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**Theorem 1** *Assume CH. Then there exists a Luzin set $X \subseteq (2^\omega)^\omega$, a Sierpinski set $Y \subseteq 2^\omega$, and a Borel function $f : (2^\omega)^\omega \times 2^\omega \to 2^\omega$ such that $f(X \times Y) = 2^\omega$.*

Proof

For $x \in 2^\omega$ define

$$G_x = \{u \in 2^\omega : \exists^\infty n \quad u \restriction [n, 2n) = x \restriction [n, 2n)\}.$$

Note that $G_x$ is a measure zero comeagre $G_\delta$ set for any $x \in 2^\omega$. Also if $x =^* y$ (equal mod finite), then $G_x = G_y$. The function $f$ is defined by:

$$f(\langle x_n : n < \omega\rangle, y) = z \text{ iff } (\forall n \;\; z(n) = 1 \text{ iff } y \in G_{x_n}).$$

As in Kunen's set theory book, define $\mathrm{Fn}(\omega \times \omega, 2)$ to be the partial order of finite partial functions from $\omega \times \omega$ into 2.

**Lemma 2** *Suppose $M$ is a countable transitive model of a sufficiently large finite fragement of ZFC and $z \in 2^\omega$ is arbitrary. Then there exists $x = \langle x_n : n < \omega\rangle$ which is $\mathrm{Fn}(\omega \times \omega, 2)$-generic over $M$ and $y$ which is random over $M$ such that $f(x, y) = z$.*

Proof

Let $u = \langle u_n : n < \omega\rangle$ be $\mathrm{Fn}(\omega \times \omega, 2)$-generic over $M$. Let $H$ be measure amoeba generic over $M[u]$. Since $H$ makes the union of all measure zero sets coded in $M[u]$ measure zero, there exist in $M[u, H]$ a perfect tree $T \subseteq 2^{<\omega}$ such that the set of infinite branchs of $T$, $[T]$, is disjoint from every measure zero set coded in $M[u]$. Note that:

- $[T] \cap G_{u_n} = \emptyset$ for every $n$, and
- every $y \in [T]$ is random over $M$.

Let $v = \langle v_n : n < \omega\rangle$ be $\mathrm{Fn}(\omega \times \omega, 2)$-generic over $M[u, H]$. An easy density argument shows that for every $n$ the set $G_{v_n}$ is dense in $[T]$ and hence comeager.

Define:

$$w_n = \begin{cases} u_n & \text{if } z(n) = 0 \\ v_n & \text{if } z(n) = 1. \end{cases}$$

It may be that $w$ is not $\mathrm{Fn}(\omega \times \omega, 2)$-generic over $M$. However, it is easy to see by the usual facts of iterated forcing that for every $N < \omega$ $\langle w_n : n < N \rangle$ is $\mathrm{Fn}(\mathrm{N} \times \omega, 2)$-generic over $M$. According to a lemma of Harvey Friedman[1], there exist $x = \langle x_n : n < \omega \rangle$ which is $\mathrm{Fn}(\omega \times \omega, 2)$-generic over $M$ and $x_n =^* w_n$ for every $n$.

We can choose

$$y \in [T] \cap \bigcap \{G_{x_n} : z(n) = 1\}$$

because these sets are all comeager in $[T]$. And hence, $f(x, y) = z$.
QED

From the Lemma and CH it is easy to construct the sets $X$ and $Y$ as required.
QED

[1]Friedman, Harvey; Large models of countable height. Trans. Amer. Math. Soc. 201 (1975), 227–239. Lemma 3.