

Models in which every nonmeager set is nonmeager in a nowhere dense Cantor set

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Abstract

We prove that it is relatively consistent with ZFC that in any perfect Polish space, for every nonmeager set A there exists a nowhere dense Cantor set C such that $A \cap C$ is nonmeager in C . We also examine variants of this result and establish a measure theoretic analog.

1 Introduction

Our starting point is the following question of Laczko:

Does there exist (in ZFC) a nonmeager set that is relatively meager in every nowhere dense perfect set?

Note that the continuum hypothesis implies the existence of a Luzin set, i.e., an uncountable set of reals which meets every nowhere dense set in a countable set. Hence, we can think of Laczko's question as asking whether one can construct a particular weak version of a Luzin set without any extra set theoretic assumptions.

Recall that a space X is Polish iff it is completely metrizable and separable. A subset of X is nowhere dense iff its closure has no interior and it is meager iff it is the countable union of nowhere dense sets. A subset of X is residual iff it is the complement of a meager set. A perfect set in

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a Polish space is a closed nonempty set without isolated points, and a Polish space is said to be perfect if it is nonempty and has no isolated points. As we shall see, the underlying space in the question of Laczkovich can be taken to be any perfect Polish space. If we ask, as is quite natural, for the nowhere dense perfect sets in the statement to be Cantor sets (i.e., sets homeomorphic to the Cantor middle third set), then we do not know whether the nature of the Polish space matters. Even for various standard incarnations of the reals (the real line, the Baire space, and so on), we have only partial results on their equivalence in this context. We answered Laczkovich's question for the Cantor set in 1997 by building a model where the answer is negative. (And of course the perfect nowhere dense sets in this case are necessarily Cantor sets.) Very shortly afterwards, we noticed the more elegant solution presented here which uses a slightly stronger variant of a statement proven consistent by Shelah in [Sh1980]. We show in Section 3 that the stronger conclusion in which, for any perfect Polish space, the perfect nowhere dense sets can be taken to be Cantor sets follows from yet another variant on the same statement. The proof of the consistency of the variants in question is similar to the proof of Shelah. Unfortunately, the proof is quite technical and the argument in [Sh1980] is only a brief sketch, so we give the argument in some detail in Section 4 in order to be clear. An alternative model for the negative answer to Laczkovich's question for the Cantor set is provided by a paper of Ciesielski and Shelah [CS]. See Remark 3.6. In the final section of the paper, we show how a measure theoretic version of our results can be deduced from results in Roslanowski and Shelah [RS]. The authors thank Ilijas Farah for helpful discussions concerning the models constructed in [RS].

Write $\text{perfect}(X)$ for a Polish space X to mean that for every nonmeager set $A \subseteq X$ there is a nowhere dense perfect set $P \subseteq X$ such that $A \cap P$ is nonmeager relative to P . Write $\text{cantor}(X)$ if moreover P can be taken to be a Cantor set. Note that $\text{perfect}(X)$ and $\text{cantor}(X)$ are trivially equivalent in spaces in which nowhere dense perfect sets are necessarily Cantor sets, e.g., 2^ω and \mathbb{R} .

We recall for emphasis the following well-known elementary fact of which we will make frequent use without mention.

Proposition 1.1 *If X is a topological space and Y is a dense subspace of X , then for any $A \subseteq Y$, A is nowhere dense in Y if and only if A is nowhere dense in X . Similarly, A is meager in Y if and only if A is meager in X . \square*

2 Relationships between various Polish spaces

We begin by showing that for any two perfect Polish spaces X and Y , $\text{perfect}(X)$ and $\text{perfect}(Y)$ are equivalent statements.

Proposition 2.1 *We have the following implications.*

- (a) *Suppose X is a perfect Polish space and $\text{perfect}(X)$ holds. Then $\text{perfect}(\omega^\omega)$ holds.*
- (b) *$\text{perfect}(\omega^\omega)$ implies $\text{perfect}(X)$ for every perfect Polish space X .*

PROOF. We will use the well-known fact that every perfect Polish space X has a dense G_δ subset Y homeomorphic to ω^ω . (To get Y , first remove the boundaries of the elements of a countable base

for X . What remains is a zero-dimensional dense G_δ . Remove a countable dense subset of this dense G_δ and call the result Y . Then Y is a perfect Polish space which is zero-dimensional and has no compact open sets and hence is homeomorphic to ω^ω .)

(a) Let Y be a residual subspace of X homeomorphic to ω^ω . Let A be a nonmeager set in Y . In X , A is nonmeager so there is a nowhere dense perfect set C so that A is nonmeager in C . By replacing C by the closure of one of its nonempty open subsets, we may assume that A is everywhere nonmeager in C . In particular, $A \cap C$ is dense in C . Note that $F = Y \cap C$ is closed relative to Y , is nonempty and has no isolated points (because it contains $A \cap C$ which is dense in C). Since F is dense in C , $A \cap C = (A \cap Y) \cap C = A \cap F$ is not meager in F . Also, because Y is dense in X and F is nowhere dense in X , F is also nowhere dense in Y .

(b) Let X be a perfect Polish space. Let Y be a residual subspace of X homeomorphic to ω^ω . Let $A \subseteq X$ be nonmeager. Then $A \cap Y$ is nonmeager in X and hence in Y as well since Y is dense. By perfect(ω^ω), there is a nowhere dense perfect set C in Y such that $A \cap C$ is nonmeager in C . If P denotes the closure of C in X , then, since C is dense in P , $A \cap C$ is nonmeager in P and hence $A \cap P$ is also nonmeager in P . P is perfect since it is the closure of a nonempty set without isolated points. P is nowhere dense since it is the closure of a set which is nowhere dense in Y and hence in X as well. \square

Part (b) holds for $\text{cantor}(\cdot)$ by an easier argument.

Proposition 2.2 $\text{cantor}(\omega^\omega)$ implies $\text{cantor}(X)$ for every perfect Polish space X .

PROOF. Similar to the proof of Proposition 2.1(b), except that this time the proof yields a nowhere dense Cantor set $C \subseteq Y$ such that $A \cap C$ is nonmeager in C and then we are done. \square

We do not know whether (a) holds for $\text{cantor}(\cdot)$.

Problem 2.3 Does $\text{cantor}(2^\omega)$ imply $\text{cantor}(\omega^\omega)$?

Problem 2.4 Does $\text{cantor}([0, 1])$ imply $\text{cantor}([0, 1] \times [0, 1])$?

Of course, $\text{cantor}([0, 1]) \equiv \text{perfect}([0, 1]) \equiv \text{perfect}(2^\omega) \equiv \text{cantor}(2^\omega)$, so these two questions have equivalent hypotheses.

We introduce one more version of $\text{perfect}(X)$ based on the following observation. Suppose that $\text{perfect}(\omega^\omega)$ holds. Then for any nonmeager set A , we have a nowhere dense perfect set P such that $A \cap P$ is nonmeager in P . Replacing P by the closure of one of its open sets, we may assume that $A \cap P$ is everywhere nonmeager in P . Then if P has a compact open subset U , then U is a Cantor set and $A \cap U$ is nonmeager in U . Otherwise, P itself is homeomorphic to ω^ω . Hence, the perfect set P in the conclusion of $\text{perfect}(\omega^\omega)$ can always be taken to be either a closed nowhere dense copy of ω^ω or a Cantor set. Let $\text{baire}(X)$ be the strengthening of $\text{perfect}(X)$ in which we require that the perfect nowhere dense sets in the definition be homeomorphic to the Baire space ω^ω . Of course a Polish space need not contain any closed copies of ω^ω , so $\text{baire}(X)$ can fail. However, when $X = \omega^\omega$ it would seem reasonable that $\text{baire}(X)$ might hold, and we will show in the next section that $\text{baire}(\omega^\omega)$ is indeed consistent. Its relationship to $\text{cantor}(\omega^\omega)$ is unclear to us.

Problem 2.5 (a) Does $\text{perfect}(\omega^\omega)$ imply that one of $\text{baire}(\omega^\omega)$ or $\text{cantor}(\omega^\omega)$ must hold? (b) Does either of $\text{baire}(\omega^\omega)$ or $\text{cantor}(\omega^\omega)$ imply the other?

3 Consistency results

We now turn to the proof of the consistency of $\text{cantor}(\omega^\omega)$ and $\text{baire}(\omega^\omega)$. We need a variation on the following result which forms part of the proof of [Sh1980, Theorem 4.7] which states that if ZFC is consistent, then so is $\text{ZFC} + 2^\omega = \omega_2 +$ “There is a universal (linear) order of power ω_1 .”

Theorem 3.1 *If ZFC is consistent, then so is ZFC + both of the following statements.*

- (a) *There is a nonmeager set in \mathbb{R} of cardinality ω_1 .*
- (b) *Let A and B be everywhere nonmeager subsets of \mathbb{R} of cardinality ω_1 . Then A and B are order-isomorphic.*

We shall need the following variant of this result.

Theorem 3.2 *If ZFC is consistent, then so is ZFC + both of the following statements.*

- (a)' *Every nonmeager set in \mathbb{R} has a nonmeager subset of cardinality ω_1 .*
- (b)' *Let A and B be everywhere nonmeager subsets of \mathbb{R} of cardinality ω_1 . Suppose we are given countable dense subsets $A_0 \subseteq A$ and $B_0 \subseteq B$. Then A and B are order-isomorphic by an order isomorphism taking A_0 isomorphically to B_0 .*

Problem 3.3 In the presence of (a), does (b) imply (b)'?

We shall in fact verify in Theorem 4.9 that in (b)' we can even ask that given pairwise disjoint countable dense subsets $A_i, i < \omega$, of A and pairwise disjoint countable dense subsets $B_i, i < \omega$, of B , the order-isomorphism of A and B takes A_i isomorphically to B_i for each $i < \omega$. As explained in the introduction, the proof is similar to the one in [Sh1980], but as the proof is quite technical and the argument in [Sh1980] is only a brief sketch, we need to give the argument in some detail in order to be clear. We do that in the next section. Here, we derive the consequences of interest to us for this paper. The definition of $\text{baire}(X)$ is given at the end of Section 2.

Theorem 3.4 *Assume (a)' and (b)'. Then $\text{cantor}(\omega^\omega)$ and $\text{baire}(\omega^\omega)$ both hold.*

PROOF. We will use the following elementary fact.

FACT 3.5 If $K, L \subseteq \mathbb{R}$ are dense and $h: K \rightarrow L$ is an order isomorphism, then h extends to an order isomorphism of \mathbb{R} . □

Suppose that $A \subseteq \mathbb{R} \setminus \mathbb{Q}$ is not meager. We wish to find a Cantor set $C \subseteq \mathbb{R} \setminus \mathbb{Q}$ such that $A \cap C$ is nonmeager relative to C . By (a)', we may assume that A has cardinality exactly ω_1 . A is everywhere nonmeager in some open interval (a, b) . Let $C \subseteq (a, b) \setminus \mathbb{Q}$ be a Cantor set, and, by (a)', let $B \subseteq C$ be a set of cardinality ω_1 which is nonmeager relative to C . Then $(A \cup \mathbb{Q}) \cap (a, b)$ and $(A \cup B \cup \mathbb{Q}) \cap (a, b)$ are both everywhere nonmeager in (a, b) and both have cardinality ω_1 . By (b)', there is an order-isomorphism $h: (A \cup \mathbb{Q}) \cap (a, b) \rightarrow (A \cup B \cup \mathbb{Q}) \cap (a, b)$ such that $h[\mathbb{Q} \cap (a, b)] = \mathbb{Q} \cap (a, b)$. Extend h

to (a, b) and denote the extension also by h . Since h is a homeomorphism, $h^{-1}[C]$ is a Cantor set and $h^{-1}[B]$ is non meager relative to $h^{-1}[C]$. Since $h^{-1}[C] \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $h^{-1}[B] \subseteq A$, we are done.

To get $\text{baire}(\omega^\omega)$, we make a different choice of C in the preceding argument. This time, choose C to be any Cantor set so that $C \cap \mathbb{Q}$ is dense in C . Then $h^{-1}[C]$ will have the same property, so $h^{-1}[C \setminus \mathbb{Q}] = h^{-1}[C] \setminus \mathbb{Q}$ is closed nowhere dense in $\mathbb{R} \setminus \mathbb{Q}$ and homeomorphic to ω^ω . \square

Remark 3.6 The reader can easily verify that a similar but simpler argument yields that (a)' and (b) imply $\text{perfect}(\mathbb{R})$. An alternative proof of the consistency of $\text{perfect}(\mathbb{R})$ can be had by using Theorem 2 of [CS] which states that the following statement is consistent relative to ZFC:

For every $A \subseteq 2^\omega \times 2^\omega$ for which the sets A and $A^c = (2^\omega \times 2^\omega \setminus A)$ are nowhere meager in $2^\omega \times 2^\omega$ there is a homeomorphism $f : 2^\omega \rightarrow 2^\omega$ such that the set $\{x \in 2^\omega : (x, f(x)) \in A\}$ does not have the Baire property in 2^ω .

(A set has the Baire property if it has the form $U \Delta M$ where U is open and M is meager.) Note that the map $2^\omega \rightarrow 2^\omega$ given by $x \mapsto (x, f(x))$ is a homeomorphism. Hence the conclusion could be stated as “ $f \cap A$ does not have the Baire property in f ”. Since $2^\omega \times 2^\omega$ is homeomorphic to 2^ω and the graph of a homeomorphism of 2^ω is a perfect nowhere dense set in $2^\omega \times 2^\omega$, the statement above implies the following special case of $\text{perfect}(2^\omega)$ (which is equivalent to $\text{perfect}(\mathbb{R})$).

For every $A \subseteq 2^\omega$ for which the sets A and A^c are both nowhere meager in 2^ω , there is a perfect nowhere dense set P such that the set $A \cap P$ is not meager in P .

To reduce $\text{perfect}(2^\omega)$ to this special case, consider a nonmeager set $A \subseteq 2^\omega$. A is everywhere nonmeager in some clopen set U . If A is comeager in some clopen set, then it contains a nowhere dense perfect set and we are done. Hence we may assume that, relative to U , A and A^c are both everywhere nonmeager. The clopen set U is homeomorphic to 2^ω , so we now find ourselves in the special case described above.

4 Order-isomorphisms of everywhere nonmeager sets

We now turn to the proof of the consistency of (a)' and (b)'. We begin by recalling the basic properties of oracle-cc forcing. See [Sh1998, Chapter IV] for the details. A version of this material is also explained in [Bu, Sections 4–6].

Definition 4.1 A sequence

$$\overline{M} = \langle M_\delta : \delta \text{ is a limit ordinal } < \omega_1 \rangle$$

is called an *oracle* if each M_δ is a countable transitive model of a sufficiently large fragment of ZFC, $\delta \in M_\delta$ and for each $A \subseteq \omega_1$, $\{\delta : A \cap \delta \in M_\delta\}$ is stationary in ω_1 .

The meaning of “sufficiently large” depends on the context. In a particular proof, some fragment of ZFC for which models can be produced in ZFC must suffice for all the oracles in the proof. The existence of an oracle is equivalent to \diamond , (see [Ku, Theorem II 7.14]) and hence implies CH. We

limit the definition of the \overline{M} -chain condition to partial orders of cardinality ω_1 . This covers our present needs.

Associated with an oracle \overline{M} , there is a filter $\text{Trap}(\overline{M})$ generated by the sets

$$\{\delta < \omega_1 : \delta \text{ is a limit ordinal and } A \cap \delta \in M_\delta\}, \quad A \subseteq \omega_1.$$

This is a proper normal filter containing all closed unbounded sets.

Definition 4.2 If P is any partial order, $P' \subseteq P$, and \mathfrak{D} is any class of sets, then we write $P' <_{\mathfrak{D}} P$ to mean that every predense subset of P' which belongs to \mathfrak{D} is predense in P .

Definition 4.3 A partial order P satisfies the \overline{M} -chain condition, or simply is \overline{M} -cc, if there is a one-to-one function $f: P \rightarrow \omega_1$ such that

$$\{\delta < \omega_1 : \delta \text{ is a limit ordinal and } f^{-1}(\delta) <_{M_{\delta,f}} P\}$$

belongs to $\text{Trap}(\overline{M})$, where $M_{\delta,f} = \{f^{-1}(A) : A \subseteq \delta, A \in M_\delta\}$.

It is not hard to verify that if P is \overline{M} -cc, then P is ccc. Also, any one-to-one function $g: P \rightarrow \omega_1$ can replace f in the definition.

Proposition 4.4 *The \overline{M} -cc satisfies the following properties.*

- (1) *If $\alpha < \omega_2$ is a limit ordinal, $\langle \langle P_\beta \rangle_{\beta \leq \alpha}, \langle \dot{Q}_\beta \rangle_{\beta < \alpha} \rangle$ is a finite-support α -stage iteration of partial orders, and for each $\beta < \alpha$, P_β is \overline{M} -cc, then P_α is \overline{M} -cc.*
- (2) *If P is \overline{M} -cc, then there is a P -name \overline{M}^* for an oracle such that for each P -name \dot{Q} for a partial order, if \Vdash_P “ \dot{Q} is \overline{M}^* -cc” then $P * \dot{Q}$ is \overline{M} -cc.*
- (3) *If $\overline{M}_\alpha, \alpha < \omega_1$, are oracles, then there is an oracle \overline{M} such that for any partial order P , if P is \overline{M} -cc, then P is \overline{M}_α -cc for all $\alpha < \omega_1$.*

We will need the following lemmas.

Lemma 4.5 *Let $\overline{M} = \langle M_\delta : \delta < \omega_1 \rangle$ be an oracle and let A and B be everywhere nonmeager subsets of \mathbb{R} . Suppose we are given pairwise disjoint countable dense subsets $A_i, i < \omega$, of A and pairwise disjoint countable dense subsets $B_i, i < \omega$, of B . Then there is a forcing notion P satisfying the \overline{M} -cc such that for every $G \subseteq P$ generic over V , $V[G] \models A$ and B are order-isomorphic by an order isomorphism taking A_i isomorphically to B_i for each $i < \omega$.*

PROOF. Fix well-orderings of A and B in type ω_1 . (CH holds because there is an oracle.) We will inductively define one-to-one enumerations $\langle a_\alpha : \alpha < \omega_1 \rangle$ of A and $\langle b_\alpha : \alpha < \omega_1 \rangle$ of B and functions $f_\delta, \delta < \omega_1$. We let $A_\delta = \{a_\alpha : \omega\delta \leq \alpha < \omega(\delta + 1)\}$ and $B_\delta = \{b_\alpha : \omega\delta \leq \alpha < \omega(\delta + 1)\}$ for $\delta < \omega_1$. For $A' \subseteq A$ and $B' \subseteq B$, let $P(A', B')$ denote the set of finite partial order-preserving maps $p: A' \rightarrow B'$ such that $p[A_\delta] \subseteq B_\delta$ for all $\delta < \omega_1$. We also use the notation

$$A \upharpoonright \alpha = \{a_\beta : \beta < \alpha\}, \quad B \upharpoonright \alpha = \{b_\beta : \beta < \alpha\}.$$

We will arrange that the following conditions hold.

- (1) The sets A_δ and B_δ are dense in \mathbb{R} .
- (2) For $\delta < \omega$, the sets A_δ and B_δ are as in the hypothesis.
- (3) For each $\delta < \omega_1$, f_δ is a bijective map of $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$ onto $\omega\delta$.
- (4) For each $\delta < \delta' < \omega_1$, $f_\delta \subseteq f_{\delta'}$.
- (5) For each infinite $\delta < \omega_1$, the predense subsets of $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$ which have the form $f_\delta^{-1}[S]$ for some $S \in \bigcup_{\eta \leq \delta} M_\eta$ remain predense in $P(A \upharpoonright \omega(\delta+1), B \upharpoonright \omega(\delta+1))$.

To do this, we proceed as follows. The construction of the functions f_δ is dictated by (4) at limit stages, and $f_{\delta+1}$ is an arbitrary extension of f_δ satisfying (3). The elements of A_δ and B_δ for $\delta < \omega$ are given by (2). For $\delta \geq \omega$, by induction on δ we choose the elements of A_δ and B_δ by alternately defining $a_{\omega\delta+n}$ and $b_{\omega\delta+n}$, beginning with $a_{\omega\delta}$ when δ is even and with $b_{\omega\delta}$ when δ is odd. Let us illustrate the construction with the case where δ is even. Fix an enumeration $\langle I_m : 0 < m < \omega \rangle$ of the nonempty open intervals with rational endpoints. The first element $a_{\omega\delta}$ is simply the least element, in the well-ordering of A fixed at the beginning of the proof, which is different from any of the elements of A chosen so far. We now choose $b_{\omega\delta}, a_{\omega\delta+1}, b_{\omega\delta+1}, a_{\omega\delta+2}, b_{\omega\delta+2}, \dots$ in that order. For $n > 0$, we pick $a_{\omega\delta+n}$ and $b_{\omega\delta+n}$ from I_n to ensure A_δ and B_δ will be dense.

To choose one of these elements, say $b_{\omega\delta+n}$, let N be a countable elementary submodel of H_θ , for a suitably large θ , such that A, B, f_δ , the sequences $\langle a_\alpha : \alpha \leq \omega\delta + n \rangle$ and $\langle b_\alpha : \alpha < \omega\delta + n \rangle$, and $\bigcup_{\eta \leq \delta} M_\eta$ are all elements of N . Choose $b_{\omega\delta+n}$ to be a member of B which is a Cohen real over N .

We must check that the construction gives (5). Let D be a predense subset of $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$ of the appropriate form. In particular, we have $D \in N$. We will show that D remains predense in $P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n + 1)$.

Remark 4.6 We are showing by induction on n that D remains predense in $P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n)$ and then in $P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n + 1)$. (This establishes (5) since each member of $P(A \upharpoonright \omega(\delta+1), B \upharpoonright \omega(\delta+1))$ belongs to $P(A \upharpoonright \omega\delta + n, B \upharpoonright \omega\delta + n)$ for some $n < \omega$.) Our current stage has the second form. Note that at the stage $n = 0$, we first consider the passage from $P(A \upharpoonright \omega\delta, B \upharpoonright \omega\delta)$ to $P(A \upharpoonright \omega\delta + 1, B \upharpoonright \omega\delta)$. But these two partial orders are equal because there is no legal target value for $a_{\omega\delta}$ until $b_{\omega\delta}$ is chosen. So the preservation of the predense sets trivially holds at that stage. In particular, it does not matter that $a_{\omega\delta}$ is not Cohen generic over the previous construction.

Let

$$p \in P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n + 1) \setminus P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n).$$

Then p has the form $q \cup \{(a, b_{\omega\delta+n})\}$ for some $q \in P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n)$ and $a \in \{a_{\omega\delta+m} : m \leq n\}$. Fix $r \in D$. The set

$$\{b \in \mathbb{R} : q \cup \{(a, b)\} \text{ is compatible with } r\} \in N$$

(“compatible with” here means only that $q \cup \{(a, b)\} \cup r$ is a finite order isomorphism) is open and hence its complement C_r is closed, as is the set $C_D = \bigcap_{r \in D} C_r$ of b for which $q \cup \{(a, b)\}$ is

incompatible with every member of D . Since p is an partial order isomorphism, there are open rational intervals J_1 and J_2 such that $J_1 \cap \text{dom } p = \{a\}$, $J_2 \cap \text{ran } p = \{b_{\omega\delta+n}\}$. Note that whenever $x \in J_1$ and $b \in J_2$, $q \cup \{(x, b)\}$ is a partial isomorphism.

Claim 4.7 C_D is nowhere dense in J_2 .

PROOF. Fix a nonempty open subinterval J of J_2 . There is an extension of q by members of $A_0 \times B_0$ —the point of using A_0 and B_0 being simply that they are dense and contained in $A(\omega\delta + n + 1)$ and $B(\omega\delta + n)$, respectively—which adds two points in $J_1 \times J$ straddling the line $x = a$. So this extension has the form

$$q' = q \cup \{(x_1, y_1), (x_2, y_2)\}, \quad x_1 < a < x_2, \quad y_1 < y_2$$

where $(x_1, x_2) \subseteq J_1$ and $(y_1, y_2) \subseteq J$. Since $q' \in P(A \upharpoonright \omega\delta + n + 1, B \upharpoonright \omega\delta + n)$, by the induction hypothesis D must have an element r compatible with this extension. Since $a \notin A(\omega\delta)$, we have $a \notin \text{dom } r$. Let x'_1, x'_2 be the closest members of $\text{dom}(q' \cup r)$ to the left and right of a , respectively. Write $y'_1 = r(x'_1)$, $y'_2 = r(x'_2)$. Then $(y'_1, y'_2) \subseteq (y_1, y_2) \subseteq J$ and for any choice of $b \in (y'_1, y'_2)$, $q \cup \{(a, b)\}$ is compatible with r . Hence (y'_1, y'_2) is disjoint from C_r and hence from C_D . This proves the claim. \square

Thus, $b_{\omega\delta+n} \notin C_D$ and hence p is compatible with some member of D . This establishes (5).

Now take $P = P(A, B)$. The fact that P forces the desired order-isomorphism of A and B is clear from (1) and (2). To see that P is \overline{M} -cc, let $f = \bigcup_{\delta < \omega_1} f_\delta: P \rightarrow \omega_1$. For any $\delta < \omega_1$ we have $f^{-1}[\omega\delta] = P(A(\omega\delta), B(\omega\delta))$ and for each $S \subseteq \omega\delta$ whenever a set D of the form $f^{-1}[S] = f_\delta^{-1}[S]$ belongs to M_δ and is predense in $P(A(\omega\delta), B(\omega\delta))$, a simple induction on δ' using (5) shows that if δ is infinite and $\delta < \delta' \leq \omega_1$, then D is predense in $P(A(\omega\delta'), B(\omega\delta'))$. In particular, D is predense in $P = P(A(\omega_1), B(\omega_1))$. For a club of $\delta < \omega_1$ we have $\omega\delta = \delta$, so this shows that P satisfies the \overline{M} -cc. \square

Lemma 4.8 Assume \diamond . Let A be a nonmeager subset of \mathbb{R} . Then there is an oracle $\overline{M} = \langle M_\delta : \delta < \omega_1 \rangle$ such that if P is any partial order satisfying the \overline{M} -cc, then \Vdash_P “ A is nonmeager”.

PROOF. This is [Sh1998, Example IV 2.2]. \square

Theorem 4.9 If ZFC is consistent, then so is ZFC + both of the following statements.

- (a) Every nonmeager set in \mathbb{R} has a nonmeager subset of cardinality ω_1 .
- (b) Let A and B be everywhere nonmeager subsets of \mathbb{R} of cardinality ω_1 . Suppose we are given pairwise disjoint countable dense subsets $A_i, i < \omega$, of A and pairwise disjoint countable dense subsets $B_i, i < \omega$, of B . Then A and B are order-isomorphic by an order isomorphism taking A_i isomorphically to B_i for each $i < \omega$.

PROOF. Start with a ground model of $V = L$. Fix a diamond sequence

$$\langle (f_\alpha, g_\alpha, h_\alpha) : \alpha < \omega_2, \text{cof}(\alpha) = \omega_1 \rangle$$

for trapping triples (f, g, h) consisting of:

- (1) A function $f: \omega_2 \rightarrow ([\omega_2]^{\leq \omega})^\omega$. The idea of f is that, with ω_2 identified with the ccc partial order we are about to build, $[\omega_2]^{\leq \omega}$ contains the maximal antichains. Thus, $([\omega_2]^{\leq \omega})^\omega$ contains a name for each real number (construed as a subset of ω). Then for any nonmeager set X in the extension, we can find a ground model function $f: \omega_2 \rightarrow ([\omega_2]^{\leq \omega})^\omega$ enumerating the names of the elements of X .
- (2) Functions $g, h: \omega_1 \rightarrow ([\omega_2]^{\leq \omega})^\omega$ intended to represent (enumerations of the names for the elements of) everywhere nonmeager sets of cardinality ω_1 with each of the sets $\{g(\omega i + n) : n < \omega\}$ and $\{h(\omega i + n) : n < \omega\}$, for $i < \omega$, dense in \mathbb{R} .

So for each $\alpha < \omega_2$ of cofinality ω_1 , $f_\alpha: \alpha \rightarrow ([\alpha]^{\leq \omega})^\omega$, and $g_\alpha, h_\alpha: \omega_1 \rightarrow ([\alpha]^{\leq \omega})^\omega$. Also, for each (f, g, h) as in (1)–(2), $\{\alpha < \omega_2 : \text{cof}(\alpha) = \omega_1, f \upharpoonright \alpha = f_\alpha, g \upharpoonright \alpha = g_\alpha \text{ and } h \upharpoonright \alpha = h_\alpha\}$ is stationary in ω_2 .

We will inductively define an ω_2 -stage finite support iteration

$$\langle \langle P_\alpha \rangle_{\alpha \leq \omega_2}, \langle \dot{Q}_\alpha \rangle_{\alpha < \omega_1} \rangle$$

as well as a P_α -names \overline{M}_α for oracles and one-to-one functions $F_\alpha: P_\alpha \rightarrow \omega_2$ for $\alpha < \omega_2$ such that the range of each F_α is an initial segment of ω_2 which includes α and for $\beta < \alpha < \omega_2$, we have $F_\beta \subseteq F_\alpha$. (At each stage, F_α is any function satisfying these conditions.)

For $\alpha < \omega_2$, we will let \dot{X}_α denote the P_α -name for the set of real numbers whose elements have the names

$$\bigcup_{n < \omega} \{n\} \times F_\alpha^{-1}(f_\alpha(\xi)(n)), \quad \xi < \alpha.$$

Similarly, we will let \dot{A}_α and \dot{B}_α denote the ω_1 -sequences of P_α -names for real numbers

$$\langle \bigcup_{n < \omega} \{n\} \times F_\alpha^{-1}(g_\alpha(\xi)(n)) : \xi < \omega_1 \rangle$$

and

$$\langle \bigcup_{n < \omega} \{n\} \times F_\alpha^{-1}(h_\alpha(\xi)(n)) : \xi < \omega_1 \rangle$$

respectively. At stage $\alpha < \omega_2$ of the construction, if $\text{cof}(\alpha) = \omega_1$ and if

$$\Vdash_{P_\alpha} \dot{X}_\alpha \text{ is not meager,}$$

then we use Lemma 4.8 to get a P_α -name \overline{M}'_α for an oracle so that if P is any forcing notion which satisfies the \overline{M}'_α -cc, then X_α remains nonmeager after forcing with P . Otherwise, in particular if $\text{cof}(\alpha) \neq \omega_1$, we let \overline{M}'_α be any P_α -name for an oracle.

For $\beta < \alpha$, let $P_{\beta\alpha}$ be the usual P_β -name for a partial order such that P_α is isomorphic to a dense subset of $P_\beta * P_{\beta\alpha}$ (see [Ba]). Let $\overline{M}_{\beta\alpha}$ be a P_α -name for an oracle such that

- (1) If \Vdash_{P_β} “ $P_{\beta,\alpha}$ is $\overline{M}_{\beta\alpha}$ -cc and $\Vdash_{P_{\beta,\alpha}} \dot{Q}_\alpha$ is $\overline{M}_{\beta\alpha}$ -cc”,
then \Vdash_{P_β} “ $P_{\beta,\alpha+1} = P_{\beta,\alpha} * \dot{Q}_\alpha$ is $\overline{M}_{\beta\alpha}$ -cc”.

(There is such an $\overline{M}_{\beta\alpha}$ by Proposition 4.4(2). In (1), $\overline{M}_{\beta\alpha}$ is actually a P_β -name for a $P_{\beta,\alpha}$ -name for an oracle. We denote the corresponding P_α -name also by $\overline{M}_{\beta\alpha}$.)

Let \overline{M}_α be a P_α -name for an oracle such that

$$(2) \quad \Vdash_{P_\alpha} \text{“If } \dot{Q}_\alpha \text{ is } \overline{M}_\alpha\text{-cc, then } \dot{Q}_\alpha \text{ is } \overline{M}'_\alpha\text{-cc and } \overline{M}_{\beta\alpha}\text{-cc for all } \beta < \alpha\text{”}.$$

(Use Proposition 4.4(3).)

Now, if $\text{cof}(\alpha) = \omega_1$ and if

\Vdash_{P_α} The ranges of $\dot{A}_\alpha, \dot{B}_\alpha$ are everywhere nonmeager and each of the sets

$$\{\dot{A}_\alpha(\omega i + n) : n < \omega\}, \{\dot{B}_\alpha(\omega i + n) : n < \omega\}, \text{ for } i < \omega, \text{ is dense in } \mathbb{R}.$$

then use Lemma 4.5 to get a P_α -name \dot{Q}_α for a partial order satisfying the \overline{M}_α -cc and forcing an isomorphism between A_α and B_α as described in the statement of the lemma. In all other cases, take \dot{Q}_α to name the partial order Q for adding one Cohen real. We have thus

$$(3) \quad \Vdash_{P_\alpha} \text{“}\dot{Q}_\alpha \text{ satisfies the } \overline{M}_\alpha\text{-cc”}.$$

Now suppose that for some P_{ω_2} -name \dot{X} we have

$$\Vdash_{P_{\omega_2}} \dot{X} \text{ is not meager.}$$

(Every nonmeager set in any extension has a name forced by the weakest condition to be nonmeager since there is always a nonmeager set.) Fix a name \dot{f} such that

$$\Vdash_{P_{\omega_2}} \dot{f} : \omega_2 \rightarrow \dot{X} \text{ is onto.}$$

Then define $f : \omega_2 \rightarrow ([\omega_2]^{\leq \omega})^\omega$ so that if

$$\tau_\xi = \bigcup_{n < \omega} \{n\} \times F^{-1}(f(\xi)(n)), \quad \xi < \omega_2,$$

then for each $\xi < \omega_2$,

$$\Vdash_{P_{\omega_2}} \dot{f}(\xi) = \tau_\xi.$$

There is a closed unbounded set $C \subseteq \omega_2$ such that for each $\alpha \in C$ of cofinality ω_1 we have:

- (i) $f \upharpoonright \alpha : \alpha \rightarrow ([\alpha]^{\leq \omega})^\omega$.
- (ii) $\forall \xi < \alpha$, τ_ξ is a P_α -name.
- (iii) $\Vdash_{P_\alpha} \{\tau_\xi : \xi < \alpha\}$ is not meager.

(For (iii), note that when α has cofinality ω_1 , each P_α -name for a meager set is a P_β -name for some $\beta < \alpha$. Thus, if M is an elementary submodel of H_θ for a suitably large θ such that $|M| = \omega_1$, $M^\omega \subseteq M$, $\langle \tau_\xi : \xi < \omega_2 \rangle \in M$ and $\alpha = M \cap \omega_2 \in \omega_2$ has cofinality ω_1 , then for each (nice) P_α -name σ for a meager Borel set set, we have $\sigma \in M$ and hence M knows about a maximal antichain of conditions each deciding a ξ for which τ_ξ is forced not to be in σ . The antichain is countable and hence contained in M . For each condition in the antichain, the least ξ which it decides is in M and hence below α . Hence $\Vdash_{P_\alpha} \text{“}\{\tau_\xi : \xi < \alpha\} \text{ is not contained in } \sigma\text{”}.$)

Choose such an α of cofinality ω_1 for which $f \upharpoonright \alpha = f_\alpha$. By (i) and (ii), the definition of τ_ξ would not change if we used f_α instead of f and F_α instead of F . Then from the definition of \dot{X}_α we get

$$\Vdash_{P_\alpha} \dot{X}_\alpha = \{\tau_\xi : \xi < \alpha\}.$$

So at stage α we chose a P_α -name \overline{M}_α and we arranged that

$$\Vdash_{P_\alpha} \text{“} P_{\alpha,\gamma} \text{ is } \overline{M}_\alpha\text{-cc”}.$$

(This follows easily by induction on $\gamma \geq \alpha$ and Propositions 4.4(1,2). (Recall that $P_{\alpha,\gamma}$ can be viewed in canonical way as an iteration: see [Ba]. At limits γ use Propositions 4.4(1). At stages $\gamma + 1$, use (3) to get \Vdash_{P_γ} “ \dot{Q}_γ satisfies the \overline{M}_γ -cc” and then use (2) and (1) with (β, α) replaced by (α, γ) .)

Hence, by the choice of \overline{M}_α ,

$$(4) \quad \Vdash_{P_\alpha} \Vdash_{P_{\alpha,\gamma}} \dot{X}_\alpha \text{ is not meager}$$

from which it follows that

$$\Vdash_{P_\alpha} \Vdash_{P_{\alpha,\omega_2}} \dot{X}_\alpha \text{ is not meager}$$

since if this failed then we would have

$$p \Vdash_{P_\alpha} q \Vdash_{P_{\alpha,\omega_2}} \dot{X}_\alpha \subseteq \dot{B}$$

for some conditions $p \in P_\alpha$, $q \in P_{\alpha,\omega_2}$ and some name \dot{B} for a meager Borel set. But then for some γ , we have $\alpha < \gamma < \omega_2$, $q \in P_{\alpha,\gamma}$ and \dot{B} is a P_γ -name and this contradicts (4).

By what we have established, there are guaranteed to be sets of cardinality ω_1 which are not meager in any extension by P_{ω_2} . Hence there are guaranteed to be everywhere nonmeager sets of cardinality ω_1 . Suppose that for some P_{ω_2} -names \dot{A} and \dot{B} for ω_1 -sequences we have

$$\Vdash_{P_{\omega_2}} \text{The ranges of } \dot{A}, \dot{B} \text{ are everywhere nonmeager and each of the sets}$$

$$\{\dot{A}(\omega i + n) : n < \omega\}, \{\dot{B}(\omega i + n) : n < \omega\}, \text{ for } i < \omega, \text{ is dense in } \mathbb{R}.$$

(By what we just said, every pair of everywhere nonmeager sets A and B of cardinality ω_1 , together with choices of countably many disjoint countable dense subsets of each one, has a name such that the weakest condition forces the desired properties.)

Define $g, h: \omega_1 \rightarrow ([\omega_2]^{\leq \omega})^\omega$ so that if

$$\sigma_\xi = \bigcup_{n < \omega} \{n\} \times F^{-1}(g(\xi)(n)), \quad \xi < \omega_1$$

and

$$\tau_\xi = \bigcup_{n < \omega} \{n\} \times F^{-1}(h(\xi)(n)), \quad \xi < \omega_1$$

then for each $\xi < \omega_1$,

$$\Vdash_{P_{\omega_2}} \dot{A}(\xi) = \sigma_\xi$$

and

$$\Vdash_{P_{\omega_2}} \dot{B}(\xi) = \tau_\xi.$$

For all large enough $\alpha < \omega_2$, we have:

- (i) $g, h: \omega_1 \rightarrow ([\alpha]^{\leq \omega})^\omega$.
- (ii) $\forall \xi < \alpha$, σ_ξ and τ_ξ are P_α -names.

Choose any such α of cofinality ω_1 . By (i) and (ii), the definitions of σ_ξ and τ_ξ would not change if we used g_α instead of g , h_α instead of h , and F_α instead of F . Then from the definitions of \dot{A}_α and \dot{B}_α we get

\Vdash_{P_α} The ranges of $\dot{A}_\alpha, \dot{B}_\alpha$ are everywhere nonmeager and each of the sets

$$\{\dot{A}_\alpha(\omega i + n) : n < \omega\}, \{\dot{B}_\alpha(\omega i + n) : n < \omega\}, \text{ for } i < \omega, \text{ is dense in } \mathbb{R}.$$

(Being everywhere nonmeager is trivially downward absolute.) Then \dot{Q}_α was chosen to add an order isomorphism between A_α and B_α of the desired type.

This completes the proof of the theorem. □

5 A measure-theoretic analog of perfect(2^ω)

A measure theoretic version of Laczkovich's question is not completely obvious because perfect sets carry many measures. We consider the following measures on 2^ω which we will call canonical. Given $P \subseteq 2^\omega$ a perfect set, define

$$T_P = \{s \in 2^{<\omega} : P \cap U(s) \neq \emptyset\},$$

where $U(s) = \{x \in 2^\omega : s \subseteq x\}$. We say that $s \in T_P$ splits iff both $s0$ and $s1$ are in T_P . The canonical measure μ_P is the one supported by P and determined by declaring $\mu_P(U(s)) = 1/2^n$ iff $s \in T_P$ and $|\{i < |s| : s \upharpoonright i \text{ splits}\}| = n$. An equivalent view is to take the natural map from $2^{<\omega}$ to the splitting nodes of T_P and the homeomorphism $h : 2^\omega \rightarrow P \subseteq 2^\omega$ induced by it and then μ_P is the measure corresponding to the product measure μ on 2^ω , i.e., $\mu_P(A) = \mu(h^{-1}(A))$.

Theorem 5.1 *It is relatively consistent with ZFC that for any set $B \subseteq 2^\omega$ which is not of measure zero, there exists a perfect set P of measure zero such that $B \cap P$ does not have measure zero in the canonical measure μ_P on P .*

PROOF. The model is the one used by Rosłanowski and Shelah in the proof of [RS, Theorem 3.2]. It is obtained by forcing over a model of CH with an ω_2 -stage countable support iteration $\langle \langle \mathbb{P}_\alpha \rangle_{\alpha \leq \omega_2}, \langle \dot{Q}_\alpha \rangle_{\alpha < \omega_2} \rangle$ of the measured creature forcing $\mathbb{Q} = \mathbb{Q}_4^{\text{mt}}(K^*, \Sigma^*, \mathbf{F}^*)$ defined in [RS, Section 2]. We use the notation of [RS] concerning this partial order. The definition involves in particular a rapidly growing sequence of powers of 2, $\langle N_i = 2^{M_i} : i < \omega \rangle$. Forcing with \mathbb{Q} gives rise to a continuous function $h: \prod_{i < \omega} N_i \rightarrow 2^\omega$. We will make use of the following result concerning this function. The measure on $\prod_{i < \omega} N_i$ in this proposition is the product of the uniform probability measures on the factors and the measure on 2^ω is the usual product measure. In the remainder of this proof, we denote both of these measures, as well as their product, by μ , letting the context distinguish them.

[RS, Proposition 2.6] Suppose that $A \subseteq \prod_{i < \omega} N_i \times 2^\omega$ is a set of outer measure one. Then, in $V^{\mathbb{Q}}$, the set

$$\{x \in \prod_{i < \omega} N_i : (x, h(x)) \in A\}$$

has outer measure one.

We shall also need to know that \mathbb{Q} is proper and that countable support iterations of \mathbb{Q} preserve Lebesgue outer measure. The former is [RS, Corollary 1.14]. The latter is explained in the proof of [RS, Theorem 3.2]. (The explanation refers the reader to some very general preservation theorems for iterated forcing. For the reader who wants to verify this without learning these general theorems, we indicate that it also follows from the special case of these theorems, preservation of \square^{random} , given in [Go] by imitating the proof in [Pa] that Laver forcing satisfies what is called there \star and by noting that \star implies preservation of \square^{random} .)

Recall that $N_i = 2^{M_i}$. We identify N_i with the set of binary sequences of length M_i . The map $h : \prod_{i < \omega} N_i \rightarrow 2^\omega$ is determined from a generic sequence of finite maps $(W(i) : N_i \rightarrow 2 : i < \omega)$ added by \mathbb{Q} . h is defined by $h(x)(i) = W(i)(x(i))$ for each i . We use the $W(i)$'s to define a perfect set $P \subseteq 2^\omega$ by the condition that $x \in P$ if and only if there exists $y \in 2^\omega$ such that x is the concatenation of the sequence $s_0, i_0, s_1, i_1, \dots$ where y is the concatenation of s_0, s_1, s_2, \dots and where each s_k has length M_k and $i_k = W(k)(s_k) \in \{0, 1\}$. P is essentially the same as the graph of h but we spell out the details to be sure the canonical measure is the one we want. Another way to define P is as follows:

- (i) Let $l_i = M_i + \sum_{j < i} (M_j + 1)$. Let $l_{-1} = -1$. The l_i , $i < \omega$, are the nonsplitting levels of the tree T_P which can be determined by the next two conditions.
- (ii) If $s \in T_P$ and $l_{i-1} < |s| < l_i$ then both $s0$ and $s1$ are in T_P .
- (iii) If $s \in T_P$ and $|s| = l_i$, then only sj in T_P where $W(i)(t) = j$ and $s = rt$ is the concatenation of r and t where $|t| = M_i$ and r has the appropriate length.
- (iv) Define

$$P = [T_P] \stackrel{\text{def}}{=} \{x \in 2^\omega : \forall n \ x \upharpoonright n \in T_P\}$$

Every time we pass a nonsplitting level l_i we lose half the measure and so P is a perfect set of measure zero for the usual measure on 2^ω .

Let $\rho : \prod_{i < \omega} N_i \times 2^\omega \rightarrow 2^\omega$ be the natural homeomorphism given by $\rho(x, z)$ is the concatenation of the sequence $x_0, z_0, x_1, z_1, \dots$ where we are identifying N_i with the set of binary sequences of length M_i .

Claim 5.2 ρ is measure-preserving.

PROOF. By a standard uniqueness theorem for the extension of a measure from an algebra to the σ -algebra it generates, it suffices to verify that $\rho^{-1}[C]$ has the same measure as C for every clopen set $C \subseteq 2^\omega$. As above, for $k < \omega$ and $s \in 2^k$, let us write $U(s) = \{x \in 2^\omega : s \subseteq x\}$. Similarly, for

$s \in \prod_{i < k} N_i$ we write $V(s) = \{x \in \prod_{i < \omega} N_i : s \subseteq x\}$. Every clopen set $C \subseteq 2^\omega$ can be partitioned into clopen sets of the form $U(r)$, where for some $k < \omega$, $s^r \in \prod_{i < k} N_i$ and $t^r \in 2^k$, r is the concatenation of $s_0^r, t_0^r, \dots, s_{k-1}^r, t_{k-1}^r$. (These are simply the basic open sets $U(r)$ for which r has length $\sum_{i < k} (M_i + 1)$ for some $k < \omega$.) Hence it suffices to verify $\mu(\rho^{-1}[U(r)]) = \mu(U(r))$ for r of this form. We have

$$\mu(\rho^{-1}[U(r)]) = \mu(V(s^r) \times U(t^r)) = \left(\prod_{i < k} 2^{-M_i}\right) 2^{-k} = 2^{-\sum_{i < k} (M_i + 1)} = \mu(U(r)).$$

This proves the claim. \square

Let $g: \prod_{i < \omega} N_i \rightarrow \prod_{i < \omega} N_i \times 2^\omega$ be the homeomorphism of $\prod_{i < \omega} N_i$ onto the graph of h given by $g(x) = (x, h(x))$. We have $\rho[h] = P$ (i.e., the graph of h corresponds to P under ρ).

Claim 5.3 For any Borel set $B \subseteq 2^\omega$

$$\mu_P(B) = \mu(g^{-1}[\rho^{-1}[B]])$$

and similarly for outer measure.

PROOF. Since the range of g is the graph of h , we have

$$\mu(g^{-1}[\rho^{-1}[B]]) = \mu(g^{-1}[\rho^{-1}[B] \cap h]) = \mu(g^{-1}[\rho^{-1}[B \cap P]]).$$

Similarly, since μ_P concentrates on P , $\mu_P(B) = \mu_P(B \cap P)$. Hence, it suffices to prove the claim for Borel subsets of P .

Given $s \in \prod_{i < k} N_i$, define $t^s \in 2^k$ by $t_i^s = W(i)(s(i))$ for all $i < k$, and write r^s for the concatenation of $s_0, t_0^s, \dots, s_{k-1}, t_{k-1}^s$. Using the notation for basic open sets from the proof of the previous claim, we have

$$\mu_P(U(r^s)) = 2^{-\sum_{i < k} M_i} = \prod_{i < k} 2^{-M_i}.$$

Also, $\rho^{-1}[U(r^s)] = V(s) \times U(t^s)$ and $g^{-1}[\rho^{-1}[U(r^s)]] = g^{-1}[V(s) \times U(t^s)] = V(s)$, so

$$\mu(g^{-1}[\rho^{-1}[U(r^s)]]) = \mu(V(s)) = \prod_{i < k} 2^{-M_i}.$$

Thus, the claim holds for basic open sets of the form $U(r^s)$. Every clopen subset of P is partitioned by such sets, so the claim holds for all clopen sets, and hence, as in the proof of Claim 5.2, for all Borel sets. This proves the claim. \square

Now we prove Theorem 5.1 in the case that $B \subseteq 2^\omega$ has outer measure one. It follows from the usual Lowenheim-Skolem arguments that if we let $B_\alpha = V^{\mathbb{P}^\alpha} \cap B$ then there will exist $\alpha < \omega_2$ such that $B_\alpha \in V^{\mathbb{P}^\alpha}$ and

$$V^{\mathbb{P}^\alpha} \models B_\alpha \text{ has outer measure one.}$$

Letting $A = \rho^{-1}(B_\alpha)$ (which has outer measure one by Claim 5.2) in [RS, Proposition 2.6] cited above and using Claim 5.3, we have that

$$V^{\mathbb{P}^{\alpha+1}} \models B_\alpha \text{ has } \mu_P \text{ outer measure one.}$$

Because \mathbb{Q} is proper, the remainder $\mathbb{P}_{\omega_2}/\mathbb{P}_{\alpha+1}$ of the forcing is isomorphic in $V^{\mathbb{P}_{\alpha+1}}$ to a countable support iteration of \mathbb{Q} and hence preserves outer measure. It follows that in the final model $V^{\mathbb{P}_{\omega_2}}$, B has μ_P outer measure one.

Now in the case that B has outer measure less than one, replace it by $B' = Q + B$ where Q is a countable dense subset of 2^ω . Then B' has outer measure one, and so we know there exists a measure zero perfect P such that B' has positive μ_P outer measure. Hence for some $q \in Q$ we have that $q + B$ has positive μ_P outer measure. But then, B has positive μ_{q+P} outer measure.

This completes the proof of the theorem. \square

Problem 5.4 Is it relatively consistent with ZFC to have simultaneously both the category property, $\text{perfect}(2^\omega)$, and its measure theory analogue, Theorem 5.1?

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Appendix.

This appendix is intended for the electronic version only. Here is our original proof obtained May 1997.

A note on second category sets

The purpose of this note is to answer the following question of M. Laczkovich.

Does there exist (in ZFC) a second category set that is relative first category in every nowhere dense perfect set?

We shall show that the answer is no. We assume that the reader is familiar with oracle chain condition constructions. The technique is explained in [S1, pp.114–133] and all the details of a typical proof are written out in [B, Sections 4,5,6]. We shall follow the notation and general setup of the proof in [B].

Theorem *It is consistent with ZFC that every second category subset of the real line has a nowhere dense subset of cardinality ω_1 which is relatively second category in its closure.*

Proof Assume $V = L$. As in [B, Section 5], we use an oracle-cc iteration of length ω_2 , and it will suffice to prove the following lemma.

Main Lemma *Let \bar{M} be an ω_1 -oracle and let A be an everywhere nonmeager subset of 2^ω . Then there is a forcing notion P satisfying the \bar{M} -cc and a P -name \dot{K} of a nowhere dense perfect set such that for every $G \subseteq P \times Q$ generic over V (where Q is Cohen forcing), there is no Borel set T in $V[G]$ such that*

- (a) $T \cap \dot{K}[G]$ is meager relative to $\dot{K}[G]$.
- (b) $A \cap \dot{K}[G] \subseteq T$.

Proof of the main lemma

Define a partial order $P = P(\langle a_\alpha : \alpha < \beta \rangle)$ where $\beta \leq \omega_1$, $a_\alpha \in A$ as follows. Let $S_\beta = \{a_\alpha : \alpha < \beta\} \cup \{x : x \text{ is eventually constant}\}$. Then the conditions in P are the pairs $p = (F_p, n_p)$ where F_p is a finite subset of S_β and $n_p < \omega$. The order on P is: $p \leq q$ if and only if $F_p \supseteq F_q$, $n_p \geq n_q$, and $x \in F_p$ implies $x \upharpoonright n_q = y \upharpoonright n_q$ for some $y \in F_q$. Let \dot{K} be a P -name for the closure of $\bigcup \{F : \text{for some } n < \omega, (F, n) \in G\}$. As in [B, Section 6], the main lemma will follow if we prove the following:

Main Claim *Let $P_\delta = P(\langle a_\alpha : \alpha < \delta \rangle)$, $\delta < \omega_1$ be given, as well as a countable M_δ , $P_\delta \in M_\delta$, a condition $(p^*, r^*) \in P_\delta \times Q$ and a $P_\delta \times Q$ -names τ_n , $n < \omega$, for relatively nowhere dense closed subsets of \dot{K} . Then we can find $a_\delta \in A$ such that, letting $P_{\delta+1} = P(\langle a_\alpha : \alpha \leq \delta \rangle)$, the following conditions hold:*

- (A) *Every predense subset of P_δ which belongs to M_δ is a predense subset of $P_{\delta+1}$.*
- (B) *There is a condition $(p', r') \in P_{\delta+1} \times Q$ extending (p^*, r^*) such that for all $n < \omega$, $(p', r') \Vdash_{P_{\delta+1} \times Q} "a_\delta \notin \tau_n"$.*

Proof of the main claim

Choose a sufficiently large regular λ and choose a countable $N \prec H_\lambda$ such that $\rho, P_\delta, \langle a_\alpha : \alpha < \delta \rangle, \langle \tau_n : n < \omega \rangle, M_\delta \in N$. Choose a Cohen real over N , $x = a_\delta \in A$, such that $x \upharpoonright n_{p^*} = y^* \upharpoonright n_{p^*}$ for some $y^* \in F_{p^*}$.

Proof of condition (A) Let $J \subseteq P_\delta$ be predense, $J \in M_\delta$. We must show that J is predense in $P_{\delta+1}$. Let $p \in P_{\delta+1}$, $p \notin P_\delta$. By the definition of $P_{\delta+1}$, $p = p(x) = (F \cup \{x\}, n)$ where $x = a_\delta$ and

$F \subseteq S_\delta$. Think of x as being generic for $2^{<\omega}$ ordered the usual way. If $p(x)$ is not compatible with any element of J , then some $s \in 2^{<\omega}$ forces this over N . We may assume $n \geq |s|$. Since S_δ is dense, there is $y \in S_\delta$ such that $y \upharpoonright n = x \upharpoonright n$. Now choose $p_1 \in P_\delta$, a common extension of $(F \cup \{y\}, n)$ and some $p_2 \in J$. If x is any Cohen real such that $x \upharpoonright n_{p_1} = y \upharpoonright n_{p_1}$, then clearly $p(x)$ is compatible with p_2 , contradiction.

Proof of condition (B) Let $p' = (F, n)$ where $F = F_{p^*} \cup \{x\}$ and $n = n_{p^*}$, and let $r' = r^*$. If $N[x]$ believes that these satisfy (B) then they do and we are done. Otherwise, choose (p'', r'') in $N[x]$ extending (p', r') and $n < \omega$ such that

$$N[x] \models (p'', r'') \Vdash_{P_{\delta+1} \times Q} "x \in \tau_n".$$

For some Cohen condition $s \in 2^{<\omega}$ we have

$$N \models s \Vdash "p'' = (F'' \cup \{x\}, n'') \text{ and } (p'', r'') \Vdash_{P_{\delta+1} \times Q} x \in \tau_n",$$

where $F'' \subseteq S_\delta$ contains a member u of $[s]$ and without loss of generality $n'' \geq |s|$. Extend $((F'', n''), r'')$ to $((F''', n'''), r''')$ in $P_\delta \times Q$ to decide a $t \in 2^{<\omega}$ such that $u \upharpoonright n'' \subseteq t$, $[t] \cap \dot{K} \neq \emptyset$ and $[t] \cap \tau_n = \emptyset$. By increasing n''' and/or t , we may arrange that $n''' = |t|$. Clearly $[t] \cap F''' \neq \emptyset$, so N models the following for $p''' = (F''' \cup \{x\}, n''')$:

$$N \models t \Vdash "p''' \leq (F''', n'''), (F'' \cup \{x\}, n'') \text{ and } (p''', r''') \Vdash_{P_{\delta+1} \times Q} x \notin \tau_n"$$

which is a contradiction. This completes the proof of the main claim and of the theorem. \square

References:

- [B] Burke M.R., *Liftings for Lebesgue measure*, Israel Math. Conf. Proc., 6 (1993) 119–150.
- [S1] Shelah S., *Proper Forcing*, Lecture Notes in Math. 940, Berlin, Springer-Verlag, 1982.