Proceedings of the International Conference on Set-theoretic Topology and its Applications, Part 2 (Matsuyama, 1994). Topology Appl. **84** (1998), no. 1-3, 91–103.
2. Engelking, Ryszard;*General topology.* Translated from the Polish by the author. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989. viii+529 pp. ISBN: 3-88538-006-4.

The appendix is not intended for final publication but for the electronic version only.

Appendix

> Suppose M is a metric space and $X \subseteq M$. Then $M(X)$ is metrizable iff X is an F_σ in M. (Engelking 5.5.2)

Proof.
Suppose X is not F_σ in M, then $Y = M \setminus X$ is closed in $M(X)$ (since the points of X are isolated, X is open). But Y is not G_δ in $M(X)$. To see, this suppose that $Y = \cap_{n\in\omega} U_n$ were each U_n is open in $M(X)$. Then there would exists V_n open in M and $X_n \subseteq X$ with $U_n = V_n \cup X_n$. But then $Y = \cap_{n\in\omega} V_n$ which contradicts Y is not G_δ in M.

For the converse, suppose X is F_σ in M and write it as the union of closed sets $X = \cup_{n<\omega} C_n$. $M(X)$ is regular so it is enough by the Bing Metrization Theorem to check that it has a σ-discrete base. Let $\mathcal{B}$ be a σ-discrete base for M. We claim that

$$\mathcal{B} \cup \{\{x\} : x \in X\}$$

which is a basis for $M(X)$ is σ-discrete in $M(X)$. $\mathcal{B}$ is σ-discrete in M so it is also σ-discrete in $M(X)$.

$$\{\{x\} : x \in X\} = \cup_{n<\omega} \mathcal{C}_n \text{ where } \mathcal{C}_n = \{\{x\} : x \in C_n\}$$

shows that it is σ-discrete, since for any n if $x \notin C_n$ then $M \setminus C_n$ is a neighborhood of x missing all elements of $\mathcal{C}_n$.

> $M(X)$ is regular paracompact, whenever M is metric. (Engelking 5.1.22)

Proof.
Regular: Given $p \in M$ if $p \in X$ then it is has the clopen neighborhood $\{p\}$, if $p \notin X$, then the neighborhoods of p in M are also a neighborhood basis in $M(X)$.

Paracompact: Let $\mathcal{U}$ be an open cover of basic open sets in $M(X)$. We may assume it has the form:

$$\mathcal{U} = \mathcal{V} \cup \{\{x\} : x \in Z\}$$

where $\mathcal{V}$ is a family of basic open sets in M and $Z = X \setminus \cup\mathcal{V}$. Since metric spaces are hereditarily paracompact, there exists a locally finite refinement $\mathcal{W}$ of $\mathcal{V}$ with $\cup\mathcal{V} = \cup\mathcal{W}$. But then $\mathcal{W} \cup \{\{x\} : x \in Z\}$ is a locally finite refinement of $\mathcal{U}$.

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