Projective subsets of separable metric spaces [1]

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Abstract

In this paper we will consider two possible definitions of projective subsets of a separable metric space X. A set $A \subseteq X$ is $\Sigma^1_1(X)$ iff there exists a complete separable metric space Y and Borel set $B \subseteq X \times Y$ such that $A = \{x \in X : \exists y \in Y \ (x,y) \in B\}$. Except for the fact that X may not be completely metrizable, this is the classical definition of analytic set and hence has many equivalent definitions, for example, A is $\Sigma^1_1(X)$ iff A is relatively analytic in X, i.e. A is the restriction to X of an analytic set in the completion of X. Another definition of projective we denote by Σ^X_1 or abstract projective subset of X. A set $A \subseteq X$ is Σ^X_1 iff there exists an $n \in \omega$ and a Borel set $B \subseteq X \times X^n$ such that $A = \{x \in X : \exists y \in X^n \ (x,y) \in B\}$. These sets can be far more pathological. While the family of sets $\Sigma^1_1(X)$ is closed under countable intersections and countable unions, there is a consistent example of a separable metric space X where Σ^X_1 is not closed under countable intersections or countable unions. This takes place in the Cohen real model. Assuming CH there exists a separable metric space X such that every $\Sigma^1_1(X)$ set is Borel in X but there exists a $\Sigma^1_1(X^2)$ set which is not Borel in X^2. The space X^2 has Borel subsets of arbitrarily large rank while X has bounded Borel rank. This space is a Luzin set and the technique used here is Steel forcing with tagged trees. We give examples of spaces X illustrating the relationship between $\Sigma^1_1(X)$ and Σ^X_1 and give some consistent examples partially answering an abstract projective hierarchy problem of Ulam.

[1] in Annals of Pure and Applied Logic, 50(1990), 53-69.
[2] Partially supported by NSF grant 8801139

1 Equivalent definitions

For general background about analytic sets the reader should consult Kuratowski [4], Rogers [10], or Moschovakis [8]. One notation we will use thruout is

$$proj_X(B) = \{x \in X : \exists y \in Y\ (x, y) \in B\}$$

i.e. the projection of $B \subseteq X \times Y$ onto X. We begin by considering the notion of $\Sigma^1_1(X)$. This notion corresponds to any of the following equivalent definitions.

Theorem 1.1 *For X a separable metric space and $A \subseteq X$ the following are all equivalent and denoted $\Sigma^1_1(X)$:*

1. *there exists a complete separable metric space Y and a Borel $B \subseteq X \times Y$ such that $A = proj_X(B) = \{x \in X : \exists y \in Y\ (x, y) \in B\}$*
2. *(relatively analytic) if $\hat{X}$ is the completion of X, then there exists $\hat{A} \subseteq \hat{X}$ a $\Sigma^1_1(\hat{X})$ set such that $A = \hat{A} \cap X$*
3. *(Souslin in X) there exists $\langle A_s : s \in \omega^{<\omega}\rangle$ where each $A_s \subseteq X$ is closed in X and $A = \bigcup_{f \in \omega^\omega} \bigcap_{n \in \omega} A_{f \restriction n}$*
4. *there exists $\langle A_s : s \in \omega^{<\omega}\rangle$ where each $A_s \subseteq X$ is Borel in X and $A = \bigcup_{f \in \omega^\omega} \bigcap_{n \in \omega} A_{f \restriction n}$*
5. *there exists a closed set $C \subseteq X \times \omega^\omega$ such that $A = proj_X(C)$*
6. *there exists a complete separable metric space Y and a closed set $C \subseteq X \times Y$ such that $A = proj_X(C)$*
7. *(truth tables) there exists $T \subseteq P(\omega)$ which is Σ^1_1 and $\langle U_n : n \in \omega\rangle$ each $U_n \subseteq X$ Borel in X and $A = \{x \in X : \{n \in \omega : x \in U_n\} \in T\}$*

proof:

(3) $\to$ (4), (5) $\to$ (6), (6) $\to$ (1) trivial.

(2) $\to$ (3) Note that by the classical theory of analytic sets in complete metric spaces

$$\hat{A} = \bigcup_{f \in \omega^\omega} \cap_{n \in \omega} A_{f \restriction n}$$

where each $A_{f\restriction n} \subseteq Y$ is closed in Y. Hence

$$A = \bigcup_{f\in\omega^\omega} \cap_{n\in\omega}(A_{f\restriction n} \cap X)$$

(3) → (5) Let

$$A = \bigcup_{f\in\omega^\omega} \cap_{n\in\omega} A_{f\restriction n}$$

where each $A_{f\restriction n}$ is closed in X. Define $C \subseteq X \times \omega^\omega$ by

$$C = \bigcap_{n\in\omega} (\cup_{s\in\omega^n}[s] \times A_s)$$

Then C is closed in $X \times \omega^\omega$ and A is the projection onto X of C.

(4) → (1) same proof as (3) → (5).

(1) → (2) Let $B \subseteq X\times Y$ be Borel. and let $\hat{B} \subseteq \hat{X}\times Y$ be Borel such that $\hat{B} \cap (X \times Y) = B$. Now if $\hat{A} = proj_{\hat{X}}(\hat{B})$ then $\hat{A}$ is $\Sigma^1_1(\hat{X})$, and $A = \hat{A} \cap X$.

(4) → (7) Let

$$T = \{Q \subseteq \omega^{<\omega} : \exists f \in \omega^\omega \forall n \in \omega\ f \restriction n \in Q\}$$

so $T \subseteq P(\omega^{<\omega})$, then

$$x \in \bigcup_{f\in\omega^\omega} \cap_{n\in\omega} A_{f\restriction n} \text{ iff } \{s \in \omega^{<\omega} : x \in A_s\} \in T$$

(7) → (1) Let $D \subseteq P(\omega) \times \omega^\omega$ be Borel such that

$$T = \{y \in P(\omega) : \exists z \in \omega^\omega\ (y, z) \in D\}$$

Then

$$Q = \{(x, y, z) : (y, z) \in D \text{ and } \forall n(n \in y \text{ iff } x \in U_n)\}$$

is Borel in $X \times \omega^\omega \times P(\omega)$ and

$$x \in A \text{ iff } \{n \in \omega : x \in U_n\} \in T \text{ iff } \exists y \exists z\ (x, y, z) \in Q$$

This completes the proof.

□

From a more abstract point of view, for example see Ulam [20], suppose we started with arbitrary countable field of subsets of a set X. We could then form the $\sigma-$algebra of subsets of X that they generated and similarly the $\sigma-$algebra of subsets of $X \times X$ generated by products of our original family and so on for all finite products X^n. Then closing under projection would give the abstract projective sets. Using the idea of Szpilrajn's characteristic function of a sequence of sets ([19]) this is basically equivalent to the following notion of Σ_1^X subset of X.

Theorem 1.2 *For X a separable metric space and $A \subseteq X$ the following are all equivalent and denoted Σ_1^X:*

1. *there exists $n \in \omega$ and a Borel set $B \subseteq X \times X^n$ such that $A = \{x \in X : \exists y \in X^n \ (x,y) \in B\}$*
2. *there exists $n \in \omega$ and a Borel set $B \subseteq X \times X^n$ and a continuous function $f : B \mapsto X$ such that $f''B = A$.*
3. *there exists $n \in \omega$ and a Borel set $B \subseteq X \times X^n$ and a Borel function $f : B \mapsto X$ such that $f''B = A$.*

proof:

(1) $\rightarrow$ (2) since projection is continuous, (2) $\rightarrow$ (3) is trivial, and (3) $\rightarrow$ (1) because the graph of f is a Borel subset of $X^n \times X$ and $f''B$ is the projection onto X of the graph of f.

$\square$

Unlike $\Sigma_1^1(X)$, any of which can be obtained by projecting a closed subset of $X \times \omega^\omega$, Σ_1^X may require projecting arbitrarily high ranking Borel subsets of $X \times X^n$. The example of Miller [5] (Theorem 43 p.259) shows this. This X has the property that there exists a $\Pi_{\alpha+1}^0$ Borel subset of X which is not the projection of any $\Sigma_{\alpha+1}^0$ set. The argument is similar to that of the last example of Section 2.

Note that it would be a mistake to consider a notion of projective which would allow arbitrary separable metric spaces Y in Theorem 1.1(6), because then every subset of X would be projective. To see this note that if $A \subseteq X$ is arbitrary, then $D = \{(x,y) \in X \times A : x = y\}$ is closed in $X \times A$ and A is the projection of D onto X.

2 Relationship between $\Sigma_1^1(X)$ and Σ_1^X

Let Borel(X) be the family of Borel subsets of X. Clearly, we always have Borel$(X) \subseteq \Sigma_1^1(X) \cap \Sigma_1^X$. In this section we give some (consistent) examples of separable metric spaces illustrating some of the possible relationships between these three families.

Example: Borel$(X) \subsetneqq \Sigma_1^1(X) = \Sigma_1^X$.

If X is an uncountable complete separable metric space, such as ω^ω, then $\Sigma_1^1(X) = \Sigma_1^X$ and Borel(X) is a proper subset of $\Sigma_1^1(X)$, i.e. Borel$(X) \subsetneqq \Sigma_1^1(X)$.

Example: Borel$(X) \subsetneqq \Sigma_1^1(X) \subsetneqq \Sigma_1^X$.

For $A \subseteq \omega^\omega$ and $n \in \omega$ let

$$(n)A = \{y \in \omega^\omega : y(0) = n \text{ and } \exists x \in A \forall m\ x(m) = y(m+1)\}$$

Let $A \subset \omega^\omega$ be a set which is not $\Sigma_1^1(\omega^\omega)$. Then $X = (0)A \cup (1)\omega^\omega$. To see that this works note that Borel$(X) \subsetneqq \Sigma_1^1(X)$ because ω^ω is a clopen subspace of X. Also because X includes ω^ω we have $\Sigma_1^1(X) \subseteq \Sigma_1^X$ (see Theorem 1.1). Also by Theorem 1.1 for every set of the form $(0)B \cup (1)C$ which is $\Sigma_1^1(X)$ we have that C is $\Sigma_1^1(\omega^\omega)$. However $(1)A$ is Σ_1^X, since it's the projection of

$$D = \{(x, y) \in X^2 : x(0) = 0, y(0) = 1, \text{ and } \forall n > 0\ x(n) = y(n)\}$$

Consequently $\Sigma_1^1(X) \subsetneqq \Sigma_1^X$.

The remaining examples are all consistent examples. The first two use Luzin sets (see Section 4).

Example: (CH) Borel$(X) = \Sigma_1^1(X) \subsetneqq \Sigma_1^X$

Let $Y \subset \omega^\omega$ be a Luzin set, so by Theorem 4.1 section 4 Borel$(Y) = \Sigma_1^1(Y)$. Let $A \subset Y$ be a set which is not $\Sigma_1^1(Y)$, and let $X = (0)Y \cup (1)A$. Since X is a Luzin set Borel$(X) = \Sigma_1^1(X)$. On the other hand $(0)A$ is Σ_1^X, so $\Sigma_1^1(X) \subsetneqq \Sigma_1^X$.

Example: If X is a generic Luzin set, then Borel$(X) = \Sigma_1^1(X) = \Sigma_1^X$.

If X is countable or every subset of X is Borel in X (for example a Q-set), then we have Borel$(X) = \Sigma_1^1(X) = \Sigma_1^X$. To get an X of cardinality the continuum we can use a generic Luzin set. By a generic Luzin set we mean that $X \subset \omega^\omega$ is produced by forcing with the partial order $\mathbb{P}$ of finite partial functions from $\omega_1 \times \omega$ into ω over some model of ZFC M. Then the $\mathbb{P}$-generic object is essentially a function $G : \omega_1 \times \omega \mapsto \omega$ and we let $X = \{x_\alpha : \alpha < \omega_1\}$ where $x_\alpha(n) = G(\alpha, n)$.

We need only show Borel$(X) = \Sigma_1^X$, since by Theorem 4.1 section 4, we already have that $\Sigma_1^1(X) =$ Borel(X). We do the argument just for the projection of Borel subsets of $X \times X$, since the argument for $X \times X^n$ is similar. Let $B \subseteq \omega^\omega \times \omega^\omega$ be a Borel set. By the countable chain condition there exists a countable set $Q \in M$ such that B has a Borel code in $M[G \restriction (Q \times \omega)]$. Let $D = \{(x, x) : x \in \omega^\omega\}$ and let $Y = \{x_\alpha : \alpha \in Q\}$ then

$$\begin{aligned} proj(B \cap (X \times X)) &= proj(B \cap (Y \times X)) \\ &\cup proj(B \cap (X \times Y)) \\ &\cup proj(B \cap D \cap X) \\ &\cup proj(B \cap (X - Y)^2 - D) \end{aligned}$$

where projection is taken onto the first coordinate. Since Y is countable and $proj(B \cap (Y \times X)) \subseteq Y$ it is Borel. Since cross sections of Borel sets are Borel and $proj(B \cap (X \times Y))$ is a countable union of cross sections, it is Borel. If we let $C = \{x : (x, x) \in B\}$, then C is Borel and $C \cap X = proj(B \cap D \cap X)$.

So it suffices to see that $proj(B \cap (X - Y)^2 - D)$ is in Borel(X). Without loss of generality we may assume that $Y = \emptyset$ and that $B \subseteq (\omega^\omega \times \omega^\omega) - D$ is coded in the ground model M (otherwise we could work over a new ground model $M[Y]$).

Let $\mathbb{Q} = \omega^{<\omega}$ the partial order for forcing a single Cohen real and let $[p] = \{x \in \omega^\omega : p \subseteq x\}$ for $p \in \mathbb{Q}$. For any two distinct $x, y \in X$ we have $x \in proj(B \cap X)$ iff there exists $y \in X$ distinct from x such that $(x, y) \in B$. But since (x, y) is $\mathbb{Q} \times \mathbb{Q}$ generic over the ground model, we have that $(x, y) \in B$ iff there exists $p, q \in \mathbb{Q}$ with $p \subset x$ and $q \subset y$ such that $(p, q) \Vdash (x, y) \in B$. But since B is a Borel set coded in the ground model $(p, q) \Vdash (x, y) \in B$ iff $([p] \times [q]) \cap B$ is comeager in $[p] \times [q]$, (see Solovay [13]). Note that X is dense, so that it is easy to check now that $x \in proj(B \cap X)$ iff $x \in X$ and $\exists p, q \in \mathbb{Q}$ $x \in [p]$ and $([p] \times [q]) \cap B$ is comeager in $[p] \times [q]$. Hence the projection of $B \cap X$ is in Borel(X).

This example can also be obtained under CH using a proof similar to that of Theorem 4.2.

Example: (the set from Miller [6]) $\Sigma_1^X \subsetneqq \Sigma_1^1(X)$.

In Miller [6] (Theorem 4 p. 177) a forcing construction is given for a set $X^* \subset \omega^\omega$ with the property that every subset of X^* is $\Sigma_1^1(X^*)$, but not every subset of X^* is in Borel(X^*). From here on we will refer to X for the X^* of [6]. The argument given in [6] that not every subset of X is Borel(X) generalizes to show that the first generic Souslin set (i.e. $A \in \Sigma_1^1(X)$) is not the projection of a Borel subset of $X \times X^n$ for any $n \in \omega$. See the last paragraph of section 3 [6]. Suppose there exists $p \in \mathbb{Q}_{\omega_2}$ and $\tau \in 2^\omega$ such that

$$p \Vdash \text{“}\forall x \in X(x \in A \text{ iff } \exists \vec{y} \in X^n\ (x, \vec{y}) \in B_\tau)\text{”} \tag{1}$$

where $B_\tau \subseteq X \times X^n$ is Σ_β^0 set with code τ. Using the countable chain condition of $p \in \mathbb{Q}_{\omega_2}$ it is easy to obtain a countable $K \subset \omega_2$ with $0 \in K$, and an α with $0 < \beta < \alpha < \omega_1$, such that K and α also satisfies $|p|(K, \alpha) = 0$, $|\tau|(K, \alpha) = 0$, and

$$\forall \delta \in K\ \forall \gamma < \alpha\ \ \{q \in \mathbb{Q}_\delta : |q|(K, \alpha) = 0\} \text{ decides “}\gamma \in Z_\delta\text{”}$$

Hence by Lemma 5 [6] $|\ |(K, \alpha)$ is a rank function with p in its domain (see [6] definition (11) p.172). Now we use the argument of the last paragraph on p.174 [6]. Let $\gamma > \alpha + \omega$ be arbitrary and extend p to p_1 by adding to $p(0)$, $p_\gamma(\emptyset) = 1$, which means that

$$p_1 \Vdash \text{“}x_\gamma \in A\text{”}$$

Since p_1 extends p by line (1)

$$p_1 \Vdash \text{“}\exists \vec{y} \in X^n\ (x_\gamma, \vec{y}) \in B_\tau\text{”}$$

So find $\vec{y} \in X^n$ and p_2 extending p_1 so that

$$p_2 \Vdash \text{“}(x_\gamma, \vec{y}) \in B_\tau\text{”}$$

Now since $(x_\gamma, \vec{y})$ is in the ground model we can think of this as a Σ_β^0 statement about τ, consequently by Lemma 2 [6] p.173, there exists a $q \in \mathbb{Q}_{\omega_2}$ with $|q|(K, \alpha) < \beta$ which is compatible with p_2 such that

$$q \Vdash \text{“}(x_\gamma, \vec{y}) \in B_\tau\text{”}$$

But now extend q to q_1 by adding to $q(0)$ that $q_\gamma(\emptyset) = 0$ (this is possible because $|q|(K,\alpha) < \beta$) but then

$$q_1 \Vdash \text{“}x_\gamma \notin A \text{ and } \exists \vec{y} \in X^n \ (x_\gamma, \vec{y}) \in B_\tau\text{”}$$

contradicting line (1) and the fact that q_1 extends p.

Problem: Give examples of X such that $\text{Borel}(X) = \Sigma^X_1 \subsetneqq \Sigma^1_1(X)$ and $\text{Borel}(X) \subsetneqq \Sigma^X_1 \subsetneqq \Sigma^1_1(X)$.

3 Closure under unions and intersections

Our first two results are simple observations.

Theorem 3.1 *For any separable metric space X the family of sets $\Sigma^1_1(X)$ is closed under countable unions and intersections.*

proof:

This is immediate from Theorem 1.1(2) since in complete metric spaces Σ^1_1 sets are closed under countable intersection and union.

□

Theorem 3.2 *For any separable metric space X the family of subsets of X, Σ^X_1, is closed under finite unions and intersections.*

proof:

Let $A_i = proj_X(B_i)$ where $B_i \subseteq X \times X^{n_i}$ is Borel for $i = 0$ or 1. By replacing B_i with $B_i \times X^{k_i}$ for a suitable k_i we may assume without loss of generality that $n_0 = n_1$. Then

$$A_0 \cup A_1 = proj(B_0 \cup B_1)$$

For intersection let $\hat{B}_0 = B_0 \times X^{n_1}$ and

$$\hat{B}_1 = \{(x,y,z) \in X \times X^{n_0} \times X^{n_1} : (x,z) \in B_0\}$$

Then

$$A_0 \cap A_1 = proj_X(\hat{B}_0 \cap \hat{B}_1)$$

□

The remainder of this section is devoted to proving the following theorem.

Theorem 3.3 *It is relatively consistent with ZFC that there exists a separable metric space X such that Σ^X_1 is closed under neither countable unions nor countable intersections.*

proof:

Fix $Y \subseteq \omega^\omega$ a set in the ground model of cardinality ω_1 and consider the following forcing notions: $\mathbb{Q}$ is the partial order of finite partial functions from Y to 2 and $\mathbb{P}$ is the direct sum of countably many copies of $\mathbb{Q}$, $\Sigma_{n\in\omega}\mathbb{Q}$. Of course both $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic to the usual way of adding ω_1 Cohen reals. We view forcing with $\mathbb{P}$ as equivalent to adding a sequence $\langle A_n : n \in \omega\rangle$ of generic subsets of Y, i.e. if G is a $\mathbb{P}$-generic filter, then for each $n \in \omega$ let $A_n = \{x \in Y : \exists p \in G \;\; p_n(x) = 1\}$. For $n \in \omega$ and $A \subseteq \omega^\omega$ recall that

$$(n)A = \{x \in \omega^\omega : x(0) = n \text{ and } \exists y \in A \, \forall m \in \omega \;\; x(m+1) = y(m)\}$$

The space X is defined by

$$X = \bigcup_{n\in\omega} (2n)Y \cup \bigcup_{n\in\omega} (2n+1)A_n$$

i.e. countably many copies of Y and one of each A_n.

Lemma 3.4 *For each $n, m \in \omega$ the set $(2m)A_n$ is Σ^X_1.*

proof:

Let $D_{nm} \subset X \times X$ be the appropriate diagonal, namely,

$$D_{nm} = \{(x, y) \in X \times X : x(0) = 2m, y(0) = 2n+1, \forall k > 0 \; x(k) = y(k)\}$$

Then D_{nm} is closed and $proj(D_{nm}) = (2m)A_n$.
□

For $k < \omega$ let $B_k = (2k)(\bigcap_{n<k} A_n)$. So B_k is Σ^X_1 by Theorem 3.2. Also let $B^*_k = B_k \cup \bigcup\{(2n)Y : n < \omega, n \neq k\}$, then B^*_k is Σ^X_1, since $\bigcup\{(2n)Y : n < \omega, n \neq k\}$ is clopen in X and hence Σ^X_1. So to prove the theorem it suffices to show R is not Σ^X_1 where R is defined by:

$$R = \bigcup_{k\in\omega} B_k = \bigcap_{k\in\omega} B^*_k$$

Now since each B_k is a clopen subset of R it suffices to prove:

Lemma 3.5 *B_{k+1} is not the projection of a Borel subset of $X \times X^k$.*

proof:

Suppose for contradiction that

$$B_{k+1} = (2k)(A_0 \cap \ldots \cap A_k) = proj(B)$$

where $B \subseteq X \times X^k$ is Borel. Decompose B as the countable union of Borel sets:

$$B = \bigcup_{n_1,\ldots,n_k \in \omega} C_{n_1,\ldots,n_k}$$

where each $C_{n_1,\ldots,n_k} \subseteq (2k)Y \times (n_1)Z_1 \times \cdots \times (n_k)Z_k$ is Borel and each Z_i is either Y or some A_j depending whether n_i is even or odd. By an easy density argument we can see that B_{k+1} must be uncountable. Hence to prove the lemma it suffices to see:

Claim: Each $proj(C_{n_1,\ldots,n_k})$ is countable.

To see this note that since there are only k Z's but $k+1$ many A_j's in the definition of B_{k+1} there must be some $j \leq k$ which does not appear as a Z_i, however $proj(C_{n_1,\ldots,n_k}) \subseteq (2k)A_j$. By the countable chain condition there exists a countable $K \subset X$ such that the Borel code for $C_{n_1,\ldots,n_k}$ and hence $C_{n_1,\ldots,n_k}$ itself is an element of $N = M[\langle A_j \upharpoonright K\rangle\langle A_i : i < \omega, i \neq j\rangle]$ where M is the ground model. It follows that $proj(C_{n_1,\ldots,n_k})$ is also in N. However $A_j \upharpoonright (Y - K)$ is generic over N, so if $proj(C_{n_1,\ldots,n_k}) \cap (2k)(Y - K)$ is infinite then $proj(C_{n_1,\ldots,n_k}) - (2k)A_j \neq \emptyset$, which would contradict the fact that $proj(C_{n_1,\ldots,n_k}) \subseteq (2k)A_j$. This proves the Claim, Lemma, and Theorem 3.3.

□

This proof also shows that it is possible that $n - \Sigma_1^X \neq (n+1) - \Sigma_1^X$ for all $n \in \omega$ where $n - \Sigma_1^X$ is the family of projections of Borel subsets of $X \times X^n$. Note also that for fixed n the family of $n - \Sigma_1^X$ sets is closed under countable union but not finite intersection. It is also true in this example that there exists a countable intersection of $1 - \Sigma_1^X$ sets which is not Σ_1^X, namely if $E_{kn} = (2n)A_k \cup \bigcup\{(2m)Y : m < \omega, m \neq n\}$ (each of which is $1 - \Sigma_1^X$), then $R = \bigcap_{n \in \omega} \bigcap_{k<n} E_{kn}$.

Problem: Can we have an example where Σ_1^X is closed under countable union but not countable intersection? Can we have an example where Σ_1^X is closed under countable intersection but not countable union?

4 Properties of Products

A separable metric space X is Luzin iff it is uncountable and every meager subset of X is countable. A set is nowhere dense iff its closure has empty interior and meager iff it is the countable union of nowhere dense sets. The following theorem is well known.

Theorem 4.1 *If X is Luzin, then every $\Sigma_1^1(X)$ set is Borel in X.*

proof:

In an arbitrary topological space the Souslin operation preserves the property of Baire (see Kuratowski [4]). Hence for any $A \in \Sigma_1^1(X)$ (by Theorem 1.1(3)) there exists open U and meager M such that $A = (U - M) \cup (M - U)$. But since meager sets are countable, clearly A is Borel.
$\Box$

Theorem 4.2 *Assume the continuum hypothesis. Then there exists a Luzin space X such that every $\Sigma_1^1(X^2)$ is Borel in X^2.*

proof:

This is true of any sufficiently generic Luzin set. Suppose that $M_\alpha \preceq (HC, \in)$ for $\alpha < \omega_1$ is an increasing sequence of countable elementary substructures whose union is all of HC, the hereditarily countable sets, and $M_\alpha \in M_{\alpha+1}$ for each α. For each $\alpha < \omega_1$ let $x_\alpha \in (2^\omega \cap M_{\alpha+1})$ be a Cohen generic real over M_α. Then $X = \{x_\alpha : \alpha < \omega_1\}$ has the required property. Suppose $A \subseteq 2^\omega \times 2^\omega$ is $\Sigma_1^1(2^\omega \times 2^\omega)$, then since it has the property of Baire, there exists a open U and meager M such that $A = (U - M) \cup (M - U)$. Let F be a meager Borel set with $M \subseteq F$. Suppose that F is coded in M_α, then for every $\beta \neq \gamma > \alpha$ we have that $(x_\beta, x_\gamma) \notin F$. To see this suppose that $\alpha < \beta < \gamma$ and note that since F is meager, for comeagerly many x, $F_x = \{y : (x, y) \in F\}$ is meager (by the Kuratowski-Ulam Theorem see Oxtoby [9]). Consequently F_{x_β} which is coded in M_γ is meager and therefore $x_\gamma \notin F_{x_\beta}$. Hence

$$A \cap \{(x_\beta, x_\gamma) : \beta \neq \gamma > \alpha\} = U \cap \{(x_\beta, x_\gamma) : \beta \neq \gamma > \alpha\}$$

Also letting $D = \{(x, x) : x \in X\}$, then since D is homeomorphic to X we have that $A \cap D$ is Borel in X. Finally for all $\beta \leq \alpha$ let $A_\beta = \{(x_\beta, x_\gamma) : \gamma <$

$\omega_1\} \cap A$ and $A^\beta = \{(x_\gamma, x_\beta) : \gamma < \omega_1\} \cap A$. Each of these is Borel in X^2, and so A is Borel in X^2.
$\Box$

This result also holds for generic Luzin sets.

Theorem 4.3 *Assume the continuum hypothesis. Then there exists a Luzin space X such that not every $\Sigma^1_1(X^2)$ is Borel in X^2.*

proof:

It suffices to construct $X, Y \subseteq 2^\omega$ Luzin sets such that there exists $A \subseteq X \times Y$ which is $\Sigma^1_1(X \times Y)$ but not (relatively) Borel in $X \times Y$. For $x, y \in 2^\omega$ let $x+y$ be pointwise addition modulo 2, i.e. $(x+y)(n) = x(n)+y(n) \bmod 2$. Let

$$A = \{(x, y) : x + y \text{ is the characteristic function of a nonwellfounded set }\}$$

More precisely let $\# : \omega^{<\omega} \mapsto \omega$ be a fixed bijection, then

$$(x, y) \in A \text{ iff } \exists f \in \omega^\omega\ \forall n \in \omega\ \ (x + y)(\#f \restriction n) = 1$$

Clearly A is $\Sigma^1_1(2^\omega \times 2^\omega)$. Lemma 4.4 will finish the proof of the theorem. Let M_α be as in the proof of Theorem 4.2 and let $\langle B_\alpha : \alpha < \omega_1\rangle$ list all Borel subsets of $2^\omega \times 2^\omega$ with B_α coded in M_α. Using Lemma 4.4 construct $X = \{x_\alpha : \alpha < \omega_1\}$ and $Y = \{y_\alpha : \alpha < \omega_1\}$ such that x_α and y_α are Cohen generic over M_α and $(x_\alpha, y_\alpha) \in (A - B_\alpha) \cup (B_\alpha - A)$. Then X and Y are Luzin sets, but $A \cap (X \times Y)$ is not Borel in $X \times Y$.

Lemma 4.4 *Suppose that M is a countable transitive model of ZFC-Power Set and $B \subseteq 2^\omega \times 2^\omega$ is a Borel set coded in M, then there exists $x, y \in 2^\omega$ Cohen generic over M such that $(x, y) \in (A - B) \cup (B - A)$.*

proof:

To prove this we use Steel forcing [14] as explained in Harrington [3]. Let $\mathbb{Q}$ be Steel forcing with tagged trees, hence

$$\mathbb{Q} = \{\langle t, h\rangle : t \subseteq \omega^{<\omega} \text{ finite subtree, } h : t \mapsto \omega_1 \cup \{\infty\} \text{ a rank function}\}$$

where rank function means that $h(\emptyset) = \infty$ and $s \subsetneqq r \in t \to h(s) < h(r)$ ($\alpha < \infty < \infty$ for $\alpha < \omega_1$). If G is $\mathbb{Q}$-generic over a model M, then G is essentially equal to (T, H) where $T \subseteq \omega^{<\omega}$ is a tree and $H : T \mapsto \omega_1 \cup \{\infty\}$ is a rank function. It has the property that if $H(s) = \infty$, then T_s is nonwellfounded ($T_s = \{t \in \omega^{<\omega} : s \,\hat{}\, t \in T\}$); and otherwise if $H(s) \in \omega_1$, then T_s is wellfounded. Let $x \in 2^\omega$ be Cohen generic real over M and let $G = (T, H)$ be $\mathbb{Q}$-generic over $M[x]$. Let $z \in 2^\omega$ be the characteristic function of some $T_{\langle n \rangle} \subseteq \omega^{<\omega}$ and let $y = x + z$.

Claim: y is a Cohen real over M.
proof:

Let $\mathbb{P} = 2^{<\omega}$ be Cohen real forcing, then iterated forcing is the same as product forcing: $\mathbb{P} \times \mathbb{Q}$ since conditions are finite. So x is $\mathbb{P}$-generic over $M[G]$ and since $z \in M[G]$ and $y = x + z$ we have that y is $\mathbb{P}$-generic over $M[G]$ and hence over M.

□

Let $\langle n \rangle$ be such that $H(\langle n \rangle) = \infty$ so that $T_{\langle n \rangle}$ is not wellfounded.

Case 1. $\langle x, y \rangle \notin B$

Since $x + y = z$ codes a nonwellfounded tree we're done, since $\langle x, y \rangle \in A - B$.

Case 2. $\langle x, y \rangle \in B$

In this case we use the main property of Steel forcing. Let $p \in G$ be a condition such that $p \Vdash \langle x, y \rangle \in B$. The statement "$\langle x, y \rangle \in B$" is a Borel proposition with code in $M[x]$ about the real z since $y = x + z$. Therefore "$\langle x, y \rangle \in B$"is equivalent to a propositional sentence in $L_{\omega_1\omega}$ built up from the atomic propositions "$s \in \hat{T}$" where $s \in \omega^{<\omega}$ and $\hat{T}$ is a name in the ground model for the generic object T. This propositional sentence is in $M[x]$ and has rank less than $\omega_1^{M[x]}$. Say it has rank γ. Then working in $M[x]$ we can find a condition $\overline{p} \in \mathbb{Q}$ such that $p(\gamma) = \overline{p}(\gamma)$ (see Harrington [3]) with the property $\overline{h}(\langle n \rangle) \in \omega_1$. By the retagging lemma $\overline{p} \Vdash \langle x, y \rangle \in B$. Hence if we take $\overline{G}$ to be $\mathbb{Q}$-generic over $M[x]$ with $\overline{p} \in \overline{G}$, then $\langle x, y \rangle \in B - A$. This proves the lemma and hence the theorem.

□

This result can also be proved for Sierpiński sets. Steel forcing has also been used effectively in Stern [15] [18] [17] [16] and Friedman [2]. This proof is a slight generalization of a classical construction due to Sierpiński [12] of a Luzin set X such that X^2 can be mapped continuously onto 2^ω. In fact we show that this set could have been used to prove Theorem 4.3.

I. Recław has pointed out the following result.

Theorem 4.5 *(Recław) For any separable metric space X if X has bounded Borel order, then X cannot be mapped continuously onto the real line.*

proof:

Theorem 12 of Bing, Bledsoe, and Mauldin [1] says that if G is a countable family of subsets of the real line closed under complementation and whose $\sigma-$algebra contains all Borel subsets of the real line, then the $\sigma-$algebra generated by G contains ω_1 distinct levels. Now suppose $f : X \mapsto \mathbb{R}$ is continuous, onto, and one-to-one. Let G smallest family of sets closed under complements and containing a basis for $\mathbb{R}$ and the image under f of a basis for X. The hierarchy generated by G must have ω_1 levels and therefore the same is true for the Borel hierarchy of X.

$\square$

Thus Recław answers a question of Miller [7] negatively, since it is impossible to map a $\sigma-$ set continuously onto the reals. The following is proved similarly to Theorem 12 of Bing et al [1].

Theorem 4.6 *Suppose G is a countable family of subsets of ω^ω closed under complementation and such that the $\sigma-$algebra generated by G, which we denote $B(G)$, contains all Borel subsets of ω^ω. Then there exists a set $X \subset \omega^\omega$ which is not in $B(G)$ but is obtained by applying the Souslin operation to sets in $B(G)$, i.e. there exists $B_s \in B(G)$ for $s \in \omega^{<\omega}$ such that $X = \bigcup_{f\in\omega^\omega} \cap_{n\in\omega} B_{f\restriction n}$*

proof:

Denote by $S(G)$ the family of sets obtained by applying the Souslin operation to sets in G. The idea of the proof is to obtain a universal set for $S(G)$. Namely there exists a map $U : \omega^\omega \mapsto S(G)$ which is onto and has the property that the diagonal $D = \{x : x \in U(x)\}$ is in $S(G)$. This will conclude the proof since D cannot be in $B(G)$, else for some $x \in \omega^\omega$ we would have $U(x) = \omega^\omega - D$ and hence for this x we would have $x \in U(x)$ iff $x \notin U(x)$.

Let $G = \{G_n : n \in \omega\}$ and let $\# : \omega^{<\omega} \mapsto \omega$ be our fixed bijection. For any $x \in \omega^\omega$ let $A^x_s = G_{x(\# s)}$ and let $U(x) = \bigcup_{f\in\omega^\omega} \cap_{n<\omega} A^x_{f\restriction n}$ We need to see that the diagonal D is in $S(G)$. For fixed $s \in \omega^{<\omega}$ let $B_s = \{x : x \in G_{x(\# s)}\}$. It is easy to see that

$$D = \{x : x \in U(x)\} = \bigcup_{f\in\omega^\omega} \cap_{n\in\omega} B_{f\restriction n}$$

Now $B_s = \bigcup_{n<\omega}\{x \in \omega^\omega : x(\#s) = n \text{ and } x \in G_n\}$ since we are assuming every clopen subset of ω^ω is in $B(G)$ we have that each B_s is in $B(G)$. Since G is closed under complementation we know that $B(G)$ is the smallest family of sets containing G and closed under countable unions and countable intersections. Two classical results of Sierpiński are that $S(S(G)) = S(G)$ and $S(G)$ is closed under countable union and countable intersection (for a proof see Rogers and Jayne [11]). So $B(G) \subseteq S(G)$ and $D \in S(G)$.
□

Note that since every uncountable complete separable metric space contains a homeomorphic Borel copy of ω^ω this result also holds for every uncountable complete separable metric space. Just as in Recław's result we have the following corollary.

Corollary 4.7 *For any separable metric space X if X can be mapped continuously onto ω^ω, then $\Sigma^1_1(X) - Borel(X)$ is nonempty.*

Problem: (Mauldin) Is it consistent to have a separable metric space X with bounded Borel order but not every $\Sigma^1_1(X)$ subset is Borel in X?

In Theorem 4.3 the Borel order of X^2 is ω_1.

5 The hierarchy of projective sets

For X a separable metric space we make the following definitions.

- Define $\Sigma^X_0 = \Pi^X_0 = \bigcup_{n\in\omega} \text{Borel}(X^n)$ (the set of all Borel subsets of finite products of X).
- Define $A \subseteq X^m$ to be Π^X_{n+1} iff $X^m - A$ is Σ^X_{n+1}.
- Define $A \subseteq X^m$ to be Σ^X_{n+1} iff there exists a $k \in \omega$ and $B \subseteq X^m \times X^k$ in Π^X_n such that $A = proj_{X^m}(B) = \{x \in X^m : \exists y \in X^k \; (x, y) \in B\}$.
- Define $\Delta^X_n = \Sigma^X_n \cap \Pi^X_n$.

Theorem 5.1 $\Delta^X_n \subseteq \Sigma^X_n \subseteq \Delta^X_{n+1}$ *and* $\Delta^X_n \subseteq \Pi^X_n \subseteq \Delta^X_{n+1}$.

proof:

Left to the reader.

□

Theorem 5.2 *Δ_n^X, Σ_n^X, and Π_n^X are closed under finite unions and intersections.*

proof:

Similar to the proof of Theorem 3.2.

□

Define the projective subsets of X to be the $\bigcup_{n\in\omega}\Sigma_n^X$ and define the projective order of X to be the least $n<\omega$ such that every projective subset of X is Σ_n^X.

Problem: (Ulam [20]) For what n does there exist a space of projective order n.

Obviously a countable space has projective order 0 and a complete uncountable space has infinite projective order.

Problem: Is it consistent with ZFC that every uncountable space has infinite projective order? In fact, I do not know if it is consistent with ZFC that every uncountable space has projective order greater than 0.

Theorem 5.3 *In the Cohen real model there exist subsets of ω^ω which have projective order 1 and 2.*

proof:

Let $X\subset\omega^\omega$ be a batch of ω_1 Cohen reals and let $A\subset X$ be a Cohen generic subset with finite conditions. Let $Y=(0)X\cup(1)A$ and let $Z=(0)X\cup(1)A\cup(2)(X-A)$. We will show that the projective order of Y is 2 and the projective order of Z is 1. We begin with the proof for Y.

An A-cylinder is one of the sets A_{in} where $1\leq i\leq n<\omega$ and $A_{in}=Y^{i-1}\times(0)A\times Y^{n-i}$. Let Σ be the smallest family of sets containing Borel(Y^n) for all n and all A-cylinders and closed under finite union and finite intersection. Our main lemma is that $\Sigma=\Sigma_1^Y$ (Lemma 5.7). The next three lemmas will be used to prove the main lemma.

Lemma 5.4 *Suppose $C \in \Sigma$ where $C \subseteq Y^n \times Y$ and there exists i, $1 \leq i \leq n$, such that for all $(\langle y_1, \ldots, y_n\rangle, y) \in C$ we have $y_i = y$, then $proj_{Y^n}(C) \in \Sigma$.*

proof:

Define $p : Y^n \mapsto Y^n \times Y$ by $p(\vec{y}) = (\vec{y}, y)$ where $y = y_i$. Then p is continuous, hence for any B Borel we have $p^{-1}(B)$ is Borel. Also for A-cylinders:

$$p^{-1}(Y^n \times (0)A) = Y^{i-1} \times (0)A \times Y^{n-i}$$

and for $j < n$:

$$p^{-1}(Y^j \times (0)A \times Y^{n-j}) = Y^j \times (0)A \times Y^{n-j-1}$$

Hence p^{-1} of elements of Σ are elements of Σ. But note that $proj_{Y^n}(C) = p^{-1}(C)$.

□

For $y \in \omega^\omega$ with $y(0) = 0$ or 1 define $\tilde{y} \in \omega^\omega$ by $\tilde{y}(0) = 1 - y(0)$ and for all $m > 0$, $\tilde{y}(m) = y(m)$.

Lemma 5.5 *Suppose $C \in \Sigma$ where $C \subseteq Y^n \times Y$ and there exists i, $1 \leq i \leq n$, such that for all $(\langle y_1, \ldots, y_n\rangle, y) \in C$ we have $y = \tilde{y}_i$, then $proj_{Y^n}(C) \in \Sigma$.*

proof:

Define $q : Y^n \mapsto Y^n \times \omega^\omega$ by $q(\vec{y}) = (\vec{y}, y)$ where $y = \tilde{y}_i$. Note that $proj_{Y^n}(C) = q^{-1}(C)$, so it is enough to check that preimages of Borel sets and A-cylinders are elements of Σ. Let $B \subseteq Y^n \times Y$ be Borel and let $\hat{B} \subseteq Y^n \times \omega^\omega$ be Borel such that $B = \hat{B} \cap (Y^n \times Y)$, then

$$q^{-1}(B) = q^{-1}(\hat{B}) \cap (Y^{i-1} \times [(0)A \cup (1)A] \times Y^{n-i})$$

This set is the intersection of a Borel set with the union of an $A-$cylindar and a clopen set, hence it is in Σ. Now we consider the preimages of $A-$cylindars, $q^{-1}(A_{jn+1})$. Suppose $1 \leq j < n+1$, then

$$q^{-1}(Y^{j-1} \times (0)A \times Y^{n+1-j}) = \{\langle y_1, \ldots, y_n\rangle : y_j \in (0)A \text{ and } y_i \in (0)A \cup (1)A\}$$

which is the union of a clopen set and an $A-$cylindar. In case $j = n+1$:

$$q^{-1}(Y^n \times (0)A) = Y^{i-1} \times (1)A \times Y^{n-i}$$

which is a clopen set. So in each case the preimage is in Σ and the lemma is proved.

□

Lemma 5.6 *Suppose $1 \leq j_1 < j_2 < \cdots < j_k \leq n+1$ (k may be zero) and $C \subseteq Y^{n+1}$ is given by*

$$C = A_{j_1 n+1} \cap A_{j_2 n+1} \cap \cdots \cap A_{j_k n+1} \cap B$$

- *$B \subseteq Y^{n+1}$ is the intersection with Y^{n+1} of a Borel subset of $(\omega^\omega)^{n+1}$ coded in $V[X \restriction \Gamma, A \restriction \Gamma]$ where Γ is a countable set indexed in the ground model V,*
- *there exists $s \in 2^{n+1}$ such that $C \subset s(0)\omega^\omega \cup \ldots \cup s(n)\omega^\omega$,*
- *there exists an equivalence relation $\approx$ on $\{0, 1, \ldots, n\}$ with the property that for all $\langle y_0, \ldots, y_n \rangle \in C$ and $i, j < n+1$*

  $$i \approx j \text{ iff } \forall m > 0 \; y_i(m) = y_j(m)$$

  and for all $i \neq n$, $i \not\approx n$,
- *there exists $t \in (\Gamma \cup \{*\})^{n+1}$ such that $t(n) = *$ and for all $i < n+1$ $t(i) \in \Gamma$ implies $y_i = s(i)\hat{\ }t(i)$ and $t(i) = *$ implies $y_i \notin (s(i))\Gamma$.*

Then $proj_{Y^n}(C) \in \Sigma$.

proof:

Define $Q \subseteq (\omega^\omega)^{n+1}$ to be the G_δ set determined by the above conditions, namely $\vec{y} \in Q$ iff for all $i, j < n+1$ $y_i(0) = s(i)$, $i \approx j$ iff $\forall m > 0$ $y_i(m) = y_j(m))$, $t(i) \in \Gamma$ implies $y_i = s(i)\hat{\ }t(i)$, and $t(i) = *$ implies $y_i \notin (s(i))\Gamma$.

Let $\mathbb{P} \subseteq (\omega^{<\omega})^{n+1}$ be the subpartial order defined by

$$\vec{p} \in \mathbb{P} \text{ iff } \exists \vec{y} \in Q \; \forall i < n+1 \;\; p_i \subseteq y_i$$

And for $\vec{p} \in \mathbb{P}$ define

$$[\vec{p}] = \{\vec{y} \in Q : \forall i < n+1 \;\; p_i \subseteq y_i\}$$

The set of $[\vec{p}]$ form a basis for Q.

Consider $V[X \restriction \Gamma, A \restriction \Gamma]$ to be the ground model. Any $\vec{y} \in Q$ determines the filter $\{\vec{p} : \forall i < n+1 \; p_i \subseteq y_i\}$ on $\mathbb{P}$. We claim that every $\vec{y} \in Y^{n+1} \cap Q$ is $\mathbb{P}$-generic over $V[X \restriction \Gamma, A \restriction \Gamma]$. To see this note that $\mathbb{P}$ is defined in $V[X \restriction \Gamma, A \restriction \Gamma]$ and the rest of X and A are generic over $V[X \restriction \Gamma, A \restriction \Gamma]$.

Since

$$C = A_{j_1 n+1} \cap \cdots \cap A_{j_k n+1} \cap B \subseteq Y^{n+1} \cap Q$$

where $B \subseteq Y^{n+1}$ is Borel and coded in the ground model $V[X \restriction \Gamma, A \restriction \Gamma]$, by genericity we have:

$$\forall \vec{y} \in C \exists \vec{p} \in \mathbb{P} \;\; \vec{y} \in [\vec{p}] \text{ and } [\vec{p}] \cap Y^{n+1} \cap Q \subseteq B$$

Let $B = Y^{n+1} \cap \hat{B}$ where $\hat{B} \subseteq Q$ is an (absolute) Borel subset of the complete metric space Q. Since Borel sets have the property of Baire, there exists an open set $U \subseteq Q$ and a meager (in Q) Borel set $F \subset Q$ such that U and F are coded in the ground model $V[X \restriction \Gamma, A \restriction \Gamma]$ and $(\hat{B} - U) \cup (U - \hat{B}) \subseteq F$. Consequently we have that $B = Y^{n+1} \cap U$. For $\vec{p} \in \mathbb{P}$ define $[\vec{p} \restriction n] \subseteq Y^n$ by $\vec{y} \in [\vec{p} \restriction n]$ iff $\forall i, j < n$, $p_i \subseteq y_i$, $y_i(0) = s(i)$, $(i \approx j$ iff $\forall m > 0 \; y_i(m) = y_j(m))$, $t(i) \in \Gamma \rightarrow y_i = s(i)\hat{\ }t(i)$, and $t(i) = * \rightarrow y_i \notin (s(i))\Gamma)$.

Claim: If $j_k < n$, then

$$proj_{Y^n}(C) = A_{j_1 n} \cap \ldots \cap A_{j_k n} \cap (\bigcup_{[\vec{p}] \subseteq U} [\vec{p} \restriction n] \cap Y^n)$$

else if $j_k = n$, then

$$proj_{Y^n}(C) = A_{j_1 n} \cap \ldots \cap A_{j_{k-1} n} \cap (\bigcup_{[\vec{p}] \subseteq U} [\vec{p} \restriction n] \cap Y^n)$$

proof:

$\subseteq$ This is clear since $B = Y^{n+1} \cap U$.

$\supseteq$ Suppose $\vec{y} = \langle y_0, \ldots, y_{n-1} \rangle \in [\vec{p} \restriction n]$ where $[\vec{p}] \subseteq U$. We need to show that $\exists y_n \in Y$ such that $(\vec{y}, y_n) \in C$. Now $A_{j_k n+1}$ may or may not be A_{nn+1} which would require $y_n \in (0)A$. But note that $t(n) = *$ so $y_n \notin (s(n))\Gamma$ and $\forall i < n$ we have $i \not\approx n$ so for all $m > 0 \; y_i(m) \neq y_n(m)$. Since A is generically chosen we can always find such a y_n.

This concludes the proof of the Claim and since the right hand sides are clearly in Σ the Lemma is proved.

□

Finally, we are ready to prove the main lemma:

Lemma 5.7 $\Sigma_1^Y = \Sigma$, *i.e. the smallest family of sets containing Borel*(Y^n) *for all n and all A-cylinders and closed under finite union and intersection.*

proof:

Recall that A-cylinders are sets of form $A_{in} = Y^{i-1} \times (0)A \times Y^{n-i}$ Each A-cylinder is in Σ_1^Y since $A_{in} = proj_{Y^n}(D_{in+1})$ where

$$D_{in+1} = \{(\vec{y}, y) \in Y^{n+1} : y_i(0) = 0, y(0) = 1, \text{ and } \forall m > 0\ y(m) = y_i(m)\}$$

Hence $\Sigma \subseteq \Sigma_1^Y$ since each Borel set in Y and each A-cylinder is in Σ_1^Y and Σ_1^Y is closed under finite unions and intersections (Theorem 5.2).

To show that $\Sigma_1^Y \subseteq \Sigma$ it is enough to show that Σ is closed under projection, i.e. if $C \in \Sigma$ and $C \subseteq Y^n \times Y$, then $proj_{Y^n}(C) \subseteq Y^n$ is in Σ. To this end for $i < n$ let $C_i = \{\vec{y} \in C : \forall m > 0\ y_i(m) = y_n(m)\}$ and define $C_n = C - \bigcup_{i<n} C_i$. Note that each C_i for $i \leq n$ is a Borel set intersected with C. Since $proj_{Y^n}(C) = \bigcup_{i \leq n} proj_{Y^n}(C_i)$ it is enough to see each $proj_{Y^n}(C_i)$ is in Σ. The case C_i for $i < n$ is handled by Lemma 5.4 and 5.5.

So without loss of generality assume $C = C_n$, i.e.

$$\forall \vec{y} \in C\ \forall i < n\ \exists m > 0\ y_i(m) \neq y_n(m) \tag{2}$$

By normal form every set in Σ which is contained in Y^{n+1} is a finite union of sets of the form: $A_{j_1 n+1} \cap \cdots \cap A_{j_k n+1} \cap B$ where B is Borel. So we can assume

$$C = A_{j_1 n+1} \cap \cdots \cap A_{j_k n+1} \cap B \tag{3}$$

where $B \subseteq Y^{n+1}$ is the intersection with Y^{n+1} of a Borel subset of $(\omega^\omega)^{n+1}$ coded in $V[X \restriction \Gamma, A \restriction \Gamma]$ where Γ is a countable set indexed in the model V. Although we will cut B down some more it will only be by intersecting it with Borel sets coded in the ground model $V[X \restriction \Gamma, A \restriction \Gamma]$. Working in this model we can write B as a union of Borel sets B_k for $k < \omega$ such that for each B_k:

$$\exists s \in 2^{n+1} \quad B_k \subset s(0)\omega^\omega \cup \ldots \cup s(n)\omega^\omega \tag{4}$$

and there exists $t \in (\Gamma \cup \{*\})^{n+1}$ such that $\forall \vec{y} \in B_k\ \forall i < n+1$

$$t(i) \in \Gamma \quad \rightarrow \quad y_i = s(i)\hat{\ }t(i) \tag{5}$$

and

$$t(i) = * \quad \rightarrow \quad y_i \notin (s(i))\Gamma \tag{6}$$

and an equivalence relation $\approx$ on $\{0, 1, \ldots, n\}$ such that $\forall \vec{y} \in B_k \forall j, i < n+1$

$$j \approx i \text{ iff } \forall m > 0\ y_i(m) = y_j(m) \tag{7}$$

Fix B_k and the t and $\approx$ given by lines (6) and (7). And let $C_k = C \cap B_k$. We claim there exists a Borel set H_k such that

$$proj_{Y^n}(C_k) = A_{j_1 n} \cap \cdots \cap A_{j_{k^*} n} \cap H_k$$

where $k^* = n - 1$ if $k = n$ and otherwise $k^* = k$. The reason is that if $t(n) \in \Gamma$, then $proj_{Y^n}(C_k)$ is the $t(n)$ cross section of C_k. Otherwise use lines (2) thru (7) to apply Lemma 5.6. Hence

$$\begin{aligned} proj_{Y^n}(C) &= proj_{Y^n}\left(\bigcup_{k<\omega} C_k\right) = \bigcup_{k<\omega} proj_{Y^n}(C_k) \\ &= \bigcup_{k<\omega}(A_{j_1 n} \cap \cdots \cap A_{j_{k^*} n} \cap H_k) = A_{j_1 n} \cap \cdots \cap A_{j_{k^*} n} \cap \bigcup_{k<\omega} H_k \end{aligned}$$

Since this set is in Σ we are done.
□

Now we prove the Theorem.

We claim the the projective order of Y is 2 where $Y = (0)X \cup (1)A$. By the Lemma 5.7 we see that $(0)A$ is not Π^Y_1, hence the projective order of Y is at least 2. Let Δ be the smallest family containing all Borel subsets of Y^n for all n and all A cylinders $(Y^i \times (0)A \times Y^j)$, and $(X - A)$ cylinders $(Y^i \times (0)(X - A) \times Y^j)$, and closed under finite union and intersection. Note that Δ is closed under complementation and $\Delta \subseteq \Delta^Y_2$.

Lemma 5.8 Δ *is closed under projection.*

proof:

Similar to Lemma 5.7.
□

Hence Δ is the set of all projective subsets of Y and the projective order of Y is 2.

Next we see that $Z = (0)X \cup (1)A \cup (2)(X - A)$ has projective order 1. Let Δ_0 be defined similarly to Δ but for Z, i.e. let Δ_0 be the smallest family containing all Borel subsets of Z^n for all n and all A-cylinders $(Z^i \times (0)A \times Z^j)$, and $(X - A)$-cylinders $(Z^i \times (0)(X - A) \times Z^j)$, and closed under finite union and intersection. Note that Δ_0 is closed under complementation and $\Delta_0 \subseteq \Delta^Z_1$.

Lemma 5.9 Δ_0 *is closed under projection.*

proof:

Similar to Lemma 5.7.

□

An easy density argument shows that $(0)A$ is not Borel in Z hence the projective order of Z is exactly 1. This ends the proof of Theorem 5.3.

□

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