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MAPPING A SET OF REALS ONTO THE REALS

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Abstract. In this paper we show that it is consistent with ZFC that for any set of reals of cardinality the continuum, there is a continuous map from that set onto the closed unit interval. In fact, this holds in the iterated perfect set model. We also show that in this model every set of reals which is always of first category has cardinality less than or equal to ω_1.

§1. Introduction. Sierpinski ([9] and [10, property C_5]) showed that assuming the continuum hypothesis there exists a set of reals of cardinality the continuum which cannot be mapped continuously onto the closed unit interval. In fact, he showed that every continuous image of a Luzin set has measure zero.

The main result of this paper is that it is consistent with ZFC that for every set of reals of cardinality the continuum there is a continuous map from that set onto the closed unit interval. This will be proved in §§3 and 4. It holds in the iterated perfect set model (see Baumgartner and Laver [2]). In this model every metric space X of cardinality the continuum can be mapped continuously onto the closed unit interval. To see this, first reduce to the separable case as follows. Let $D \subseteq X$ be a dense set of minimal cardinality. In this model the continuum is ω_2, so if the cardinality of D is greater than ω_1, X contains a closed discrete subset of cardinality the continuum, and so may be mapped continuously onto the entire real line. Otherwise let $D = \{d_\alpha : \alpha < \omega_1\}$ and let X_α be the closure of $\{d_\beta : \beta < \alpha\}$. Then since $X = \bigcup_{\alpha<\omega_1} X_\alpha$ it must be that X contains a closed separable subspace of cardinality ω_2. So we may assume X is separable. We may also assume X is zero dimensional, since otherwise the metric on X gives a continuous map whose image contains an interval. But any separable zero dimensional metric space is homeomorphic to a set of reals (in fact, a subset of the Cantor set) and we are done.

It should be remarked that one can always find two sets of reals of cardinality the continuum such that neither can be mapped continuously onto the other. In fact, Lindenbaum showed that there are always 2^c sets of reals of cardinality c none of which is a continuous image of another (see Kuratowski [6, §35]).

Isbell [3] showed that if the real line is partitioned into countably many pieces, then one of the pieces can be mapped continuously onto the closed unit interval. He notes that since it is consistent that the continuum be $\aleph_{\omega_1}$ one cannot in general improve this result by replacing "countably" by "fewer than continuum".

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THEOREM. *It is consistent with any cardinal arithmetic that the real line can be partitioned into* ω_1 *pieces none of which can be mapped continuously onto the closed unit interval.*

PROOF. Suppose a Cohen real x is added to a model M; then for any continuous map $f: \boldsymbol{R} \to \boldsymbol{R}$ coded in M we have that $f''(\boldsymbol{R} \cap M) \subseteq \boldsymbol{R} \cap M$. Now suppose we let $\boldsymbol{P} = \{p: \mathrm{dom}(p) \in [\omega_1]^{<\omega}$ and range $p = 2\}$. Let G be $\boldsymbol{P}$-generic over M and for $\alpha < \omega_1$ let $X_\alpha = \boldsymbol{R} \cap M[G \upharpoonright \alpha]$. Then in $M[G]$, $\boldsymbol{R} = \bigcup_{\alpha<\omega_1} X_\alpha$ and no X_α can be mapped continuously onto the closed unit interval. ■

In a related paper Isbell [4] showed that assuming the continuum hypothesis there is a set of reals X such that X cannot be mapped continuously onto the closed unit interval, but X^2 can be. This result can also be proved using Martin's axiom, and generalized, for example, to show that there exists a set of reals X such that X^2 cannot be mapped continuously onto the closed unit interval, but X^3 can be.

§2. Singular spaces. A set of reals X is a Luzin set iff for every first category subset N of the reals $N \cap X$ is countable. A set of reals X is a Sierpinski set iff for every measure zero set M of the reals $M \cap X$ is countable. Neither a Luzin set nor a Sierpinski set can be mapped continuously onto the unit interval. This is implied by the following easy proposition.

THEOREM. *Suppose that I is a class of Borel sets with the property that one cannot find continuum many disjoint Borel sets not in I. Suppose X is a set of reals which intersects each element of I in a set of cardinality less than the continuum. Then X cannot be mapped by any Borel map onto the reals.*

PROOF. Any Borel map on X extends to a map with domain $\boldsymbol{R}$. Suppose f: $\boldsymbol{R} \to \boldsymbol{R}$ is a Borel map. Let $\{B_\alpha: \alpha < c\}$ be a family of disjoint Borel sets each of cardinality c. By hypothesis, for some $\alpha < c$ the set $f^{-1}(B_\alpha)$ is in I. Since $X \cap f^{-1}(B_\alpha)$ has cardinality less than c, B_α is not included in $f''X$. ■

Assuming the continuum hypothesis there are Luzin sets and Sierpinski sets of cardinality the continuum (see Kuratowski [6, §40]). By defining generalized Luzin sets or generalized Sierpinski sets appropriately (see Kunen [5] or Laver [7]) this result generalizes to Martin's axiom.

In contrast to the last theorem, we have the following result.

THEOREM. *Assuming the continuum hypothesis, there exists an uncountable set of reals X such that every uncountable subset of X can be mapped continuously onto the entire real line.*

PROOF. By a well-known theorem of Sierpinski (see Sierpinski [10, p. 12] or Bagemihl and Sprinkle [1]), there are countably many functions $f_n: \omega_1 \to \omega_1$ such that for every $Y \in [\omega_1]^{\omega_1}$ there is an $n < \omega$ (in fact, for all but finitely many $n < \omega$) such that $f_n''Y = \omega_1$. Let $\boldsymbol{R} = \{x_\alpha: \alpha < \omega_1\}$ and $D = \{E_n: n < \omega\}$ be a family of subsets of ω_1 such that for every interval with rational endpoints I and $n < \omega$, the set $f_n^{-1}\{\alpha: x_\alpha \in I\}$ is in D. Define the map $f: \omega_1 \to 2^\omega$ by $f(\alpha)(n) = 0$ iff $\alpha \in E_n$. It is easy to show that $X = f''\omega_1$ has the desired property. ■

Kunen remarks that if we let $X = \{y_\alpha: \alpha < \omega_1\} \subseteq \boldsymbol{R}^\omega$ be defined by $(y_\alpha)_n = x_{f_n(\alpha)}$, then X has the property that for all $Y \in [X]^{\omega_1}$ all but finitely many projections of Y are onto $\boldsymbol{R}$.

Assuming Martin's axiom plus the continuum is a successor cardinal, we can show that there is a set of reals X of cardinality the continuum such that every subset of X of cardinality the continuum can be mapped onto $\boldsymbol{R}$ by a Borel map. If c carries a c-saturated c-ideal, then there can be no such X as above. The next theorem implies that there can be no such X in the Cohen or random real models. It is probably a known result, but it has not been published.

THEOREM. *If ω_2 Cohen reals are added to a model of CH, then in the extension every set of reals of cardinality ω_2 contains a subset of cardinality ω_2 which is the continuous image of a Luzin set. The analogous statement is true when "Cohen reals" and "Luzin set" are replaced by "random reals" and "Sierpinski set".*

LEMMA. *Suppose M is a model of ZFC and $x \in 2^\omega$ is either a Cohen real or a random real over M. Then for any $y \in M[x] \cap 2^\omega$ there exists a continuous function f coded in M such that $f(x) = y$.*

PROOF. First assume x is a Cohen real. This means that x is obtained by forcing with $\boldsymbol{P} = \{p|\text{dom}(p) \in [\omega]^{<\omega} \text{ and } \text{range}(p) = 2\}$. For each $n < \omega$ choose $D_n = D_n^0 \cup D_n^1$ a maximal antichain in $\boldsymbol{P}$ so that for each $p \in D_n^i$ $p \Vdash$ "$y(n) = i$". For $p \in \boldsymbol{P}$ let $[p] \subseteq 2^\omega$ be defined to be the set of $z \in 2^\omega$ such that $z \upharpoonright \text{dom}(p) = p$. Let $H \subseteq 2^\omega$ be the set of all $z \in 2^\omega$ such that for all $n < \omega$ there exist $p \in D_n$ such that $z \in [p]$. Define f: $H \to 2^\omega$ by $f(z)(n) = i$ iff there exists $p \in D_n^i$ such that $z \in [p]$.

Now suppose that x is random over M. This means that x is obtained by forcing with Borel subsets of 2^ω of positive measure. For each $n < \omega$ let B_n be a Borel set such that $B_n \Vdash$ "$y(n) = 0$" and $(2^\omega - B_n) \Vdash$ "$y(n) = 1$". Define f: $2^\omega \to 2^\omega$ by $f(z)(n) = 0$ iff $z \in B_n$. By Luzin's theorem (see Royden [8, p. 72]) for any Borel set of positive measure B, there exists a closed set of positive measure $C \subseteq B$ such that $f \upharpoonright C$ is continuous. ∎

We say that G: $\Sigma \to 2$ is Cohen over a model M if G is $\boldsymbol{P}$-generic over M where $\boldsymbol{P}$ is the set of p: $E \to 2$ for $E \in [\Sigma]^{<\omega}$. Similarly G: $\Sigma \to 2$ is random over M if it is obtained by forcing with Borel subsets of 2^Σ of positive measure which are coded in M.

We will assume, without proof, the following three facts. Suppose M is a model of ZFC, $\Sigma \in M$, and G: $\Sigma \to 2$ is Cohen over M (respectively, random over M).

(A) If $\Sigma = \Sigma_0 \cup \Sigma_1$ is a partition of Σ in M, then $G \upharpoonright \Sigma_0$ is Cohen (random) over M and $G \upharpoonright \Sigma_1$ is Cohen (random) over $M[G \upharpoonright \Sigma_0]$.

(B) If x is a Cohen real (random real) over M, then x is not in any first category Borel set (measure zero Borel set) coded in M.

(C) Working in M, if x is a term for an element of 2^ω, then one can find, uniformly in x, a countable set $A \subseteq \Sigma$ such that x is realized in $M[G \upharpoonright A]$.

Finally, let us prove the theorem. Suppose that M is a model of CH and G: $\omega_2 \to 2$ is either Cohen (or random) over M. Suppose that $\Vdash$ "$\{x_\alpha : \alpha < \omega_2\} \subseteq 2^\omega$ are distinct". Working in M, by property C, find $A_\alpha \in [\omega_2]^{\leq\omega}$ for $\alpha < \omega_2$ such that for each $\alpha < \omega_2$, x_α is realized in $M[G \upharpoonright A_\alpha]$. Since M is a model of CH, by the Δ-system lemma, there is a $\Sigma \in [\omega_2]^{\omega_2}$ and a $B \subseteq \omega_2$ such that for every two distinct α, $\beta \in \Sigma$ we have that $A_\alpha \cap A_\beta = B$. Let $M' = M[G \upharpoonright B]$. Since B is countable it is easy to see that M' is a model of CH. By property A we may assume without loss of generality that $M = M'$ and $B = \emptyset$. Working in M for each $\alpha \in \Sigma$ choose a bijection F_α: $\omega \to A_\alpha$ and define $z_\alpha \in 2^\omega$ by $z_\alpha = G \circ F_\alpha$. By using all three properties it is easy to prove that each z_α is Cohen over M (random over

M) and $\{z_\alpha: \alpha \in \Sigma\}$ is a Luzin set (Sierpinski set). We also know that for each $\alpha \in \Sigma$, x_α is realized in $M[z_\alpha]$. By the lemma, there is a continuous function f_α coded in M such that $f_\alpha(z_\alpha) = x_\alpha$. Since there are only ω_1 continuous functions coded in M, one of them is used by ω_2 of the α in Σ. ■

Most of the following definitions and results can be found in §40 of Kuratowski [6].

A set of reals X is always of first category iff it is first category relative to every perfect set. A set of reals X is a λ-set (equivalently is rarefied) iff every countable subset of X is a G_δ relative to X. A λ-set is always of first category. Luzin showed that λ-sets of cardinality ω_1 always exist. Note that continuous image of a Luzin set cannot be a λ-set unless it is countable. To see this just note that we can assume that it is a one-to-one image, but the one-to-one preimage of a λ-set is a λ-set. Luzin also showed that there is a one-to-one continuous function which maps every Luzin set onto a set which is always of first category. Therefore in the Cohen model there is a set of reals of cardinality ω_2 which is always of first category. Note also that a Sierpinski set is a λ-set.

A set of reals X has universal measure zero iff $\mu(X) = 0$ for all nonatomic Borel measures μ on X. Hausdorff showed that there exists a universal measure zero set of cardinality ω_1 (see Laver [7] for the proof). Laver showed that in the random real model, or when Sacks reals are added (this probably means the iterated perfect set model), there are no universal measure zero sets of cardinality ω_2 (this is reported in Laver [7]). This extended an unpublished result of Baumgartner on strong measure zero sets. For random reals, this follows from the above theorem since any uncountable continuous image of a Sierpinski set cannot have universal measure zero. Note also that a Luzin set has universal measure zero.

In §5 we show that in the iterated perfect set model every set of reals which is always of first category has cardinality less than or equal to ω_1.

§3. Some technical lemmas. We will use the same notation and definitions as Baumgartner and Laver [2]. In addition we make the following definition. For p an (F, n)-determined element of $\boldsymbol{P}_\alpha$ and $q \in \boldsymbol{P}_\alpha$ we say that $(q, n) \geq_F (p, n)$ iff $q \geq p$ and every $\sigma: F \to 2^n$ consistent with p is consistent with q. This says that q retains the splitting that p already has as far as (F, n) is concerned. The following lemma allows us to extend $p|\sigma$ while retaining the splitting we already have.

LEMMA 1. *Given $p \in \boldsymbol{P}_\alpha$ which is (F, n)-determined, $\sigma: F \to 2^n$ consistent with p, and $r \geq p|\sigma$, there exists $q \in \boldsymbol{P}_\alpha$ such that $(q, n) \geq_F (p, n)$ and $q|\sigma = r$.*

PROOF. Define the term $q(\beta)$ as follows. If $(p \restriction \beta)|\sigma \restriction (F \cap \beta)$ is false, then $q(\beta) = p(\beta)$. If $(p \restriction \beta)|\sigma \restriction (F \cap \beta)$ is true and $\beta \notin F$, then $q(\beta) = r(\beta)$. If $(p \restriction \beta)|\sigma \restriction (F \cap \beta)$ is true and $\beta \in F$, then $q(\beta)$ is a term such that $q(\beta)_s = p(\beta)_s$ for all $s \in 2^n - \{\sigma(\beta)\}$ and $q(\beta)_{\sigma(\beta)} = r(\beta)$. ■

One thing we know about a set of reals of cardinality ω_2 is that lots of its elements are not in V.

LEMMA 2. *Suppose $p \Vdash$ "$\tau \notin V$ and $\tau \in 2^\omega$" and p is (F, n)-determined. Then for any finite $Y \subseteq 2^\omega$ there exists a finite set $X \subseteq 2^\omega$ disjoint from Y, such that for any $k < \omega$ there is a q such that $(q, n) \geq_F (p, n)$ and $q \Vdash$ "$\exists x \in X\ \tau \restriction k = x \restriction k$".*

PROOF. Let $\{\sigma_i: i < N\}$ be all maps from F into 2^n consistent with p. Begin by constructing p_{i+1} for $i < N$ so that $(p_{i+1}, n) \geq_F (p_i, n)$ where $p_0 = p$, as follows. Given p_i choose $r \geq p_i|\sigma_i$ so that there exists $t \in 2^{<\omega}$ such that $r \Vdash$ "$\check{t} \subseteq \tau$" and for all $y \in Y$, y does not extend t. This can be done since $p \Vdash$ "$\tau \notin Y$". Applying Lemma 1, let p_{i+1} be such that $p_{i+1}|\sigma_i \geq r$ and $(p_{i+1}, n) \geq_F (p_i, n)$. Let $q_0 = p_N$ and build an infinite chain q_i for $i < \omega$ and x_i for $i < N$ so that $(q_{i+1}, n) \geq_F (q_i, n)$ and for each $i < N$ there are infinitely many $k < \omega$ such that $q_k|\sigma_i \Vdash$ "$\check{x}_i \restriction k = \tau \restriction k$". Let $X = \{x_i: i < N\}$. ■

LEMMA 3. *Suppose $p \Vdash$ "$\tau \notin V$ and $\tau \in 2^\omega$" and p is (F, n)-determined. Then there exist q such that $(q, n) \geq_F (p, n)$ and a family of disjoint clopen sets $\{C_s: s \in p(0) \cap 2^n\}$ such that for each $s \in q(0) \cap 2^n$, $q_s \Vdash$ "$\tau \in C_s$". (q_s is abusive notation for $q|\sigma$ where domain of σ is $\{0\}$ and $\sigma(0) = s$.)*

PROOF. We may as well assume $0 \in F$. Let $\{s_i: i < N\} = p(0) \cap 2^n$. By induction on $i < N$ and Lemma 2, find disjoint finite sets X_i for $i < N$ so that for each $i < N$ and $k < \omega$ there exists a q such that $(q, n) \geq_F (p_{s_i}, n)$ and $q \Vdash$ "$\exists x \in X_i\ \tau \restriction k = x \restriction k$". Choose $k < \omega$ such that if $C_i = \{y \in 2^\omega: \exists x \in X_i\ x \restriction k \subseteq y\}$, then the C_i are disjoint. For each $i < N$ choose q_i such that $(q_i, n) \geq_F (p_{s_i}, n)$ and $q_i \Vdash$ "$\tau \in C_i$". Now let q be defined by $q(0)_{s_i} = q_i(0)$ and $q(\beta) = q_i(\beta)$ if $q_i(0)$ for $\beta > 0$. ■

By the standard fusion argument, we now get τ to completely determine the first Sacks real.

LEMMA 4. *Suppose $p \Vdash$ "$\tau \notin V$ and $\tau \in 2^\omega$", then there exists $q \geq p$ such that for arbitrarily large $n < \omega$ there is a family of disjoint clopen sets $\{C_s: s \in q(0) \cap 2^n\}$ such that for every $s \in q(0) \cap 2^n$, $q_s \Vdash$ "$\tau \in C_s$".*

PROOF. Construct a sequence $(p_{n+1}, k_{n+1}) >_{F_n} (p_n, k_n)$ with $p_0 = p$, $k_n < \omega$ strictly increasing, F_n increasing with p_{n+1} (F_n, k_n)-determined, and $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} \mathrm{dom}(p_n)$, as follows. Given p_n, k_n, and F_n find q and $m < \omega$ such that $(q, m) >_{F_n} (p_n, k_n)$ and q is (F_n, k_n)-determined, using Lemma 2.3 of Baumgartner and Laver [2]. Using Lemma 3 find r such that $(r, k_n) \geq_{F_n} (p_n, k_n)$ and disjoint clopen sets $\{C_s: s \in r(0) \cap 2^{k_n}\}$ such that for every $s \in r(0) \cap 2^{k_n}$, $r_s \Vdash$ "$\tau \in C_s$". Apply Lemma 2.3 again to find p_{n+1} and k_{n+1} such that p_{n+1} is (F_n, k_n)-determined and $(p_{n+1}, k_{n+1}) >_{F_n} (r, k_n)$. Now just let q be the fusion of the p_n. ■

Lemma 4 is all that will be needed for §4. To get the result of §5 a little more work will be needed. The next lemma allows τ to continuously determine which $p|\sigma$ is in the generic filter.

LEMMA 5. *Suppose $p \in \boldsymbol{P}_\alpha$ and τ is a $\boldsymbol{P}_\alpha$ term such that $p \Vdash$ "$\tau \in 2^\omega$ and $\forall \beta < \alpha$ $\tau \notin V[G_\beta]$". Suppose p is (F, n)-determined and $\Sigma = \{\sigma: F \to 2^n | \sigma$ is consistent with $p\}$. Then there exist $q \in \boldsymbol{P}_\alpha$ and a family of disjoint clopen sets $\{C_\alpha: \sigma \in \Sigma\}$ such that $(q, n) \geq_F (p, n)$ and for every $\sigma \in \Sigma$, $q|\sigma \Vdash$ "$\tau \in C_\sigma$".*

PROOF. The proof is by induction on the cardinality of F and the minimum element of F, for every α and over every ground model. Note that if F is empty, the lemma is trivial since Σ is vacuous. If $0 \notin F$, then let β be the minimal element of F. Note that if we take $V[G_\beta]$ as our ground model with $p \restriction \beta \in G_\beta$, then the hypothesis of the lemma is true for $\boldsymbol{P}_{\beta\alpha}$. By relabeling $\boldsymbol{P}_{\beta\alpha}$ as some $\boldsymbol{P}_\gamma$ in $V[G_\beta]$ and F as some F', we have that $0 \in F'$. Consequently the lemma holds and we can find $q \in \boldsymbol{P}_\beta$ such that $q \geq p \restriction \beta$, disjoint clopen sets $\{C_\sigma: \sigma \in \Sigma\}$ in V, and a term r such that $q \Vdash$ "$r \in \boldsymbol{P}_{\beta\alpha}$, $(r, n) \geq_F (p \restriction [\beta, \alpha), n)$, and for all $\sigma \in \Sigma$ $r|\sigma \Vdash$ '$\tau \in C$'". Now just put q and r together to get a condition with the required property.

Now suppose $0 \in F$. Apply Lemma 3 to get q such that $(q, n) \geq_F (p, n)$ and disjoint clopen sets $\{C_s: s \in q(0) \cap 2^n\}$ such that for each $s \in q(0) \cap 2^n$, $q_s \Vdash$ "$\tau \in C_s$". Let $F_0 = F - \{0\}$ and for each $s \in q(0) \cap 2^n$ let $\Sigma_s = \{\sigma \in \Sigma: \sigma(0) = s\}$. Since F_0 is smaller than F, by induction, for each $s \in q(0) \cap 2^n$ there exist $q^s \in \boldsymbol{P}_\alpha$ and disjoint clopen sets $\{C_\sigma: \sigma \in \Sigma_s\}$ such that $(q^s, n) \geq_F (q_s, n)$ and for each $\sigma \in \Sigma_s$, $q^s | \sigma \Vdash$ "$\tau \in C_\sigma$". Clearly we may assume that $C_\sigma \subseteq C_{\sigma(0)}$. Now define $\hat{q} \in \boldsymbol{P}_\alpha$ by $\hat{q}(0)_s = q^s(0)$ for each $s \in q(0) \cap 2^n$ and $\hat{q}(\beta) = q^s(\beta)$ if $q^s(0)$, for $\beta > 0$. ■

We say that $q \in \boldsymbol{P}_\alpha$ is determined iff for every $H \in [\mathrm{dom}(q)]^{<\omega}$ and $k < \omega$ there are $m > n > k$ and $F \supseteq H$ with $F \in [\mathrm{dom}(q)]^{<\omega}$ such that q is (F, n)-determined and $(q, m) >_F (q, n)$. Note that the fusion of a sequence is determined. We will say that $q \in \boldsymbol{P}_\alpha$ is canonical for τ iff q is determined and for arbitrarily large (F, n), q is (F, n)-determined and if $\Sigma = \{\sigma: F \to 2^n | \sigma$ is consistent with $q\}$, then there is a family of disjoint clopen sets $\{C_\sigma: \sigma \in \Sigma\}$ such that for all $\sigma \in \Sigma$, $q | \sigma \Vdash$ "$\tau \in C_\sigma$", and in addition, for each $\sigma \in \Sigma$ there exists $s \in 2^n$ such that for all $x \in C_\sigma$, $s \subseteq x$ (i.e. $q | \sigma$ decides $\tau \upharpoonright n$).

LEMMA 6. *Suppose $p \in \boldsymbol{P}_\alpha$ and $p \Vdash$ "$\tau \in 2^\omega$ and $\forall \beta < \alpha\ \tau \notin V[G_\beta]$". Then there exists $q \geq p$, $q \in \boldsymbol{P}_\alpha$ such that q is canonical for τ.*

PROOF. The proof is similar to Lemma 4. Use Lemma 2.3 of Baumgartner and Laver [2] together with our Lemma 5 and then take the fusion. Deciding $\tau \upharpoonright n$ along the way may be done as in Theorem 3.3 of the above paper. ■

REMARK. If $q \in \boldsymbol{P}_\alpha$ is canonical for τ and $\beta < \alpha$, then $q \upharpoonright \beta \Vdash$ "$q \upharpoonright [\beta, \alpha)$ is canonical for τ".

REMARK. If p is canonical for τ and x_α for $\alpha \in \mathrm{dom}(p)$ is the αth Sacks real, then it is easy to see that $x_\alpha \in V[\tau]$. If $q \Vdash$ "$\tau \in 2^\omega$, $\tau \in V[G_\gamma]$, and $\forall \beta < \gamma\ \tau \in V[G_\beta]$", then for every $\beta < \gamma$ $q \Vdash$ "$x_\beta \in V[\tau]$". To see this just find $\bar{\tau}$ a $V[G_\gamma]$ term and $\bar{q} \geq q$ such that $\bar{q} \Vdash$ "$\tau = \bar{\tau}$". Then for any $\beta < \gamma$ one can find $r \geq \bar{q}$ such that $r | \gamma$ is canonical for $\bar{\tau}$ and $\beta \in \mathrm{dom}(r)$. It is easy to see from this remark that the degrees of constructibility of reals in $V[G_{\omega_2}]$ if $V = L$ have order type ω_2.

§4. The main theorem.

In this section we prove the main theorem.

THEOREM. *In the iterated perfect set model every set of reals of cardinality ω_2 can be mapped continuously onto the closed unit interval.*

PROOF. We may reduce to the case $X \subseteq 2^\omega$ and we are trying to map X continuously onto 2^ω. We will actually find a continuous map $f: 2^\omega \to 2^\omega$ such that $f''X = 2^\omega$. Suppose, for contradiction, for each continuous f one can find $F(f) \in 2^\omega$ such that $F(f) \notin f''X$. By a kind of Lowenheim-Skolem argument, like the proof of Theorem 4.5 of Baumgartner and Laver [2], one can find $\beta < \omega_2$ such that F restricted to the continuous functions coded in $V[G_\beta]$ is an element of $V[G_\beta]$. Replacing V by $V[G_\beta]$, which we may do without loss of generality by Theorem 2.5 of the above paper, we claim that $X \subseteq V$. Suppose that $p \Vdash$ "$\tau \in 2^\omega$ and $\tau \notin V$".

From Lemma 4 of §3 find $q \geq p$, X an infinite subset of ω, and for each $n \in X$ a family of disjoint clopen sets $\{C_s: s \in q(0) \cap 2^n\}$ such that for each $s \in q(0) \cap 2^n$, $q_s \Vdash$ "$\tau \in C_s$". Clearly we may assume that if $s \subseteq t$ (and C_s and C_t are defined), then $C_t \subseteq C_s$. Now let P be the fusion of the family of C_s's, i.e.

$$P = \bigcap_{n \in X} \bigcup_{s \in q(0) \cap 2^n} C_s.$$

Let $E = \{x \in 2^\omega\colon \forall n\ x \upharpoonright n \in q(0)\}$ and let $f^*\colon P \to E$ be defined by

$$f^*(x) = \bigcup\{s\colon x \in C_s\}.$$

It is easily checked that f^* is continuous, and since P is perfect, by the Tietze extension theorem, there is a continuous $f\colon 2^\omega \to E$ such that $f \upharpoonright P = f^*$.

In effect, f maps τ continuously to the first Sacks real. Construct $g\colon E \to 2^\omega$ by first mapping E homeomorphically to $2^\omega \times 2^\omega$ and then projecting onto the first coordinate, and let $h = g \circ f$.

Suppose that $x \in 2^\omega$, then $g^{-1}(x)$ is a perfect subset of E, and consequently there is an $r \in \boldsymbol{P}$ with $r \geq q(0)$ such that $g^{-1}(x) = \{y \in 2^\omega\colon \forall n\ y \upharpoonright n \in r\}$. If we define $\hat{q}$ by $\hat{q}(0) = r$ and $\hat{q}(\beta) = q(\beta)$ for $\beta > 0$, then $\hat{q} \Vdash$ "$h(\tau) = x$".

Therefore, if $x = F(h)$, then $\hat{q} \Vdash$ "$\tau \notin X$". Since τ and p were arbitrary, we have that $X \subseteq V$.

REMARK. By using a Δ-system argument, one can prove that in $V[G_{\omega_2}]$ for every $X \in [2^\omega]^{\omega_2}$ there is a continuous map $f\colon 2^\omega \to 2^\omega$ *coded in* V such that $f''X = 2^\omega$.

REMARK. It is not hard to show that the theorem fails if Sacks reals are added side-by-side.

Question. Can one show that if $|2^\omega| > \omega_2$, then there exists $X \subseteq 2^\omega$, $|X| = |2^\omega|$, which cannot be mapped continuously onto 2^ω?

§5. Sets always of first category. In this section we prove the following theorem.

THEOREM. *In the iterated perfect set model every set of reals which is always of first category has cardinality less than or equal to* ω_1.

Recall that we say that $p \in \boldsymbol{P}_\alpha$ is determined iff for arbitrarily large (F, n), p is (F, n)-determined and there is $m > n$ such that $(p, m) >_F (p, n)$. Given a determined p, say with domain A, we associate with it a perfect subset $E_p \subseteq (2^\omega)^A$ as follows. Let E_p be the set of all $\langle x_\beta\colon \beta \in A\rangle$ such that for all (F, n) if p is (F, n)-determined, then there is $\sigma\colon F \to 2^n$ consistent with p such that for all $\beta \in F$ $\sigma(\beta) \subseteq x_\beta$. We are going to show that given any determined $p \in \boldsymbol{P}_\alpha$ and $C \subseteq E_p$ first category relative to E_p, there exists a determined $q \geq p$ in $\boldsymbol{P}_\alpha$ such that $E_q \cap C = \emptyset$.

Let us see that this will prove the theorem. Proceeding as in the last section we may assume $X \subseteq 2^\omega$ has the property that for any perfect set P coded in V, one can find (working in V) a set C of first category relative to P such that $X \cap P \subseteq C$. It must be that $X \subseteq V$. To see this note that given any p and τ such that $p \Vdash$ "$\tau \in 2^\omega$ and $\tau \notin V$" one can extend p and find $\alpha < \omega_2$ and $\bar{\tau}$ a $\boldsymbol{P}_\alpha$-term so that $p \Vdash$ "$\tau = \bar{\tau}$ and $\forall \beta < \alpha\ \bar{\tau} \notin V[G_\beta]$", and $\bar{p} = p \upharpoonright \alpha$ is canonical for $\bar{\tau}$. Define a function $f\colon E_{\bar{p}} \to 2^\omega$ by requiring that for any $s \in 2^{<\omega}$, $s \subseteq f(x)$ iff there is some $\sigma\colon F \to 2^n$ which extends x (i.e. $\forall \beta \in F\ \sigma(\beta) \subseteq x_\beta$) such that $\bar{p}|\sigma \Vdash$ "$s \subseteq \bar{\tau}$". Since $\bar{p}$ is canonical for $\bar{\tau}$, the map f is continuous, one-to-one, and for any determined $q \geq \bar{p}$ one has that $q \Vdash$ "$\tau \in f''E_q$".

Let $P = f''E_{\bar{p}}$ and suppose $X \cap P \subseteq C$ where C is first category in P. Then $f^{-1}(C)$ is first category in $E_{\bar{p}}$. Therefore, there exists $q \geq \bar{p}$ such that $E_q \cap f^{-1}C = \emptyset$. Thus any $r \geq p$ with $r \upharpoonright \alpha \geq q$ forces "$\tau \notin X$".

For $p \in \boldsymbol{P}$ the set E_p is equal to $\{x \in 2^\omega : \forall n\, x \upharpoonright n \in p\}$. If C is nowhere dense in E_p, in general, it is not true that for any $q \geq p$, C is nowhere dense in E_q, in fact, maybe $E_q \subseteq C$. However, if there exists $n < \omega$ such that for all $s \in q \cap 2^n$ $q_s = p_s$, then it will remain true that C is nowhere dense in E_q. This motivates the following definition for any $p, q \in \boldsymbol{P}_\alpha$:

$q \geq^w p$ iff $q \geq p$, $\mathrm{dom}(q) = \mathrm{dom}(p)$, and there exists (F, n) such that p and q are (F, n)-determined and for any $\sigma: F \to 2^n$ which is consistent with q we have that $q|\sigma = p|\sigma$.

We say that (F, n) witnesses that $q \geq^w p$.

LEMMA 1. *Let p, q, and r be elements of $\boldsymbol{P}_\alpha$.*

(A) *If $p \geq^w q$ is witnessed by (F_0, n_0), $F \supseteq F_0$, $n \geq n_0$, and q is (F, n)-determined, then p is (F, n)-determined and (F, n) witnesses that $p \geq^w q$. If, in addition, $(q, m) >_F (q, n)$, then $(p, m) >_F (p, n)$.*

(B) *If $p \geq^w q$ and q is determined, then p is determined.*

(C) *If $p \geq^w q$, $q \geq^w r$, and r is determined, then $p \geq^w r$.*

PROOF. (A) The proof of this is by induction on α. For α a limit it is easy. So assume $\alpha = \beta + 1$, and note that $p \upharpoonright \beta \geq^w q \upharpoonright \beta$ is witnessed by $(F_0 \cap \beta, n_0)$. By induction, we have that $p \upharpoonright \beta$ is $(F \cap \beta, n)$-determined, $(p \upharpoonright \beta, m) >_{F \cap \beta} (p \upharpoonright \beta, n)$, and $p \upharpoonright \beta \geq^w q \upharpoonright \beta$ is witnessed by $(F \cap \beta, n)$. Suppose $\sigma: F \to 2^n$ is such that $\sigma \upharpoonright (F \cap \beta)$ is consistent with $p \upharpoonright \beta$. Let $\sigma_0: F_0 \to 2^{n_0}$ be defined by requiring that $\sigma_0(\gamma) \subseteq \sigma(\gamma)$ for each $\gamma \in F_0$. We have that $(p|\sigma) \upharpoonright \beta = (q|\sigma) \upharpoonright \beta$ and, since q is (F, n)-determined, $(q|\sigma) \upharpoonright \beta$ forces "$\sigma(\beta) \in q(\beta)$" or it forces "$\sigma(\beta) \notin q(\beta)$". Also $(q|\sigma_0) \upharpoonright (F_0 \cap \beta) \upharpoonright \beta$ forces "$q(\beta)_{\sigma_0(\beta)} = p(\beta)_{\sigma_0(\beta)}$" if $\beta \in F_0$ or it forces "$q(\beta) = p(\beta)$" if $\beta \notin F_0$. It follows that p is (F, n)-determined and (F, n) witnesses $p \geq^w q$. To see that $(p, m) >_F (p, n)$, just note that it is enough to see that for every $\sigma: (F \cap \beta) \to 2^n$ consistent with $p \upharpoonright \beta$ we have that

$$(p|\beta)|\sigma \Vdash \text{“}(p(\beta), m) > (p(\beta), n)\text{”}.$$

This is true since the $(p|\beta)|\sigma$ form a maximal antichain beneath $p \upharpoonright \beta$.

(B) This is immediate from (A).

(C) Suppose (F_0, n_0) witnesses $q \geq^w r$ and (F_1, n_1) witnesses $p \geq^w q$. Let $F \supseteq F_0 \cup F_1$ and $n \geq n_0, n_1$ be such that r is (F, n)-determined. Then by (A) (F, n) witnesses both $p \geq^w q$ and $q \geq^w r$. It follows that for all $\sigma: F \to 2^n$ if σ is consistent with p, then $p|\sigma = q|\sigma = r|\sigma$. ■

Weak extension was defined exactly to make the following lemma true.

LEMMA 2. *Suppose p is determined and C is nowhere dense in E_p. If $q \geq^w p$, then C is nowhere dense in E_q.*

PROOF. By Lemma 1, q is determined. Let (F, n) witness $q \geq^w p$, and let $\Sigma = \{\sigma: F \to 2^n | \sigma \text{ is consistent with } q\}$. Then $E_q = \bigcup \{E_{q|\sigma}: \sigma \in \Sigma\}$. But C is nowhere dense in each $E_{q|\sigma} = E_{p|\sigma}$, and so C is nowhere dense in E_q. ■

The next lemma is analogous to Lemma 1 of §3 and proved the same way.

LEMMA 3. *If p is determined, (F, n)-determined, $\sigma: F \to 2^n$ is consistent with p, and $r \geq^w p|\sigma$, then there exists q such that $(q, n) \geq^w{}_F (p, n)$ and $q|\sigma = r$.*

PROOF. Let $q(\beta)$ be a term denoting the following. If $(p \upharpoonright \beta)|\sigma \upharpoonright (F \cap \beta)$ does not hold, then $q(\beta) = p(\beta)$. If $(p \upharpoonright \beta)|\sigma \upharpoonright (F \cap \beta)$ does hold and $\beta \notin F$, then

$q(\beta) = r(\beta)$. Otherwise $q(\beta)$ denotes the term such that $q(\beta)_s = p(\beta)_s$ for all $s \in 2^n - \{\sigma(\beta)\}$ and $q(\beta)_{\sigma(\beta)} = r(\beta)$. As before $(q, n) \geq_F (p, n)$ and we need only verify that $q \geq^w p$. Let (F_0, n_0) witness $r \geq^w p | \sigma$. Choose (F_1, n_1) such that $F_1 \supseteq F \cup F_0$, $n_1 \geq n, n_0$, and p is (F_1, n_1)-determined. By Lemma 1, since $p|\sigma$ is (F_1, n_1)-determined, it follows that (F_1, n_1) witnesses that $r \geq^w p|\sigma$. I claim that (F_1, n_1) witnesses that $q \geq^w p$. It is not hard to prove by induction on α that $(F_1 \cap \alpha, n_1)$ witnesses $q \restriction \alpha \geq^w p \restriction \alpha$. ■

LEMMA 4. *Suppose p is determined and C is nowhere dense in E_p. Then there exists $q \geq^w p$ such that $E_q \cap C = \emptyset$.*

PROOF. Find (F, n) such that p is (F, n)-determined and $\sigma: F \to 2^n$ consistent with p such that $E_{p|\sigma} \cap C = \emptyset$. This is possible since such $E_{p|\sigma}$ form a basis for E_p. Now let $q = p|\sigma$. ■

LEMMA 5. *If p is determined, (F, n)-determined, and C is nowhere dense in E_p, then there exists q such that $(q, n) \geq^w{}_F (p, n)$ and $E_q \cap C = \emptyset$.*

PROOF. Let $\{\sigma_1: i < N\}$ be all maps from F into 2^n consistent with p. Build a chain $(p_{i+1}, n) \geq^w_F (p_i, n)$ using Lemmas 3 and 4, so that $p_0 = p$ and $E_{p_{i+1}|\sigma_i} \cap C = \emptyset$. Now let $q = p_N$. ■

As we have indicated at the beginning of this section, the theorem is proved once we have the next lemma.

LEMMA 6. *Suppose p is determined and C is first category in E_p, then there exists a determined $q \geq p$ such that $E_q \cap C = \emptyset$.*

PROOF. Let $C = \bigcup\{C_n: n < \omega\}$ where each C_n is nowhere dense in E_p. Construct a sequence p_n, F_n, and k_n, for $n < \omega$ such that:

(1) $p_0 = p$;
(2) k_n are increasing;
(3) F_n are increasing and $\bigcup\{F_n: n < \omega\} = \mathrm{dom}(p)$;
(4) $(p_{n+1}, k_{n+1}) >^w_{F_n} (p_n, k_n)$;
(5) p_n is (F_n, k_n)-determined;
(6) $E_{p_{n+1}} \cap C_n = \emptyset$.

The condition q will be the fusion of this sequence. Suppose p_n, F_n, and k_n have already been found. Since $p_n \geq^w p$, p_n is determined, has the same domain as p, and C_n is nowhere dense in E_{p_n}. By Lemma 4 let p_{n+1} be such that

$$(p_{n+1}, k_n) \geq^w_{F_n} (p_n, k_n)$$

and $E_{p_{n+1}} \cap C_n = \emptyset$. Since p_{n+1} is determined, one can find arbitrarily large (F_{n+1}, k_{n+1}) such that p_{n+1} is (F_{n+1}, k_{n+1})-determined and

$$(p_{n+1}, k_{n+1}) >^w_{F_{n+1}} (p_{n+1}, k_n).$$

It follows that

$$(p_{n+1}, k_{n+1}) >^w_{F_n} (p_n, k_n). \quad ■$$

REFERENCES

[1] F. BAGEMIHL and H. SPRINKLE, *On a proposition of Sierpinski's which is equivalent to the continuum hypothesis*, ***Proceedings of the American Mathematical Society***, vol. 5 (1954), pp. 726–728.

[2] J. BAUMGARTNER and R. LAVER, *Iterated perfect set forcing*, ***Annals of Mathematical Logic***, vol. 17 (1979), pp. 271–288.

[3] J. Isbell, *Spaces without large projective subspaces*, ***Mathematica Scandinavica,*** vol. 17 (1965), pp. 89–105.

[4] ———, *A set whose square can map onto a perfect set*, ***Proceedings of the American Mathematical Society,*** vol. 20 (1969), pp. 254–255.

[5] K. Kunen, Doctoral Dissertation, Stanford University, 1968.

[6] K. Kuratowski, ***Topology,*** vol. 1, Academic Press, New York, 1966.

[7] R. Laver, *On the consistency of Borel's conjecture*, ***Acta Mathematica,*** vol. 137 (1976), pp. 151–169.

[8] H.L. Royden, ***Real analysis,*** Macmillan, New York, 1968.

[9] W. Sierpinski, *Sur un ensemble non dénombrable donte toute image continue est de mesure null*, ***Fundamenta Mathematical,*** vol. 11 (1928), p. 304.

[10] ———, ***Hypothèse du continu,*** Monografie Matematyczne, Tom IV, Warsaw, 1934.

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