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A MINIMAL DEGREE WHICH COLLAPSES ω_1

TIM CARLSON, KENNETH KUNEN AND ARNOLD W. MILLER

Abstract. We consider a well-known partial order of Prikry for producing a collapsing function of minimal degree. Assuming $MA + \neg CH$, every new real constructs the collapsing map.

Let $\omega_1^{<\omega}$ be the tree of finite sequences from ω_1 . Define the partial order \mathbf{P} to be the set of all nonempty subtrees T of $\omega_1^{<\omega}$ which satisfy: for all $s \in T$ there exists $t \geq s$ such that $\{\alpha: t \hat{\ } \alpha \in T\}$ is uncountable. The ordering on \mathbf{P} is inclusion. This partial order was first considered by Prikry, who also showed that it gives a minimal collapsing function (see Abraham (198?)).

THEOREM. *Suppose $M \models \text{“ZFC} + MA + \neg CH\text{”}$. Then for any G \mathbf{P} -generic over M ,*

- (1) $M[G] \models \text{“}\omega_1^M \text{ is countable”}$; and
- (2) for every real $x \in M[G]$, $x \in M$ or $G \in M[x]$.

Note that (1) and (2) are impossible if $M \models \text{“}CH\text{”}$. This is because collapsing the continuum to ω always introduces Cohen reals, random reals, etc.

Let us give some definitions. For $p \in \mathbf{P}$ we say that $s \in p$ is a splitting node of p iff $\{\alpha: s \hat{\ } \alpha \in p\}$ is uncountable. We say that $s \in p$ is a level n node iff $\{t: t \leq s \text{ and } t \text{ is a splitting node of } p\}$ has size n . We say that $p \leq_n q$ iff $p \leq q$ and all level n nodes of q are still in p . The standard fusion argument shows that if $p_{n+1} \leq_n p_n$ for each $n < \omega$, then the fusion $(\bigcap_{n < \omega} p_n)$ is an element of \mathbf{P} . For any $p \in \mathbf{P}$ and $s \in p$ define $p_s = \{t \in p \mid t \leq s \text{ or } s \leq t\}$.

Now suppose $q \in \mathbf{P}$ and τ is a term such that $q \Vdash \text{“}\tau \in 2^\omega\text{”}$.

LEMMA 1. *There exist $p \leq q$ and $F: p \rightarrow 2^{<\omega}$ such that*

- (a) for all $n < \omega$, $F''(p \cap \omega_1^n) \subseteq 2^n$, and
- (b) for all $s \in p$, $p_s \Vdash \text{“}F(s) \subseteq \tau\text{”}$.

PROOF. This is an easy fusion argument. Given $q \in \mathbf{P}$ and $s \in q \cap \omega_1^n$ which we want to retain as a splitting node, simply extend each $q_{s \hat{\ } x}$ to decide $\tau \upharpoonright n$, then ω_1 of $q_{s \hat{\ } x}$ decide $\tau \upharpoonright n$ the same way. So build a sequence $q_{n+1} \leq_n q_n \leq q$ such that for every level n node s of q_{n+1} , $(q_{n+1})_s$ decides $\tau \upharpoonright \text{length}(s)$. The fusion of the q_n 's is p . \square

From now on assume that $p \Vdash \text{“}\tau \notin M\text{”}$ and p and $F: p \rightarrow 2^{<\omega}$ are from Lemma 1.

LEMMA 2. *Suppose $p_\alpha \leq p$ for $\alpha < \omega_1$. Then there exist $q_\alpha \leq p_\alpha$ and $C_\alpha \subseteq 2^\omega$ closed*

for each $\alpha < \omega_1$ such that the $\{C_\alpha: \alpha < \omega_1\}$ are disjoint and for each $\alpha < \omega_1$:

$$q_\alpha \Vdash \text{“}\tau \in C_\alpha\text{”}.$$

PROOF. Define the partial order \mathbf{Q}_α by $(T_1, T_2, n) \in \mathbf{Q}_\alpha$ iff

- (1) T_1 is a finite subtree of $p_\alpha \cap \omega_1^{\leq n}$ with every branch of length n ;
- (2) T_2 is a finite subtree of $2^{\leq n}$ with every branch of length n ; and
- (3) for all $s \in T_1, F(s) \in T_2$.

We define $(\hat{T}_1, \hat{T}_2, \hat{n}) \leq (T_1, T_2, n)$ iff

- (1) $\hat{n} \geq n$;
- (2) $\hat{T}_1 \supseteq T_1$; and
- (3) \hat{T}_2 is an end extension of T_2 (i.e. $\hat{T}_2 \cap 2^{\leq n} = T_2$).

It is easy to see that \mathbf{Q}_α has the countable chain condition, since if $T_2^p = T_2^q$, then p and q are compatible. Now let \mathbf{Q} be the direct sum of $\{\mathbf{Q}_\alpha: \alpha < \omega_1\}$. Since each \mathbf{Q}_α has property K (in fact is σ -centered), \mathbf{Q} has the c.c.c. A partial order has property K if every subset of cardinality ω_1 contains a subset of ω_1 pairwise compatible elements.

It is not hard to see that the product of two partial orders with property K has property K , and the direct sum of such orders has the property. Also $\text{MA} + \neg\text{CH}$ implies that every c.c.c. order has property K .

CLAIM 1. Given $q \in \mathbf{Q}_\alpha$ and $r \in \mathbf{Q}_\beta$ there exist $\hat{q} \leq q$ and $\hat{r} \leq r$ with the same n and $T_2^{\hat{q}} \cap T_2^{\hat{r}} \cap 2^n = \emptyset$.

PROOF. This is where $p \Vdash \text{“}\tau \notin M\text{”}$ is used. For each $s \in T_1^q$ let $x_s: \omega \rightarrow \omega_1$ be a branch of p extending s and let $y_s: \omega \rightarrow 2$ be $\bigcup \{F(x_s \upharpoonright n): n < \omega\}$. (I.e. so $p_{x_s \upharpoonright n} \Vdash \text{“}\tau \upharpoonright n = y_s \upharpoonright n\text{”}$.) Since $p \Vdash \text{“}\tau \notin M\text{”}$, there exists for each $s \in T_1^q$ some $\hat{s} \supseteq s$ such that $F(\hat{s})$ is incompatible with all of the y_s 's. Now it is easy to prove Claim 1. \square

For any G a \mathbf{Q}_α filter let $q_\alpha = \bigcup \{T_1^p: p \in G\}$ and let $\hat{C}_\alpha = \bigcup \{T_2^p: p \in G\}$.

CLAIM 2. There are ω_1 dense subsets of \mathbf{Q}_α such that if G is any \mathbf{Q}_α filter meeting them all, then $q_\alpha \in \mathbf{P}$.

PROOF. For any $s \in p$ and $\beta < \omega_1$ let $D_\beta^s = \{q \in \mathbf{Q}_\alpha: s \in T_1^q \text{ and there exists } t \in T_1^q, t \supseteq s, \text{ and } \text{range}(t) \text{ contains some } \gamma > \beta\}$. It is easy to see that D_β^s is dense beneath the set of q such that $s \in T_1^q$. Consequently if we let

$$E_\beta^s = D_\beta^s \cup \{q: q \Vdash \text{“}s \notin T_1^q\text{”}\},$$

then E_β^s is dense in \mathbf{Q}_α . If G meets each E_β^s for $s \in p$ and $\beta < \omega_1$, then $q_\alpha \in \mathbf{P}$. \square

Note that $q_\alpha \Vdash \text{“}\forall n \tau \upharpoonright n \in \hat{C}_\alpha\text{”}$. The lemma follows easily from the claims and $\text{MA} + \neg\text{CH}$. \square

Using Lemma 2 and a fusion argument, find $q \leq p$ such that for all $s \in q$ there exists $\langle C_\alpha^s: s \hat{\alpha} \in q \rangle$, a family of disjoint closed sets, such that $q_{s \hat{\alpha}} \Vdash \text{“}\tau \in C_\alpha^s\text{”}$. Thus $q \Vdash \text{“}G \in M[\tau^G]\text{”}$ and the theorem is proved. \square

REMARKS. Assume that $M \models \text{“}\text{MA} + \neg\text{CH}\text{”}$ and G is Prikrý collapsing generic over M . Then for $f \in M[G] \cap \omega^\omega$ there exists $g \in M \cap \omega^\omega$ such that for every $n < \omega, f(n) < g(n)$. Also for every $X \in M[G] \cap [\omega]^\omega$ there exists $Y \in M \cap [\omega]^\omega$ such that $Y \subseteq X$ or $X \cap Y = \emptyset$. And every meager set (measure zero set) coded in $M[G]$ is covered by one coded in M . All of these properties are true when G is Sacks generic over M . The proofs are similar here with the addition of a suitable forcing notion to apply Martin's axiom.

The fact that ω_1 is collapsed but every element of ω^ω is dominated by a ground model element of ω^ω implies that in the ground model the Boolean algebra associated with the Prikry collapse is (ω, ω) -weakly distributive but not (ω, ω_1) -weakly distributive. This is also true of Namba forcing (see Namba (1972)).

Of course, in our theorem we only needed that $M \models \text{“MA}(K)\text{”}$, since we only did property K forcing. If, in addition, $M \models \text{“there are no Souslin trees”}$, then for every set of ordinals $X \in M[G]$, $X \in M$ or $G \in M[X]$. Since a branch through a Souslin tree cannot be minimal, this assumption is necessary. The proof is left as an exercise for the reader.

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