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RIGID BOREL SETS AND BETTER QUASIORDER THEORY

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Abstract: A topological space is rigid iff the only autohomeomorphism it has is the identity. We show that no zero dimensional infinite Borel set is rigid. Our proof is partly based on the well foundedness of the Borel Wadge degrees. We generalize this theorem of Martin using the theory of better quasiorders. In addition we give a simple proof of one of the main lemmas of BQO theory, namely the minimal bad array lemma.

1. Lipschitz and Wadge degrees for 2^ω.

In Van Wesep [1978], Lipschitz and Wadge games and degrees for Baire space ω^ω are discussed; here, we need these notions for the Cantor set 2^ω. The Lipschitz game $G_\ell(A,B)$ where $A,B \subset 2^\omega$, is played as follows: I and II alternate playing 0 or 1; in the end, player I has played $\alpha \in 2^\omega$, player II has played $\beta \in 2^\omega$.

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I	II
α_0	β_0
α_1	β_1
.	.
.	.
.	.
α	β

II wins this run of the game $G_\ell(A,B)$ iff $[\alpha \in A \Longleftrightarrow \beta \in B]$.

The Wadge game $G_w(A,B)$ is defined exactly the same, except that II also has the option of passing (provided he makes infinitely many moves). Define

$A \leq_\ell B$ iff II has a winning strategy in $G_\ell(A,B)$.

Similarly we can define $A \leq_w B$. It can be shown that $A \leq_w B$ iff there is a continuous $f:2^\omega \to 2^\omega$ such that $A = f^{-1}[B]$, and $A \leq_\ell B$ iff there is such f which also satisfies $d(f(\alpha),f(\alpha')) \leq d(\alpha,\alpha')$ for all $\alpha,\alpha' \in 2^\omega$, with d the usual Baire space metric.

Define $A \equiv_\ell B$ iff $A \leq_\ell B$ and $B \leq_\ell A$.

Define $A <_\ell B$ iff $A \leq_\ell B$ and not $B \leq_\ell A$.

Similarly define $\equiv_w$ and $<_w$. Both $\leq_\ell$ and $\leq_w$ are quasiorders, i.e. reflexive and transitive relations, and clearly $\leq_\ell$ refines $\leq_w$. The w-degree a of a set $A \subseteq 2^\omega$ is defined to be $a = \{B \subseteq 2^\omega | A \equiv_w B\}$.

Lemmas can all be proved similarly to the results in Van Wesep [1978b] which studies $\leq_\ell$ and $\leq_w$ for subsets of ω^ω.

1.1 Lemma. (AD) (Wadge) For any $A,B \subseteq 2^\omega$ either $A \leq_\ell B$ or $B \leq_\ell (2^\omega \setminus A)$.

1.2 Lemma. (AD) (Martin) There does not exist an infinite descending $<_\ell$ sequence.

1.3 Lemma. (AD) (Wadge) If $A \leq_w 2^\omega \setminus A$ then A contains a nonempty relatively clopen subset B with $B <_w A$.

1.4 Lemma. (AD) If not $A \leq_w 2^\omega \setminus A$ then for any $B \leq_w A$ we have that $B \leq_\ell A$.

This follows from 1.1 since if not $B \leq_\ell A$, then $A \leq_\ell (2^\omega \setminus B)$ and since $B \leq_w A$ implies $(2^\omega \setminus B) \leq_w (2^\omega \setminus A)$; so $A \leq_w (2^\omega \setminus A)$ contradiction.

Note that $A \leq_\ell B$ implies $2^\omega \setminus A \leq_\ell 2^\omega \setminus B$. Hence the first Lemma implies there cannot be three sets $\{A,B,C\}$ which are mutually $\leq_\ell$ incomparable, since if not $A \leq_\ell B$ and not $B \leq_\ell A$ then $A \equiv_\ell (2^\omega \setminus B)$. So if we identify sets and their complements and mod out by $\equiv_\ell$ we get a linear order. The second Lemma says this order is a well order. In Section 3 we will generalize this result. The third Lemma says self dual A have simpler clopen subsets. The fourth Lemma says that non self dual Wadge degrees consist of a single Lipschitz degree.

All of these Lemmas are true locally for determined classes. ZFC proves that they are true for all Borel A and B. Projective determinacy implies that they are true for all projective A and B.

2. Rigid subsets of 2^ω.

If $\Gamma \subset P(2^\omega)$, let $b\Gamma$ denote the set of all finite boolean combinations of sets in Γ. We first show that non-trivial rigid spaces do not appear at the lower levels of the Borel hierarchy in 2^ω.

2.1 Lemma. No non-trivial, i.e. having more than one point, rigid subset of 2^ω is in $b\underset{\sim}{\Pi}^0_2$.

Proof. In van Engelen [1985], topological properties P^2_{-1} = "σ-compact", and P_{4k}, P_{4k+1}, P^1_{4k+2}, P^1_{4k+3}, P^2_{4k+2}, P^2_{4k+3}, for $k < \omega$, were defined, such that if we order these properties by

$$P^2_{-1} < P_{4k} < P_{4k+1} < P^1_{4k+2} < P^1_{4k+3} < P^2_{4k+2} < P^2_{4k+3} < P_{4(k+1)}$$

for each $k < \omega$, then up to homeomorphism, there exists exactly one subset $X^{(i)}_n$ of 2^ω which is $P^{(i)}_n$, and nowhere $P^{(j)}_m$ (i.e. no non-empty clopen subset of the space is $P_m^{(j)}$) for all $P_m^{(j)} < P_n^{(i)}$. These spaces $X_n^{(i)}$ are all homogeneous, i.e. any two points can be interchanged by an autohomeomorphism.

Furthermore, the elements of $b\underset{\sim}{\Pi}_2^0$ are exactly those spaces that have one of the above properties.

Now suppose that some non-trivial rigid subset A of 2^ω is in $b\underset{\sim}{\Pi}_2^0$; then A has at most one isolated point, since interchanging two isolated points of A and leaving the other points fixed is a non-trivial autohomeomorphism of A. If $p \in A$ is isolated, then $A\setminus\{p\} \in b\underset{\sim}{\Pi}_2^0$ is dense in itself; thus, some rigid dense in itself subset of 2^ω is $P_n^{(i)}$ for some i,n, and hence there is such a subset B such that $P_n^{(i)}$ is minimal. Since each non-empty clopen subset of B is also a rigid subset of 2^ω without isolated points, B is $P_n^{(i)}$, and nowhere $P_m^{(j)}$ for all $P_m^{(j)} < P_n^{(i)}$; so if $n < \omega$, then $B \approx X_n^{(i)}$, so B is homogeneous, a contradiction. Hence, $n = -1$, i.e. B is σ-compact. If B is nowhere compact and nowhere countable, then $B \approx \mathbb{Q} \times 2^\omega$ ($\mathbb{Q}$ is the space of rationals) by a result of Alexandroff and Urysohn [1928]; but $\mathbb{Q} \times 2^\omega$ is not rigid, so some non-empty clopen subset C of B is compact or countable. If C is compact, then C is a compact zero-dimensional space without isolated points, so $C \approx 2^\omega$ (Brouwer [1910]), and if C is countable, then $C \approx \mathbb{Q}$ (Sierpiński [1920]); in both cases, we have the required contradiction. □

For $i \in \{0,1\}$, let

$$\theta_i = \{z \in 2^\omega : \exists n \forall m \geq n \quad z(m) = i\}.$$

If $x \in 2^\omega\setminus(\theta_0 \cup \theta_1)$, then x consists of blocks of zeros separated by blocks of ones; define $\varphi : 2^\omega\setminus(\theta_0 \cup \theta_1) \to 2^\omega$ by $\varphi(x)(n) = 0$ iff the n^{th} block of zeros or ones has even length. Let $\ell : 2^{<\omega} \to 2^{<\omega}$ be such that $\varphi(x) = \bigcup_{n<\omega} \ell(x \upharpoonright n)$. For $A \subset 2^\omega$, put

$$A^* = \varphi^{-1}[A] \cup \theta_0.$$

Some more notation: if $s \in 2^{<\omega}$, let $[s] = \{x \in 2^\omega : s$ is an initial segment of $x\}$. In a game $G_W(A,B)$, if τ is a strategy for II, and I plays α, II plays β following τ, then we write $\beta = \tau(\alpha)$, i.e. we also consider τ as a (continuous) function from 2^ω to 2^ω. If τ is a winning strategy for player II, then $A = \tau^{-1}[B]$.

2.2 Lemma. (AD) Let $A \subset 2^\omega$ be such that $A \notin b\underset{\sim}{\Pi}^0_2$, and $A \cap [s] \equiv_W A$ for each $s \in 2^{<\omega}$. Then $A^* \leq_W A$.

Proof. The proof resembles that of Van Wesep [1978b], Theorem 3.1; it esentially appeared in Steel [1977]. If $\neg(A^* \leq_W A)$, then $2^\omega \setminus A \leq_W A^*$ by the Wadge Lemma; let g_0 be a winning strategy for II in $G_W(2^\omega \setminus A, A^*)$. Furthermore, let g_1 be the strategy telling II to copy I's moves, and for $s \in 2^{<\omega}$, let τ_s be a winning strategy for II in $G_W(A, A \cap [s])$, which we may assume starts by writing down s.

Let $n < \omega$, and for each $i \leq n$, let $\sigma_{2i} = \tau_{s_i}$ for some $s_i \in 2^{<\omega}$, and for each $i < n$, let $\sigma_{2i+1} \in \{g_0, g_1\}$.

Consider the finite diagram

$$\begin{array}{cccccc} \sigma_{2n} & \sigma_{2n-1} & \cdots & \sigma_1 & \sigma_0 & \\ A_{2n} & A_{2n-1} & A_2 & A_1 & A_0 & . \end{array}$$

The <u>filling in for</u> $\langle \sigma_i : i \leq 2n \rangle$ consists of filling in a finite column x_i of zeros and ones below A_i $(i \leq 2n)$, such that

(1) $s_n = x_{2n}$;

(2) for each $m < n$, x_{2m} is the response of τ_{s_m} to x_{2m+1};

(3) for each $m < n$, if $\tilde{x}_{2m+1}$ is the response of σ_{2m+1} to x_{2m+2}, then $x_{2m+1} = \ell(\tilde{x}_{2m+1})$ if $\sigma_{2m+1} = g_0$ or $x_{2m+1} = \tilde{x}_{2m+1}$ if $\sigma_{2m+1} = g_1$.

CLAIM. Suppose that $s_i \in 2^{<\omega}$, $\sigma_{2i} = \tau_{s_i}$ for $i < n$. Then there exists $s \in 2^{<\omega}$ such that if $\sigma_{2n} = \tau_s$, then for all $\langle \sigma_{2m+1} : m < n \rangle \in \{g_0, g_1\}^n$, the filling in for $\langle \sigma_i : i \leq 2n \rangle$ has at least n entries below A_0.

Indeed, for $j < 2n$, put $\tilde{\sigma}_j = \varphi\circ\sigma_j$ if j is odd and $\sigma_j = g_0$, $\tilde{\sigma}_j = \sigma_j$ otherwise. If for some $x \in 2^\omega$, $\tilde{\sigma}_j \circ \cdots \circ \tilde{\sigma}_{2n-1}(x)$ is defined for each $j < 2n$, and each $\langle\sigma_{2m+1}:m<n\rangle \in \{g_0,g_1\}^n$, then it is clear that a sufficiently long initial segment of x does the job. If this is not the case, then for each $x \in 2^\omega$ there exist $\langle\sigma_{2m+1}: m<n\rangle \in \{g_0,g_1\}^n$ and $j < 2n$ such that $\tilde{\sigma}_j \circ \cdots \circ \tilde{\sigma}_{2n-1}(x)$ is not defined, i.e. $h_j(x) = \sigma_j \circ \tilde{\sigma}_{j+1} \circ \cdots \circ \tilde{\sigma}_{2n-1}(x) \in \theta_0 \cup \theta_1$. Since for each $i < n$,

(i) $y \in A \iff \tilde{\sigma}_{2i}(y) \in A$;

(ii) $y \in A \iff \tilde{\sigma}_{2i+1}(y) \in A$ if $\sigma_{2i+1} = g_1$;

(iii) $y \notin A \iff g_0(y) \in A^* \iff \varphi g_0(y) \in A$ or $g_0(y) \in \theta_0$

$\iff \tilde{\sigma}_{2i+1}(y)$ is defined and $\tilde{\sigma}_{2i+1}(y) \in A$,

it is clear that in the described situation we have:

$x \in A$ if and only if for some $\langle\sigma_{2m+1}:m<n\rangle \in \{g_0,g_1\}^n$,

$h_j(x) \in \theta_0$ and $|\{i \geq j:\sigma_i = g_0\}|$ is even, or

$h_j(x) \in \theta_1$ and $|\{i \geq j:\sigma_i = g_0\}|$ is odd.

Thus, A is a finite union of sets of the form $h_j^{-1}[\theta_0]$ and $h_j^{-1}[\theta_1]$, which are easily seen to be $b\Pi^0_2$; but $A \notin b\underset{\sim}{\Pi}^0_2$, a contradiction. This proves the claim.

By the claim, we can find an infinite sequence $\langle s_i:i < \omega\rangle$ of elements from $2^{<\omega}$ such that for each $a \in \{0,1\}^\omega$, and each $n < \omega$, the filling in for $\langle\sigma_i:i \leq 2n\rangle$, where $\sigma_{2i} = \tau_{s_i}(i \leq n)$, $\sigma_{2i+1} = g_{a(i)}(i < n)$ has at least n entries in the 0^{th} column. The union of all these fillings is a complete filling in of the infinite diagram

$$\begin{array}{cccccccccc} & \sigma_{2n} & & \sigma_{2n-1} & & \cdots & \sigma_1 & & \sigma_0 & \\ \cdots A_{2n+1} & & A_{2n} & & A_{2n-1} & \cdots & A_2 & A_1 & & A_0, \end{array}$$

i.e. below each A_i we can put $y_i^a \in 2^\omega$ such that for each $m < \omega$,

$$y^a_{2m} = \tau_{s_m}(y^a_{2m+1}),$$

$$\text{and if } a(m) = \begin{matrix} 0 \\ 1 \end{matrix} \text{ then } y^a_{2m+1} = \begin{matrix} (\varphi\circ g_0)(y^a_{2m+2}) \\ y^a_{2m+2} \end{matrix}.$$

Put $M = \{a \in 2^\omega : y^a_0 \in A\}$. By AD, M has the property of Baire, so there exists $s \in 2^{<\omega}$ such that $M \cap [s]$ is either meager or comeager relative to $[s]$, say meager. Let $k = \min\{n : n \notin \mathrm{dom}(s),\ n \text{ odd}\}$, and consider the map $T : [s] \to [s]$ defined

$$\text{by } T(a)(k) = 1 - a(k),$$
$$T(a)(j) = a(j) \text{ if } j \neq k.$$

Clearly, $T[M]$ is of the same category as M, and thus meager. But $y^a_n = y^{T(a)}_n$ for $n > k$, so $y^a_n \in B$ iff $y^{T(a)}_n \notin B$ for $n \leq k$ (use (i),(ii),(iii) from the proof of the claim above), and thus $T[M \cap [s]] = [s]\backslash M$, and we have a contradiction. □

Let $\Gamma \subset \mathscr{P}(2^\omega)$, and put $\Gamma^d = \{2^\omega\backslash B : B \in \Gamma\}$. A subset A of 2^ω is <u>everywhere properly</u> Γ if $A \cap [s] \in \Gamma\backslash\Gamma^d$ for each $s \in 2^{<\omega}$. Γ is <u>reasonably closed</u> if for each $A \in \Gamma$, we have

(1) $A^* \in \Gamma$;

(2) if $B \leq_w A$, then $B \in \Gamma$.

The following theorem is due to Steel [1980].

2.3 Theorem. (AD) If $\Gamma \subset P(2^\omega)$ is reasonably closed, and $A, B \subset 2^\omega$ are everywhere properly Γ, and either both meager or both comeager, then A and B are homeomorphic.

We are now ready to prove the main theorem.

2.4 Theorem. (AD) No non-trivial subset of 2^ω is rigid.

Proof. Suppose there is such a subset. As in the proof of lemma 2.1, we can show that there must also be such a subset which is dense in itself; and using the property of Baire, there is such a subset which is also meager or comeager. So put $\mathscr{A} = \{B \subset 2^\omega : B$ rigid, dense in 2^ω, and meager or comeager$\}$, and let

$$a = \min\{\text{w-degree of } B : B \in \mathscr{A}\}.$$

say a is the w-degree of $A \in \mathcal{A}$.

If $s \in 2^{<\omega}$, and $f_s : 2^\omega \to 2^\omega$ is defined by $f_s(x) = s ^\frown x$, then $A_s = f_s^{-1}[[s] \cap A] \in \mathcal{A}$; since $A_s \equiv_w [s] \cap A$, and $[s] \cap A \leq_w A$, we have $A_s \leq_w A$, whence $A_s \equiv_w A$ by minimality of a. So $A \cap [s] \equiv_w A$ for each $s \in 2^{<\omega}$, and $A \notin b\Pi^0_2$ by lemma 2.1. Thus, by lemma 2.2, $A^* \leq_w A$. Again by minimality of a, and by Lemma 1.3, a is non self-dual, and hence $A \cap [s] \in \Gamma \setminus \Gamma^d$ for each $s \in 2^{<\omega}$, i.e. A is everywhere properly Γ.

Since a is non self-dual by Lemma 1.4 if $B \leq_w A$ then $B \leq_\ell A$. It is easy to see that $B \leq_\ell A$ implies $B^* \leq_\ell A^*$. Player II's strategy in $G_\ell(B^*, A^*)$ is to simply copy player I's moves except when player I switches from playing the digit i to playing $1-i$; at these times a simulated play in $G_\ell(B,A)$ has taken place, so after consulting player II's strategy in that game he plays either one i or two i's to adjust the parity correctly. Hence $\Gamma = \{B \mid B \leq_w A\}$ is reasonably closed. Then with f_s as above, we have that $A_0 = f^{-1}_{\langle 0\rangle}[[0] \cap A]$, $A_1 = f^{-1}_{\langle 1\rangle}[[1] \cap A]$ are also everywhere properly Γ, and both of the same category as A, so $[0] \cap A \approx A_0 \approx A_1 \approx [1] \cap A$; thus, interchanging $[0] \cap A$, $[1] \cap A$ yields a non-trivial autohomeomorphism of A, a contradiction. □

For Theorem 2.3, we only need AD for games with payoff set in $b\Gamma$. Thus, analyzing the above results, we see that restricting the determinacy hypothesis yields Theorem 2.4 for a restricted class of subsets of 2^ω, e.g. PD implies no projective subset of 2^ω is rigid, and the following theorem of ZFC.

2.5 Theorem. No non-trivial Borel subset of 2^ω is rigid.

This answers a question of Eric van Douwen who independently showed that no zero dimensional Borel set which is the union of a $\underset{\sim}{\Pi}^0_2$ and a $\underset{\sim}{\Sigma}^0_2$ is rigid.

An alternative proof of Theorem 2.5 can be deduced from the results in van Engelen [1985] and [1986], as follows.

Put

$$\mathcal{A}_0 = \{A \subset 2^\omega : A \in \underset{\sim}{\Delta}^0_3,\ A \text{ rigid, dense in itself}\},$$

and

$$\mathcal{A}_1 = \{A \subset 2^\omega : A \text{ Borel}, \ A \notin \Delta^0_3, \ A \text{ rigid, dense in itself}\}.$$

In van Engelen [1985], for each limit $\alpha < \omega_1$, and each $n < \omega$, a closed-hereditary topological property $P_{\alpha+n}$ is defined such that, up to homeomorphism, there exists exactly one subset of 2^ω

- X^2_ω which is P_ω, and nowhere $P^{(i)}_n$ for all properties $P^{(i)}_n$ in Lemma 2.1;
- X^2_α which is P_α, and nowhere P_β for all $\beta \in [\omega,\alpha)$ $(\alpha>\omega)$;
- $X^1_{\alpha+n}$ which is $P_{\alpha+n}$, nowhere $P_{\alpha+n-1}$, and which contains no closed copy of $X^2_{\alpha+n-1}$ $(n \in \mathbb{N})$.
- $X^2_{\alpha+n}$ which is $P_{\alpha+n}$, nowhere $P_{\alpha+n-1}$, and such that every non-empty clopen subset contains a closed copy of $X^2_{\alpha+n-1}$ $(n \in \mathbb{N})$.

These spaces are all homogeneous; furthermore, the elements of Δ^0_3 are exactly those spaces that have one of the properties $P_{\alpha+n}$.

Suppose that $\mathcal{A}_0 \neq \emptyset$; then there is a $\beta \in [\omega,\omega_1)$ such that some $A \in \mathcal{A}_0$ is P_β, but no $B \in \mathcal{A}_0$ is P_γ for some $\gamma \in [\omega,\beta)$. Since $\mathcal{A}_0$ is closed with respect to taking non-empty clopen subsets of its elements, A is nowhere P_γ for $\gamma \in [\omega,\beta)$.

If β is a limit, then $A \approx X^2_\beta$ if $\beta > \omega$, and if $\beta = \omega$, then $A \approx X^2_\omega$ by lemma 2.1. So $\beta = \alpha+n$ for some limit $\alpha < \omega_1$, and some $n \in \mathbb{N}$, and A is $P_{\alpha+n}$, nowhere $P_{\alpha+n-1}$. If each non-empty clopen subset of A contains a closed copy of $X^2_{\alpha+n-1}$, then $A \approx X^2_{\alpha+n}$, and if some non-empty clopen subset C of A contains no closed copy of $X^2_{\alpha+n-1}$, then $C \approx X^1_{\alpha+n}$. In both cases, A contains a non-empty clopen homogeneous subset, a clear contradiction. Thus, $\mathcal{A}_0 = \emptyset$.

To show that $\mathcal{A}_1 = \emptyset$, we use the results of van Engelen [1985]. Note that if $B \in \mathcal{A}_1$, and V is clopen in B and non-empty, then $V \in \mathcal{A}_1$ since $\mathcal{A}_0 = \emptyset$; thus, as in the proof of Theorem 2.4, it follows that if $\mathcal{A}_1 \neq \emptyset$, then there exists $A \in \mathcal{A}_1$ such that $A \cap [s] \equiv_W A$ for each $s \in 2^{<\omega}$, and A is meager or comeager, while moreover the w-degree a of A is minimal. By van Engelen [1986], proofs of Lemma 3.3 and 4.1, either $\Gamma = [A]$ is reasonably closed and $\Gamma \neq \Gamma^d$, or A contains a non-empty clopen subset of w-degree strictly less than a; thus by minimality of a, Γ is reasonably closed, and A is everywhere properly Γ. We then obtain a contradiction as in the proof of Theorem 2.4. □

Using the axiom of choice it is possible to construct a non-trivial rigid subset of 2^ω (see Kuratowski [1925]).

2.6 Theorem. If $V = L$ then there exist a $\underset{\sim}{\Pi}^1_1$ subset of 2^ω so that both it and its complement are rigid.

Proof. Define L_α to be point definable iff the Skolem-hull of (L_α,ϵ) is isomorphic to (L_α,ϵ). Note that if L_α is point definable so is $L_{\alpha+\omega}$. It is well known that there are unboundedly many $\alpha < \omega_1$ such that L_α is point definable (see Boolos and Putnam (1968) or Mansfield and Weitkamp (1985)). For example, if (L_δ,ϵ) is an elementary substructure of (L_{ω_1},ϵ), then for the first $\alpha > \delta$ such that $L_\alpha \models$ "δ is countable", L_α is point definable, since δ is definable in $L_{\alpha-1}$ as the first uncountable ordinal and an elementary substructure X would have that $\delta \subseteq X$ and hence would collapse to say L_λ which satisfied "δ is countable". By the minimality of α we would have $\alpha = \lambda$. Since L has built-in Skolem functions point definable L_α have the property that there exists $E \subseteq \omega \times \omega$ recursive in $\mathrm{Th}(L_\alpha,\epsilon)$ such that $(\omega,E) \simeq (L_\alpha,\epsilon)$. Since the first order theory of L_α appears in say $L_{\alpha+2}$, we have that for any point definable L_α there exists $E \subseteq \omega \times \omega$, $E \in L_{\alpha+3}$, and $(\omega,E) \simeq (L_\alpha,\epsilon)$.

Define $X \subseteq 2^\omega$ to be the set of all $x \in 2^\omega$ such that there exists a limit ordinal α such that L_α is point definable, there exists $E \subseteq \omega \times \omega$ recursive in x such that $(\omega,E) \simeq (L_\alpha,\epsilon)$, and x is the first element of 2^ω constructed not in L_α and satisfying these two conditions for α. X is

Π^1_1 since $x \in X$ iff there exist a model M hyperarithmetic in x such that $M \simeq (L_{\alpha+\omega}, \epsilon)$ and $M \models$ "$x \in X$" (i.e. the definition above). Let $C \subseteq 2^\omega$ be the set of accumulation points of X, i.e. $x \in C$ iff every open neighborhood of x contains uncountably many points of X. Clearly C is homeomorphic to 2^ω. We show that $X \cap C$ and $C \setminus X$ are rigid. For $x \in X$ write $x = x_\alpha$ if x is in X because of α.

Lavrentiev's Theorem implies that for any homeomorphism $h:A \to B$ with $A,B \subseteq 2^\omega$ arbitrary there exists a homeomorphism $k:G \to H$ of $\underset{\sim}{\Pi}^0_2$ sets G and H with $G \supseteq A$, $H \supseteq B$, and $k \restriction A = h$ (see Kuratowski (1966) p. 429). Note that if k is a continuous function with $\underset{\sim}{\Pi}^0_2$ domain coded in L_α for some limit α and $x \in \mathrm{dom}(k) \cap L_\alpha$, then $k(x) \in L_\alpha$. This is true since $k(x)$ is recursive in x and a code for k. Now suppose for contradiction that $h:X \cap C \to X \cap C$ is a nontrivial autohomeomorphism and k it's extension above.

Since every point of $X \cap C$ is an accumulation point of $X \cap C$ there are uncountably many $x \in X \cap C$ such that $k(x) \neq x$. Since $V = L$ there exists $\gamma < \omega_1$ with k, k^{-1}, H, G all coded in L_γ. Hence there exists α, β point definable with $\gamma < \alpha < \beta$ and either $k(x_\alpha) = x_\beta$ or $k^{-1}(x_\alpha) = x_\beta$. But then $x_\beta \in L_\beta$, contradiction. Now let us show that $C \setminus X$ is rigid. Suppose for contradiction $f:C\setminus X \to C\setminus X$ is a nontrivial homeomorphism and $k:G \to H$ a homeomorphism extending f with $G,H \subseteq C$ $\underset{\sim}{\Pi}^0_2$ sets.

Note that for $x \in G \cap X$ $k(x) \in X$ and for $x \in H \cap X$, $k^{-1}(x) \in X$. Hence it is enough to see that there are uncountably many $x \in X \cap G$ such that $k(x) \neq x$. But k is nontrivial so for some $u \in G$, $k(u) = v$ and $u \neq v$. Choose $n < \omega$ so that $k([u\restriction n] \cap G) \subseteq [v\restriction n] \cap H$ and $u\restriction n \neq v\restriction n$ where $[u\restriction n] = \{x \in 2^\omega | x\restriction n = u\restriction n\}$. So it is enough to see that $[u\restriction n] \cap G \cap X$ is uncountable. But if it were countable, then $X \cap C \cap [u\restriction n]$ would be a Borel set so $X \cap C$ would not be rigid. □

In Theorem 2.6 we could have in fact found a lightface Π^1_1 set X. (The boldface parameter only coming in because of the $\underset{\sim}{\Pi}^0_1$ set C.) One way to do it is to demand that if $\alpha = \omega^2 \cdot \beta + \omega \cdot n$ is point definable, then x_α is in the n^{th} clopen subset of 2^ω. This ensures that every point of 2^ω is an accumulation point of X.

Zero dimensionality is important in Theorem 2.5 because of the results of de Groot and Wille (1958) who show that there is a nontrivial compact subset of the plane which is rigid. There cannot be a nontrivial Borel rigid subset of the real line, since such a set cannot contain an interval, hence must be zero dimensional and so embeddable into 2^{ω}.

Call a set $B \subseteq \mathbb{R}$, where $\mathbb{R}$ is the real line with its usual order, order rigid iff the identity is the only bijection $f \cdot B \to B$ which preserves the order on B inherited from $\mathbb{R}$. Note that the positive integers are order rigid. Now we describe an uncountable order rigid Borel set. Here is its order type. Let the rationals $\mathbb{Q}$ be listed $\{q_n : n < \omega\}$. Replace the n^{th} rational with the n element linear order L_n. Let $X = (\mathbb{R}\backslash\mathbb{Q}) \cup \bigcup_{n<\omega} L_n$ with the obvious order. X is order rigid since for all n, L_n must be mapped to itself. It is not hard to see that X has the same order type as a closed subset of $\mathbb{R}$. (Inductively choose L_n of size n so that for every $m < n$, $\max(L_m) < \min(L_n)$ if $q_m < q_n$ and $\max(L_n) < \min(L_m)$ if $q_n < q_m$.) However we can ask:

Question. Does there exist an order rigid Borel set $B \subseteq \mathbb{R}$ without isolated points?

3. Better quasiorder theory.

A quasi order is a transitive, reflexive but not necessarily antisymmetric binary relation $\leq$. We define $x \equiv y$ to mean $x \leq y$ and $y \leq x$. If we mod out by $\equiv$ then we get a partial order. Hence Wadge reducibility $\leq_w$ (and most other reducibilities) are natural examples of quasiorders. A well quasiorder (WQO) is a quasiorder which has no infinite descending chains or infinite antichains (where antichain here means pairwise incomparable set). It is easy to see using Ramsey's Theorem that a quasiorder $(Q,\leq)$ is a well quasiorder iff for any sequence $\langle a_n : n < \omega\rangle$ from Q there exists $n < m$ with $a_n \leq a_m$. Martin's Theorem that the Wadge degrees of Borel sets are well founded implies that $\leq_w$ is a well-quasiorder when restricted to the Borel sets. Better quasiorders were introduced by Nash-Williams (1968). It is a stronger condition than well-quasi-ordering. Here we will use the definition of better-quasiorder from Simpson (1985). Let $[\omega]^{\omega}$ be the set of all infinite subsets of ω with the inherited product topology ($[\omega]^{\omega} \subseteq P(\omega) = 2^{\omega}$). For any set Q a Q-array is a map $f : [X]^{\omega} \to Q$ where

$X \in [\omega]^\omega$, the range of f is countable, and for every $q \in Q$, $f^{-1}(q)$ is Borel. For any $X \in [\omega]^\omega$ let $X^* = X \setminus \{\min X\}$, i.e. all of X except its least element. For $(Q,\leq)$ a quasiorder a good Q-array is a Q-array $f : [X]^\omega \to Q$ such that there exist $Y \in [X]^\omega$ such that $f(Y) \leq f(Y^*)$.

An array is bad iff it is not good. A quasiorder $(Q,\leq)$ is better-quasiordered (BQO) iff every Q-array is good. BQO implies WQO, since if we are given $\langle a_n : n \in \omega \rangle$ a sequence from Q, then just consider the array $f : [\omega]^\omega \to Q$ defined by $f(X) = a_{\min(X)}$.

One of the main technical lemmas of BQO theory is the minimal bad array lemma (see Simpson (1985) 9.17).

We pause to give a short simple proof of it. If $(Q,\leq)$ is a quasiorder, then a partial ranking of Q is a well founded partial ordering $\leq^*$ of Q such that $q \leq^* p$ implies $q \leq p$. A minimal bad array is a bad array $f : [X]^\omega \to Q$ such that every array $g : [Y]^\omega \to Q$, with $Y \in [X]^\omega$ and for all $Z \in [Y]^\omega$ $g(Z) <^* f(Z)$, is good.

3.1 Theorem. Suppose $(Q,\leq)$ is a quasiorder and $\leq^*$ a partial ranking. Let $f_0 : [X_0]^\omega \to Q$ be a bad array. Then there exists a minimal bad array $f : [X]^\omega \to Q$ with $X \in [X_0]^\omega$ and $f(Z) \leq^* f_0(Z)$ for all $Z \in [X]^\omega$.

Proof.

3.1.1 Lemma. Suppose $X_\alpha \in [\omega]^\omega$ for $\alpha < \omega_1$ and for all $\alpha < \beta < \omega_1$, $X_\beta \subseteq^* X_\alpha$ (inclusion mod finite, i.e. $X_\beta \setminus X_\alpha$ finite). Then there exists $Z \in [\omega]^\omega$ and $\Sigma \in [\omega_1]^\omega$ such that $Z \subseteq \bigcap_{\alpha\in\Sigma} X_\alpha$.

Proof. Construct $F_n \in [\omega_1]^n$, $s_n \in [\omega]^n$, and $A_n \in [\omega_1]^{\omega_1}$ so that $F_n \subseteq F_{n+1}$, $s_n \subseteq s_{n+1}$, and $A_n \supseteq A_{n+1}$; and for every $\alpha \in F_n \cup A_n$, $s_n \subseteq X_\alpha$. Given s_n, F_n, and A_n; let $F_{n+1} = F_n \cup \{\alpha\}$ for any $\alpha \in A_n$, let $Q = (\bigcap_{\alpha\in F_{n+1}} X_\alpha) \setminus s_n$, and let $m_\alpha \in Q \cap X_\alpha$ for each $\alpha \in A_n$. Then there exists m and $A_{n+1} \in [A_n]^{\omega_1}$ such that for all $\alpha \in A_{n+1}$, $m_\alpha = m$. So let $s_{n+1} = s_n \cup \{m\}$. To finish the proof let $Z = \bigcup_{n<\omega} s_n$ and $\Sigma = \bigcup_{n<\omega} F_n$. □

We can also give a metamathematical proof of the lemma. First assume $MA + \neg CH$. Then there exist $Z \in [\omega]^{\omega}$ such that for all $\alpha < \omega_1$, $Z \subseteq^* X_\alpha$. Suppose $Z \setminus n_\alpha \subseteq X_\alpha$. Then for infinitely many α, $n_\alpha = n$, so $Z \setminus n$ works. But now for some B a c.c.c. complete boolean algebra, and some $\delta < \omega_1$,

$$V^B \models \text{"}\exists z \in [\omega]^{\omega} \ \exists \Sigma \in [\delta]^{\omega} \ \forall \alpha \in \Sigma \ \ Z \subseteq X_\alpha\text{"}.$$

But this is a $\underset{\sim}{\Sigma}^1_1$ sentence, so it must be true in V.

3.1.2 Lemma. Suppose there is no minimal bad array beneath f_0. Then there exists an ω_1 sequence of bad arrays $\langle f_\alpha : \alpha < \omega_1 \rangle$ with $f_\alpha : [X_\alpha]^{\omega} \to Q$ and for every $\alpha < \beta < \omega_1$ $X_\beta \subseteq^* X_\alpha \subseteq X_0$ and for all $Z \in [X_\alpha \cap X_\beta]^{\omega}$, $f_\beta(Z) <^* f_\alpha(Z)$.

Proof. We construct the f_α by induction on α. For successor steps $\alpha+1$ since f_α is not a minimal bad array (since $X_\alpha \subseteq X_0$) we can choose $f_{\alpha+1}$ with $X_{\alpha+1} \in [X_\alpha]^{\omega}$ as required. Now suppose $\delta < \omega_1$ is a limit ordinal and we have already got $\langle f_\alpha : \alpha < \delta \rangle$. First note that for any $Z \in [\omega]^{\omega}$ $\{\alpha < \delta : Z \subseteq X_\alpha\}$ is finite. Otherwise if $Z \subseteq X_{\alpha_n}$ where $\alpha_n < \alpha_{n+1}$, then $f_{\alpha_0}(Z) \overset{*}{>} f_{\alpha_1}(Z) \overset{*}{>} \cdots \overset{*}{>} f_{\alpha_n}(Z) \overset{*}{>} \cdots$ is an infinite descending sequence contradicting the well-foundedness of $\leq^*$. Let $X \in [\omega]^{\omega}$ be such that $X \subseteq^* X_\alpha$ for every $\alpha < \delta$ and $X \subseteq X_0$. This is easy to get because δ is countable. Define $g : [X]^{\omega} \to Q$ by $g(Z) = f_\alpha(Z)$ where $\alpha = \max\{\beta : Z \subseteq X_\beta\}$.

CLAIM. g is an array.

Proof. Clearly the range of g is countable. For any $q \in Q$, $Z \in g^{-1}(q)$ iff

$$\exists \alpha < \delta \ \ [Z \subseteq X_\alpha \wedge Z \in f_\alpha^{-1}(q) \wedge \forall \beta \ \ (\alpha < \beta < \delta \to Z \not\subseteq X_\beta)].$$

□

CLAIM. g is bad.

Proof. Suppose $g(Z) \leq g(Z^*)$ and $g(Z) = f_\alpha(Z)$ and $g(Z^*) = f_\beta(Z^*)$. Since $Z^* \subseteq Z$, it must be that $\alpha \leq \beta$, and hence $f_\alpha(Z^*) \geq^* f_\beta(Z^*)$. But then $f_\alpha(Z) \leq f_\alpha(Z^*)$ contradicting the badness of f_α. □

Now apply the successor step argument to g and get $f_\delta : [X_\delta]^\omega \to Q$ bad with $f_\delta(Z) <^* g(Z)$ all $Z \in [X_\delta]^\omega$. Note that for all $Z \in [X \cap X_\alpha]^\omega$ $g(Z) = f_\alpha(Z)$ or $g(Z) <^* f_\alpha(Z)$, so for all $Z \in [X_\delta \cap X_\alpha]^\omega$ $f_\delta(Z) <^* f_\alpha(Z)$. This finishes the proof of Lemma 3.1.2. □

The lemmas immediately imply the theorem, because if $Z \subseteq X_{\alpha_n}$ for $\alpha_n < \alpha_{n+1} < \omega_1$, then $\langle f_{\alpha_n}(Z) : n \in \omega \rangle$ is an infinite $<^*$ descending chain.

□

Suppose $(Q,\leq)$ is a quasiorder. Let $Q^* = \{\ell : \omega^\omega \to Q | \ell \text{ is Borel}\}$ where ℓ is Borel means that the range of ℓ is countable and for all $q \in Q$, $\ell^{-1}(q)$ is Borel. Define $\ell_1 \leq^* \ell_2$ iff there exists a continuous map $\sigma : \omega^\omega \to \omega^\omega$ such that for all $x \in \omega^\omega$ $\ell_1(x) \leq \ell_2(\sigma(x))$.

3.2 Theorem. If $(Q,\leq)$ is BQO, then $(Q^*,\leq^*)$ is BQO.

Proof. Given $\ell_1, \ell_2 \in Q^*$ consider the game $G(\ell_1,\ell_2)$ where player I writes down $x \in \omega^\omega$ and player II writes down $y \in \omega^\omega$ in alternating moves as in the Lipschitz game. Player II wins the run of the game (x,y) iff $\ell_1(x) \leq \ell_2(y)$. Since this is a Borel game one of the two players has a winning strategy. Now suppose for contradiction that Q^* is not BQO and let $\langle \ell_X : X \in [\omega]^\omega \rangle$ be a bad Q^*-array. So for any X we have that $\ell_X \not\leq^* \ell_{X^*}$ and so there is a winning strategy σ for player I in the game $G(\ell_X, \ell_{X^*})$, i.e. $\forall u \in \omega^\omega$ $\ell_X(\sigma(u)) \not\leq \ell_{X^*}(u)$. Now we will produce a bad array $f : [\omega]^\omega \to Q$ contradicting Q is BQO. Let $X \in [\omega]^\omega$ be arbitrary and let $X_0 = X$ and $X_{n+1} = X_n^*$. For each n let σ_n be the canonical winning strategy for player I in the game $G(\ell_{X_n}, \ell_{X_{n+1}})$. (By canonical we mean that there are only countably many distinct ℓ_X so we only need countable many strategies for player I and which strategy we take for $G(\ell_X, \ell_Y)$ depends only on which of countably many Borel sets X is in and which one Y is in.) Given such a set up we can construct the usual infinite game diagram.

$$
\begin{array}{ccccccccc}
 & \sigma_0 & & \sigma_1 & & \sigma_2 & & & \\
X_0 & & X_1 & & X_2 & & X_3 & \cdots \\
\hline
x_0^0 & & x_0^1 & & x_0^2 & & x_0^3 & \cdots \\
x_1^0 & & x_1^1 & & x_1^2 & & x_1^3 & \cdots \\
x_2^0 & & x_2^1 & & x_2^2 & & x_2^3 & \cdots \\
x_3^0 & & x_3^1 & & x_3^2 & & x_3^3 & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & \\
\overline{x^0} & & \overline{x^1} & & \overline{x^2} & & \overline{x^3} &
\end{array}
$$

In this diagram $x_i^j \in \omega$ and $x^j \in \omega^\omega$ and they are determined as follows. Each x_0^j is σ_j's first move in the game $\sigma(\ell_{X_j}, \ell_{X_{j+1}})$. And each x_k^j is the strategy σ_j's k^{th} move in this game given that player II has played $\langle x_\ell^{j+1} : \ell < k \rangle$. Hence the diagram is filled out row by row by transferring information from right to left as indicated by the arrows. Finally we define $f(X) = \ell_X(x^0)$ (where $X_0 = X$). It is easy to see that the graph of f is Σ_1^1, i.e. $f(X) = q$ iff there exist a game diagram with $X = X_0$ and $\ell_{X_0}(x^0) = q$. But since f is total it is Borel, and hence a Q-array. But note that the game diagram for X^* is the same as the diagram for X minus the first column. Hence $f(X^*) = \ell_{X_1}(x^1)$ where x^1 is the second column of the diagram for X. But $\sigma_0(x^1) = x^0$ and G_0 is a winning strategy for player I so it is not the case that $\ell_{X_0}(x^0) \le \ell_{X_1}(x^1)$. So f is a bad Q-array. □

This result generalizes Martin's Theorem that the Wadge ordering is well founded, let $Q = \{0,1\}$ where 0 and 1 are incomparable. If we let Q be a well ordering, then we can think of this result as generalizing part of the first periodicity theorem. The reducibility map could have been taken to be

Lipschitz since this is what the game gives us. Also under AD we can drop the assumption that our labelings are Borel.

Carlson and Laver in unpublished work have considered the following quasiorder. Let $\approx_1$ and $\approx_2$ be two Borel equivalence relations on ω^ω. Define $\approx_1 \leq \approx_2$ iff there exits a continuous function $f:\omega^\omega \to \omega^\omega$ such that for all $x,y \in \omega^\omega$ $x \approx_1 y$ iff $f(x) \approx_2 f(y)$. Laver has proved that for any $n < \omega$ the set of Borel equivalence relations with n equivalence classes is better-quasiordered by $\leq$. We arrived at 3.2 by generalizing the statement and proof of this result. For any equivalence relation $\approx$ with $\leq n$ equivalence classes let $\ell:\omega^\omega \to n$ be any Borel map such that $\ell(x) = \ell(y)$ iff $x \approx y$. Give n the trivial quasiorder in which nothing is comparable to anything else. Then $\ell_1 \leq^* \ell_2$ implies $\approx_1 \leq \approx_2$.

<u>Question</u> Does $\leq$ better-quasiorder all Borel equivalence relations?

Theorem 3.2 is not true for well-quasiorders, since it is easy to see that Q^* WQO implies Q^ω is WQO and there are Q WQO such that Q^ω is not WQO (see Rado (1954) or Laver (1976)).

An ordering on Q^* more relevant to section 2 is to demand that the reducibility be continuous and one-to-one. More precisely define $\ell_1 \leq_1^* \ell_2$ for $\ell_1, \ell_2 \in Q^*$ iff there exists a continuous one-to-one map $\sigma:\omega^\omega \to \omega^\omega$ such that for all $x \in \omega^\omega$, $\ell_1(x) \leq \ell_2(\sigma(x))$. We are unable to show that the complete analogue of Theorem 3.2 is true but we can verify a portion of it. We say that $\ell \in Q^*$ is $\underset{\sim}{\Sigma}_2^0$ iff for all $q \in Q$, $\ell^{-1}(q)$ is $\underset{\sim}{\Sigma}_2^0$.

3.3 Theorem. If $(Q,\leq)$ is BQO then the set of $\underset{\sim}{\Sigma}_2^0$ ℓ in Q^* is BQO by $\leq_1^*$.

Proof. The theorem follows immediately from the next two lemmas.

3.3.1 Lemma. The set of $\underset{\sim}{\Pi}_1^0$ ℓ in Q^* is BQO by $\leq_1^*$.

Proof. Here we use a result of Laver (1978) that labeled trees are BQO under tree embedding. Define a one-to-one map $\sigma:\omega^{<\omega} \to \omega^{<\omega}$ to be a tree embedding iff for all $s,t \in \omega^{<\omega}$ $s \subseteq t \leftrightarrow \sigma(s) \subseteq \sigma(t)$, and

$\sigma(s \cap t) = \sigma(s) \cap \sigma(t)$ where $s \cap t$ is the largest common initial segment of s and t. For $(Q,\leq)$ any quasiorder and let $T_Q = \{\ell \mid \ell : \omega^{<\omega} \to Q\}$ and for $\ell_1, \ell_2 \in T_Q$, $\ell_1 \leq \ell_2$ iff there exists a tree embedding $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ such that for every $s \in \omega^{<\omega}$ $\ell_1(s) \leq \ell_2(\sigma(s))$. A special case of Laver's Theorem is that if $(Q,\leq)$ is BQO, then $(T_Q,\leq)$ is BQO. (Unlabeled versions of this result were first proved by Nash-Williams.) Now let $\ell : \omega^\omega \to Q$ be a $\underset{\sim}{\Pi}^0_1$ element of Q^*. Suppose the range of ℓ is $\{q_n : n < \omega\}$. Define $\hat{\ell} \in T_Q$ as follows: $\hat{\ell}(s) = q_n$ where n is the least such that $[s] \cap \ell^{-1}(q_n)$ is nonempty where $[s] = \{x \in \omega^\omega \mid s \subseteq x\}$. Note that since each $\ell^{-1}(q)$ is closed, $\ell(x) = q$ iff there are infinitely many n such that $\hat{\ell}(x|n) = q$ iff for all but finitely many n $\hat{\ell}(x|n) = q$. Consequently, for any ℓ_1, ℓ_2 $\underset{\sim}{\Pi}^0_1$ elements of Q^* if there exist a tree embedding $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ such that for every $s \in \omega^{<\omega}$ $\hat{\ell}_1(s) \leq \hat{\ell}_2(\sigma(s))$, then for every $x \in \omega^\omega$ $\ell_1(x) \leq \ell_2(\bigcup_{n<\omega} \sigma(x|n))$. Therefore the map $h : \omega^\omega \to \omega^\omega$ defined by $h(x) = \bigcup_{n<\omega} \sigma(x|n)$ is clearly a continuous one-to-one map showing $\ell_1 \leq^*_1 \ell_2$. □

3.3.2 Lemma. Suppose for every BQO, $(Q,\leq)$, the $\underset{\sim}{\Pi}^0_\alpha$ elements of Q^* are BQO by $\leq^*_1$. Then for every BQO $(Q,\leq)$ the $\underset{\sim}{\Sigma}^0_{\alpha+1}$ elements of Q^* are BQO by $\leq^*_1$.

Proof. Let $\ell : \omega^\omega \to Q$ be a $\underset{\sim}{\Sigma}^0_{\alpha+1}$ element of Q^*. For each $q \in Q$ let $f^{-1}(q) = \bigcup_{m<\omega} X^m_q$ where $X^m_q \in \underset{\sim}{\Pi}^0_\alpha$ and $\{X^m_q : m \in \omega\}$ are pairwise disjoint. Define $\hat{\ell} : \omega^\omega \to (Q \times \omega)$ by $\hat{\ell}(x) = (\ell(x), m)$ where $x \in X^m_{\ell(x)}$. Put the trivial quasiorder on ω, i.e. all elements are comparable. So $Q \times \omega$ is BQO, hence the $\underset{\sim}{\Pi}^0_\alpha$ elements of $(Q \times \omega)^*$ are BQO. But $\hat{\ell}_1 \leq^*_1 \hat{\ell}_2$ implies $\ell_1 \leq^*_1 \ell_2$. □

The proof of our next result is an easy modification of the Laver and Nash-William Theorem (see Laver 1978) that BQO labeled trees are BQO under tree embeddability. Suppose $(L, <_L)$ is a scattered linear order, i.e. it fails to contain an isomorphic copy of the rationals. A tree embedding preserving the lexicographical order is a one-to-one map $\sigma: L^{<\omega} \to L^{<\omega}$ such that for every s and $t \in L^{<\omega}$ $s \subseteq t$ implies $\sigma(s) \subseteq \sigma(t)$ and $\sigma(s \cap t) = \sigma(s) \cap \sigma(t)$, and for every $n < \omega$ and $s, t \in L^{<\omega}$ if $s|n = t|n$ and $s(n) <_L t(n)$, then there exists $m < \omega$ such that $\sigma(s)|m = \sigma(t)|m$ and $\sigma(s)(m) <_L \sigma(t)(m)$. Given $(Q, \leq)$ a quasiorder let $\mathcal{L}_Q = \{\ell \mid \ell : L^{<\omega} \to Q\}$ and $\ell_1 \leq^* \ell_2$ iff there exists $\sigma: L^{<\omega} \to L^{<\omega}$ a tree embedding preserving the lexicographical order such that for all $s \in L^{<\omega}$ $\ell_1(s) \leq \ell_2(\sigma(s))$.

3.4 Theorem. For any scattered linear order L and quasiorder $(Q, \leq)$ if Q is BQO, then $\mathcal{L}_Q$ is BQO.

Proof. The only additional ingredient to the theorem is lexicographical order. The following witnessing Lemma is all that is needed. Let $(Q, \leq)$ be a quasiorder and define $\text{Scat}_Q = \{\ell \mid \ell : L \to Q, L$ is a scattered linear order$\}$. Define $\ell_1 \leq^* \ell_2$ iff there exists an order preserving map $h: L_1 \to L_2$ such that $\ell_1(a) \leq \ell_2(h(a))$ for all $a \in L_1$.

3.4.1 Lemma. Suppose $(Q, \leq)$ is any quasiorder and $\langle \ell_X : X \in [\omega]^\omega \rangle$ is a bad Scat_Q-array. Then there exists $Y \in [\omega]^\omega$ and $a_X \in L_X$ for $X \in [Y]^\omega$ such that $\langle \ell_X(a_X) : X \in [Y]^\omega \rangle$ is a bad Q-array.

The proof of this Lemma for ordinals instead of scattered types can be found in Simpson (1985) Theorem 9.19. We need the Hausdorff characterization of scattered types. Let S_0 be the class of one point orders and for any ordinal $\rho > 0$, let S_ρ be the class of linear ordered sets L which are isomorphic to either a well ordered sum

$$L_0 + L_1 + \dots + L_\beta + \dots \qquad (\beta < \alpha)$$

or a converse well ordered sum

$$\dots + L_\beta + \dots + L_1 + L_0 \qquad (\beta < \alpha)$$

where each L_β belongs to $\cup\{S_\gamma : \gamma < \rho\}$. The rank of a scattered order is the least α such that it is in S_α. We give Scat_Q the partial ranking $\ell_1 < \ell_2$ iff $L_1 \subseteq L_2$, $\mathrm{rank}(L_1) < \mathrm{rank}(L_2)$, and $\ell_1 = \ell_2 | L_1$. Now let $\langle \ell'_X : X \in [Y]^\omega \rangle$ be a minimal bad array with $\ell'_X \leq \ell_X$ all $X \in [Y]^\omega$. By the Galvin-Prikry Theorem (1973) we may assume that either for all $X \in [Y]^\omega$ $\mathrm{rank}(L'_X) = 0$ or for all $X \in [Y]^\omega$ $\mathrm{rank}(L'_X) > 0$. It is enough now to get a contradiction from the latter. Again by appling the Galvin-Prikry Theorem, we may assume each L'_X is either well-ordered sum of sets of smaller rank or it is always a conversely ordered sum of sets of smaller rank. So suppose the former and each

$$L'_X = L^0_X + L^1_X + \ldots + L^\beta_X + \ldots \qquad (\beta < \alpha_X)$$

where each L^β_X has strictly smaller rank. But now by applying the witness lemma for ordinal sequences to $\langle\langle\langle L^\beta_X, \ell'_X \upharpoonright L^\beta_X\rangle : \beta < \alpha_X\rangle : X \in [Y]^\omega\rangle$ there exists $Z \in [Y]^\omega$ and $\beta_X < \alpha_X$ for each $X \in [Z]^\omega$ such that

$$\langle\langle L^{\beta_X}_X, \ell'_X\rangle : X \in [Z]^\omega\rangle$$

is a bad Scat_Q array. This contradicts the minimality of

$$\langle \ell'_X : X \in [Y]^\omega\rangle \ . \qquad \square$$

The rest of the proof of the theorem is the same as Laver (1971) or Laver (1978). The witness lemma is also true for labeled scattered trees, i.e. trees which fail to contain $2^{<\omega}$. It seems an interesting technical question whether or not the witness lemma is true for labeled countable orders or for labeled $2^{<\omega}$ trees. Next we give two applications of Theorem 3.4.

Given a quasiorder $(Q,\leq)$ let $\mathrm{LIN}_Q = \{(L,\ell) : L$ is a countable linear order and $\ell : L \to Q\}$ and define $(L_1,\ell_1) \leq^* (L_2,\ell_2)$ iff there exists $\sigma : L_1 \to L_2$ one-to-one, order preserving, continuous, and for all $a \in L$, $\ell_1(a) \leq \ell_2(\sigma(a))$. Only the condition that σ be continuous is new, otherwise the following result is already known to Laver (1971).

3.5 Theorem. If $(Q,\leq)$ is BQO, then $(\mathrm{LIN}_Q,\leq^*)$ is BQO.

Proof. Since every countable linear order embeds into $\mathbb{Q}$ as a closed set (so order topology is the same as subspace topology) we may as well assume $\mathrm{LIN}_Q = \{\ell \mid \ell:\mathbb{Q} \to Q\}$. Let $L = \omega + \omega^*$ where ω^* is the converse of ω and consider $\mathcal{L}_Q$. Define $s \leq^* t$ for $s,t \in L^{<\omega}$ by $s <^* t$ iff either there exists n such that $s|n = t|n$ and $s(n) <_L t(n)$ or $(s = t|n$ and $t(n) \in \omega^*)$ or $(t = s|n$ and $s(n) \in \omega)$. So for any $s \in L^{<\omega}$ we have $s^\frown\langle n\rangle <^* s <^* s^\frown\langle m\rangle$ for any $n \in \omega$ and $m \in \omega^*$. Hence

$$(L^{<\omega},\leq^*) \simeq (\mathbb{Q},\leq) .$$

CLAIM. Suppose $\sigma:L^{<\omega} \to L^{<\omega}$ is a tree embedding which preserves the lexicographical order. Then σ is $\leq^*$ order preserving and continuous in the $\leq^*$ order topology.

Proof. If $s,t \in L^{<\omega}$ are $\subseteq$ incomparable it is clear that $s \leq^* t \to \sigma(s) \leq^* \sigma(t)$. But note that since

$$\sigma(s^\frown\langle x\rangle) \cap \sigma(s^\frown\langle y\rangle) = \sigma(s) ,$$

and σ is lexicographical order preserving, it must be that $x \in \omega$ implies $\sigma(s^\frown\langle x\rangle) <^* \sigma(s)$ and $y \in \omega^*$ implies $\sigma(s) <^* \sigma(s^\frown\langle y\rangle)$. Continuity is easy to check since the intervals $(s^\frown\langle n\rangle, s^\frown\langle m\rangle)$ for $n \in \omega$ and $m \in \omega^*$ form a neighborhood basis for s. □

But now our result follows immediately from Theorem 3.4. □

Let $(Q,\leq)$ be any quasiorder and define $D_Q^2 = \{\ell:\mathbb{R} \to Q \mid$ range of ℓ is countable and for every $q \in Q$, $\ell^{-1}(q)$ is $\underset{\sim}{\Sigma}_2^0\}$. For $\ell_1,\ell_2 \in D_Q^2$ define $\ell_1 \leq \ell_2$ iff there exists a one-to-one order preserving map $\sigma:\mathbb{R} \to \mathbb{R}$ such that σ is continuous at every irrational number and for every $a \in \mathbb{R}$ $\ell_1(a) \leq \ell_2(\sigma(a))$, and for every $a \in \mathbb{R}$, a is rational iff $\sigma(a)$ is rational.

3.6 Theorem. If $(Q,\leq)$ is BQO then so is $(D_Q^2,\leq)$.

Proof. Let $L = \omega^* + \omega = \mathbb{Z}$. Embed $L^{<\omega}$ into $\mathbb{Q}$ so as to preserve the lexicographical order, i.e. $s <_{lex} t$ iff $s \subsetneq t$ or there exists n $s|n = t|n$ and $s(n) <_L t(n)$. Let $t: L^{<\omega} \to \mathbb{Q}$ be an order preserving bijection so that say the distance from $t(s\hat{\ }\langle n\rangle)$ and $t(s\hat{\ }\langle n+1\rangle)$ is less than $1/k$ for all $s \in L^k$. Given a tree embedding $\sigma: L^{<\omega} \to L^{<\omega}$ which preserves the lexicographical order define $\sigma^*: \mathbb{R} \to \mathbb{R}$ by

$$\sigma^*(x) = \sup\{t(\sigma(r)) : t(r) \leq x\}$$

CLAIM. σ^* is one-to-one, order preserving, continuous except possibly on $\mathbb{Q}$, and maps $\mathbb{Q}$ to $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ to $\mathbb{R}\backslash\mathbb{Q}$.

Proof. The embedding t shows that $(\mathbb{R},\leq)$ is isomorphic to $(Z^{\leq\omega},\leq_{lex})$ with $\mathbb{Q}$ being mapped to $Z^{<\omega}$. Everything is easy to check except continuity on Z^{ω}. But clearly from tree embedding we get continuity for the product topology (where $\mathbb{Z}$ has the discrete topology). So we need only see that the product topology on Z^{ω} is the same as the lex-order topology. This means that for every $x_n \in Z^{\omega}$ $x_n \to x_0$ in lex order iff $x_n \to x_0$ in product topology. This is easy. □

Using 3.4 we have that the $\mathbb{Q}$ labeled $Z^{<\omega}$ trees are BQO under tree embeddings which preserve lex order and so by the same argument as Lemma 3.3.1 we have that $D'_{\mathbb{Q}}$ (the $\underset{\sim}{\Pi}^0_1$ elements of $D^2_{\mathbb{Q}}$) is BQO. Also by the same argument as Lemma 3.3.2 we have that $D^2_{\mathbb{Q}}$ is BQO. □

It is easy to see we cannot demand that σ be continuous at every point. We would conjecture that 3.6 is true for the set of all Borel maps $\ell: \mathbb{R} \to \mathbb{Q}$. Another conjecture we have is a Borel version of Fraïssé's conjecture. Let BORLIN be the set of all $(L,\leq)$ such that $L \subseteq \mathbb{R}$ is Borel and $\leq\ \subseteq \mathbb{R}\times\mathbb{R}$ is a Borel linear order. Define $(L_1,\leq_1) \leq (L_2,\leq_2)$ iff there exists $\sigma: L_1 \to L_2$ which is one-to-one and order preserving, i.e. for all $x,y \in L$, $x \leq_1 y \Leftrightarrow \sigma(x) \leq_2 \sigma(y)$. We conjecture that (BORLIN, $\leq$) is BQO. We can get other versions by demanding that σ be continuous or by looking at Borel labelings.

Laver (1978) shows that countable trees whose initial segments are well ordered is BQO under tree embedding. Is the set of all countable partially ordered sets $(T, \leq)$ which are tree-like (i.e. for every $s \in T$ $\{t | t \leq s\}$ is linearly ordered) BQO under tree embedding?

BIBLIOGRAPY

Alexandroff, P. and Urysohn, P. (1928) "Uber null-dimensionale Punktmengen", Math. Ann. 98, 86-106.

Boolos, G and Putnam, H. (1968) "Degrees of Unsolvability of Constructible Sets of Integers", Journal of Symbolic Logic 33, 497-513.

Brouwer, L. E. J. (1910) "On the structure of perfect sets of points", Proc. Akad. Amsterdam 12, 785-794.

de Groot, J. and Wille, R. J. (1958) "Rigid continua and topological group pictures", Archiv. der Math. 9, 441-446.

van Engelen, F. (1985) "Homogeneous Borel sets of ambiguous class two", Trans. AMS 290, 1-39.

van Engelen, F. (1986) "Homogeneous Borel sets", Proc. AMS (to appear)

Galvin, F. and Prikry, K. (1973) "Borel sets and Ramsey's theorem", J. Symb. Logic 38, 193-198.

Kuratowski, K. (1925) "Sur la puissance de ℓ'ensemble des 'nombres de dimension' au sens de M. Fréchet", Fund. Math. 8, 201-208.

Kuratowski, K. (1966) Topology, Vol. 1, Academic Press.

Laver, R., (1971) "On Fraïssé's order type conjecture", Ann. of Math. 93, 89-111.

Laver, R., (1976) "Well-quasi-orderings and sets of finite sequences", Math. Proc. Cambridge Phil. Soc. 79, 1-10.

Laver, R., (1978) "Better-quasi-orderings and a class of trees, Studies in Foundations and Combinatorics", Adv. in Math. Supplementary Studies 1, 31-48.

Mansfield, R. and Weitkamp, G. (1985) Recursive Aspects of Descriptive Set Theory, Oxford Logic Guids 11, Oxford University Press.

Moschovakis, Y. (1980) Descriptive set theory, North-Holland.

Nash-Williams, C. St. J. A., (1965), "On well-quasi-ordering infinite trees", Proc. Phil. Soc. 61, 697-720.

Nash-Williams, C. St. J. A. (1968) "On better-quasi-ordering transfinite sequences", Proc. Camb. Phil. Soc. 64, 273-290.

Rado, R. (1954) "Partial well ordering of sets of vectors", Mathematatika 1, 89-95.

Sierpiński, W. (1920) "Sur une propriété topologique des ensembles denombrables denses en soi", Fund. Math. 1, 11-16.

Simpson, S. (1985) "BQO Theory and Fraïssé's Conjecture", Chapter 9 of Mansfield and Weitkamp.

Steel, J. (1977) Determinateness and subsystems of analysis, thesis, Berkeley.

Steel, J. (1980) "Analytic sets and Borel isomorphisms", Fund. Math. 108, 83-88.

Van Wesep, R. (1978a) Subsystems of second-order arithmetic, and descriptive set theory under the axiom of determinateness, thesis, Berkeley.

Van Wesep, R. (1978b) "Wadge degrees and descriptive set theory", CABAL Seminar 76-77, Lect. Notes in Math. 689, 151-170.

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