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A MINIMAL DEGREE WHICH COLLAPSES ω_1

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Abstract. We consider a well-known partial order of Prikry for producing a collapsing function of minimal degree. Assuming MA + $\neg$CH, every new real constructs the collapsing map.

Let $\omega_1^{<\omega}$ be the tree of finite sequences from ω_1. Define the partial order $\mathbf{P}$ to be the set of all nonempty subtrees T of $\omega_1^{<\omega}$ which satisfy: for all $s \in T$ there exists $t \geq s$ such that $\{\alpha : t^\frown\alpha \in T\}$ is uncountable. The ordering on $\mathbf{P}$ is inclusion. This partial order was first considered by Prikry, who also showed that it gives a minimal collapsing function (see Abraham (198?)).

THEOREM. *Suppose $M \models$ "ZFC + MA + $\neg$CH". Then for any G $\mathbf{P}$-generic over M,*

(1) *$M[G] \models$ "ω_1^M is countable"; and*

(2) *for every real $x \in M[G]$, $x \in M$ or $G \in M[x]$.*

Note that (1) and (2) are impossible if $M \models$ "CH". This is because collapsing the continuum to ω always introduces Cohen reals, random reals, etc.

Let us give some definitions. For $p \in \mathbf{P}$ we say that $s \in p$ is a splitting node of p iff $\{\alpha : s^\frown\alpha \in p\}$ is uncountable. We say that $s \in p$ is a level n node iff $\{t : t \subseteq s$ and t is a splitting node of $p\}$ has size n. We say that $p \leq_n q$ iff $p \leq q$ and all level n nodes of q are still in p. The standard fusion argument shows that if $p_{n+1} \leq_n p_n$ for each $n < \omega$, then the fusion $(\bigcap_{n<\omega} p_n)$ is an element of $\mathbf{P}$. For any $p \in \mathbf{P}$ and $s \in p$ define $p_s = \{t \in p \mid t \subseteq s \text{ or } s \subseteq t\}$.

Now suppose $q \in \mathbf{P}$ and τ is a term such that $q \Vdash$ "$\tau \in 2^\omega$".

LEMMA 1. *There exist $p \leq q$ and $F: p \to 2^{<\omega}$ such that*

(a) *for all $n < \omega$, $F''(p \cap \omega_1^n) \subseteq 2^n$, and*

(b) *for all $s \in p$, $p_s \Vdash$ "$F(s) \subseteq \tau$".*

PROOF. This is an easy fusion argument. Given $q \in \mathbf{P}$ and $s \in q \cap \omega_1^n$ which we want to retain as a splitting node, simply extend each $q_{s^\frown\alpha}$ to decide $\tau \restriction n$, then ω_1 of $q_{s^\frown\alpha}$ decide $\tau \restriction n$ the same way. So build a sequence $q_{n+1} \leq_n q_n \leq q$ such that for every level n node s of q_{n+1}, $(q_{n+1})_s$ decides $\tau \restriction \text{length}(s)$. The fusion of the q_n's is p. $\square$

From now on assume that $p \Vdash$ "$\tau \notin M$" and p and $F: p \to 2^{<\omega}$ are from Lemma 1.

LEMMA 2. *Suppose $p_\alpha \leq p$ for $\alpha < \omega_1$. Then there exist $q_\alpha \leq p_\alpha$ and $C_\alpha \subseteq 2^\omega$ closed*

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for each $\alpha < \omega_1$ *such that the* $\{C_\alpha : \alpha < \omega_1\}$ *are disjoint and for each* $\alpha < \omega_1$:

$$q_\alpha \Vdash \text{“}\tau \in C_\alpha\text{”}.$$

PROOF. Define the partial order $\mathbf{Q}_\alpha$ by $(T_1, T_2, n) \in \mathbf{Q}_\alpha$ iff

(1) T_1 is a finite subtree of $p_\alpha \cap \omega_1^{\leq n}$ with every branch of length n;

(2) T_2 is a finite subtree of $2^{\leq n}$ with every branch of length n; and

(3) for all $s \in T_1$, $F(s) \in T_2$.

We define $(\hat{T}_1, \hat{T}_2, \hat{n}) \leq (T_1, T_2, n)$ iff

(1) $\hat{n} \geq n$;

(2) $\hat{T}_1 \supseteq T_1$; and

(3) $\hat{T}_2$ is an end extension of T_2 (i.e. $\hat{T}_2 \cap 2^{\leq n} = T_2$).

It is easy to see that $\mathbf{Q}_\alpha$ has the countable chain condition, since if $T_2^p = T_2^q$, then p and q are compatible. Now let $\mathbf{Q}$ be the direct sum of $\{\mathbf{Q}_\alpha : \alpha < \omega_1\}$. Since each $\mathbf{Q}_\alpha$ has property K (in fact is σ-centered), $\mathbf{Q}$ has the c.c.c. A partial order has property K if every subset of cardinality ω_1 contains a subset of ω_1 pairwise compatible elements.

It is not hard to see that the product of two partial orders with property K has property K, and the direct sum of such orders has the property. Also MA + $\neg$CH implies that every c.c.c. order has property K.

CLAIM 1. *Given* $q \in \mathbf{Q}_\alpha$ *and* $r \in \mathbf{Q}_\beta$ *there exist* $\hat{q} \leq q$ *and* $\hat{r} \leq r$ *with the same* n *and* $T_2^{\hat{q}} \cap T_2^{\hat{r}} \cap 2^n = \varnothing$.

PROOF. This is where $p \Vdash$ “$\tau \notin M$” is used. For each $s \in T_1^q$ let $x_s : \omega \to \omega_1$ be a branch of p extending s and let $y_s : \omega \to 2$ be $\bigcup\{F(x_s \upharpoonright n) : n < \omega\}$. (I.e. so $p_{x_s \upharpoonright n} \Vdash$ “$\tau \upharpoonright n = y_s \upharpoonright n$”.) Since $p \Vdash$ “$\tau \notin M$”, there exists for each $s \in T_1^r$ some $\hat{s} \supseteq s$ such that $F(\hat{s})$ is incompatible with all of the y_s's. Now it is easy to prove Claim 1. □

For any G a $\mathbf{Q}_\alpha$ filter let $q_\alpha = \bigcup\{T_1^p : p \in G\}$ and let $\hat{C}_\alpha = \bigcup\{T_2^p : p \in G\}$.

CLAIM 2. *There are* ω_1 *dense subsets of* $\mathbf{Q}_\alpha$ *such that if* G *is any* $\mathbf{Q}_\alpha$ *filter meeting them all, then* $q_\alpha \in \mathbf{P}$.

PROOF. For any $s \in p$ and $\beta < \omega_1$ let $D_\beta^s = \{q \in \mathbf{Q}_\alpha : s \in T_1^q$ and there exists $t \in T_1^q$, $t \supseteq s$, and range(t) contains some $\gamma > \beta\}$. It is easy to see that D_β^s is dense beneath the set of q such that $s \in T_1^q$. Consequently if we let

$$E_\beta^s = D_\beta^s \cup \{q : q \Vdash \text{“}s \notin T_1^q\text{”}\},$$

then E_β^s is dense in $\mathbf{Q}_\alpha$. If G meets each E_β^s for $s \in p$ and $\beta < \omega_1$, then $q_\alpha \in \mathbf{P}$. □

Note that $q_\alpha \Vdash$ “$\forall n\ \tau \upharpoonright n \in \hat{C}_\alpha$”. The lemma follows easily from the claims and MA + $\neg$CH. □

Using Lemma 2 and a fusion argument, find $q \leq p$ such that for all $s \in q$ there exists $\langle C_\alpha^s : s^\frown\alpha \in q\rangle$, a family of disjoint closed sets, such that $q_{s^\frown\alpha} \Vdash$ “$\tau \in C_\alpha^s$”. Thus $q \Vdash$ “$G \in M[\tau^G]$” and the theorem is proved. □

REMARKS. Assume that $M \vDash$ “MA + $\neg$CH” and G is Prikry collapsing generic over M. Then for $f \in M[G] \cap \omega^\omega$ there exists $g \in M \cap \omega^\omega$ such that for every $n < \omega$, $f(n) < g(n)$. Also for every $X \in M[G] \cap [\omega]^\omega$ there exists $Y \in M \cap [\omega]^\omega$ such that $Y \subseteq X$ or $X \cap Y = \varnothing$. And every meager set (measure zero set) coded in $M[G]$ is covered by one coded in M. All of these properties are true when G is Sacks generic over M. The proofs are similar here with the addition of a suitable forcing notion to apply Martin's axiom.

The fact that ω_1 is collapsed but every element of ω^ω is dominated by a ground model element of ω^ω implies that in the ground model the Boolean algebra associated with the Prikry collapse is (ω, ω)-weakly distributive but not (ω, ω_1)-weakly distributive. This is also true of Namba forcing (see Namba (1972)).

Of course, in our theorem we only needed that $M \models$ "MA(K)", since we only did property K forcing. If, in addition, $M \models$ "there are no Souslin trees", then for every set of ordinals $X \in M[G]$, $X \in M$ or $G \in M[X]$. Since a branch through a Souslin tree cannot be minimal, this assumption is necessary. The proof is left as an exercise for the reader.

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