Irredundant Generators

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Abstract

We say that $\mathcal{I}$ is an irredundant family if no element of $\mathcal{I}$ is a subset mod finite of a union of finitely many other elements of $\mathcal{I}$. We will show that the minimum size of a maximal irredundant family is consistently bigger than both $\mathfrak{d}$ and $\mathfrak{u}$, this answers a question of Donald Monk.

Let $\mathfrak{s}_{mm}$ be the minimal cardinality of a maximal irredundant ideal generator, i.e., an infinite family $\mathcal{B} \subseteq [\omega]^{\omega}$ such that no element of $\mathcal{B}$ is a subset mod finite of a union of finitely many other elements of $\mathcal{B}$ and $\mathcal{B}$ is maximal with respect to this property, i.e., for any $X \in [\omega]^{\omega} \setminus \mathcal{B}$, it cannot be added to $\mathcal{B}$ and still be irredundant. This means that there is $F \in [\mathcal{B}]^{\omega}$ such that either $X \subseteq^* \bigcup F$ or there is $B \in \mathcal{B} \setminus F$ with $B \subseteq^* X \cup \bigcup F$. This concept also occurs in Monk [12] where the terminology "ideal independent" is used instead of "irredundant generator". We will compare $\mathfrak{s}_{mm}$ with the ultrafilter number and the dominating number, for the definition and basic properties of the usual cardinal invariants see Blass [3].

In May 2013 at a conference at the Ben-Gurion University of the Negev Donald Monk [11] asked if $\mathfrak{s}_{mm}$ was equal to $\mathfrak{u}$ (this question was communicated to Arnold Miller by Juris Steprans). The next proposition answers this question negatively. In the rational perfect set model $\mathfrak{d} = \omega_2$ and $\mathfrak{u} = \omega_1$, see Miller [9] and Blass-Shelah [2].

Proposition 1 $\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{s}_{mm}$.

Proof
Given a maximal irredundant family $\mathcal{I}$, it is easy to see that the following family of sets is a reaping family:

$$\{A \setminus \bigcup F : F \in [\mathcal{I}]^{<\omega} \wedge A \in \mathcal{I} \setminus F\}$$

It remains to prove that $\mathfrak{d} \leq \mathfrak{s}_{mm}$. Assume otherwise that $\mathfrak{s}_{mm} < \mathfrak{d}$, and let $\mathcal{A}$ be a witness for this. Note that $\omega =^* \bigcup \mathcal{A}$, so we can assume that indeed the equality holds. Let $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ such that its union is ω. Define $C_0 = A_0$ and $C_{n+1} = A_{n+1} \setminus \bigcup_{i \leq n} A_i$. For each $F \in [\mathcal{A}]^{<\omega}$ and $B \in \mathcal{A} \setminus (F \cup \{A_i : i < \omega\})$, define a function as follows:

$$\varphi_{F,B}(n) = \min\{k \in \omega : (\exists j \geq n)(C_j \cap B \cap k \setminus \bigcup F \neq \emptyset)\}$$

Since the family $\mathcal{A}$ is irredundant, the functions $\varphi_{F,B}$ are always well defined. Let h_0 be an increasing function not dominated by

$$\{\varphi_{F,B} \ : \ F \in [\mathcal{A}]^{<\omega}, \ B \in \mathcal{A} \setminus (F \cup \{A_i : i < \omega\})\}.$$

Define $D_n = C_n \setminus h_0(n)$. Now for each $F \in [\mathcal{A}]^{<\omega}$, whenever it is possible, define a function as follows:

$$\tilde{\varphi}_F(n) = \min\{k \in \omega : (\exists j \geq n)(D_j \cap k \setminus \bigcup F \neq \emptyset)\}$$

This is defined for n, otherwise

$$\bigcup_{j \geq n} D_j = \bigcup_{j \geq n} (C_j \setminus h_0(j)) \subseteq \bigcup F$$

But then for some $j \geq n$ such that $A_j \notin F$ we would have

$$A_j \subseteq^* \bigcup_{i<j} A_i \cup \bigcup F$$

which contradicts irreducibility.

Let $h_1 > h_0$ be an increasing function not dominated by any totally defined $\tilde{\varphi}_F$ for $F \in [\mathcal{A}]^{<\omega}$ and such that $C_n \cap [h_0(n), h_1(n))$ is nonempty for all n.

Let

$$Y = \bigcup_{n \in \omega} (C_n \cap [h_0(n), h_1(n))) = \bigcup_{n \in \omega} D_n \cap h_1(n)$$

Let's see that $\mathcal{A} \cup \{Y\}$ is an irredundant family.

Claim 1. For all $F \in [\mathcal{A}]^{<\omega}$, $Y \not\subseteq^* \bigcup F$.

If the function $\tilde{\varphi}_F$ is not defined, then $Y \cap \bigcup F$ is finite. Otherwise, by the definition of the function $\tilde{\varphi}_F$, if $\tilde{\varphi}_F(n) \leq h_1(n)$, then for some $j \geq n$ we have $D_j \cap \tilde{\varphi}_F(n) \setminus \bigcup F \neq \emptyset$, which implies

$$\emptyset \neq D_j \cap h_1(n) \setminus \bigcup F \subseteq D_j \cap h_1(j) \setminus \bigcup F \subseteq Y.$$

Since this happens for infinitely j and the family $\{D_j : j \in \omega\}$ is disjoint, we are done.

Claim 2. For any $F \in [\mathcal{A}]^{<\omega} \setminus \{\emptyset\}$ and $B \in \mathcal{A} \setminus F$, we have $B \not\subseteq^* Y \cup \bigcup F$.

If $B = A_n$ for some n this is clear. Otherwise, by the definition of $\varphi_{F,B}$ and the choice of h_0, we have that if $\varphi_{F,B}(n) \leq h_0(n)$, then for some $j \geq n$,

$$\emptyset \;\neq\; C_j \cap B \cap \varphi_{F,B}(n) \setminus \bigcup F \;\subseteq\; C_j \cap B \cap h_0(j) \setminus \bigcup F.$$

If $m \in C_j \cap B \cap h_0(j) \setminus \bigcup F$, then $m \notin Y \cup \bigcup F$. Since this happens infinitely many times, we are done.
QED

We will now show some results related to irredundant families.

Proposition 2 *If $\mathcal{I}$ is an ideal generated by a strictly ascending mod finite sequence $A_\alpha \subseteq \omega$ for $\alpha < \omega_1$, then $\mathcal{I}$ is not generated by an irredundant family.*

Proof
So $\mathcal{I} = \{B \;:\; \exists\alpha < \omega_1 \;\; B \subseteq^* A_\alpha\}$. Suppose $\mathcal{B} \subseteq \mathcal{I}$ generates $\mathcal{I}$. For each α choose $F_\alpha \subseteq \mathcal{B}$ finite with $A_\alpha \subseteq^* \cup F_\alpha$. Suppose F_α for $\alpha \in \gamma$ is a delta system for γ uncountable. We may find $\alpha < \beta$ in γ with $\cup F_\alpha \subseteq^* A_\beta$. Since $A_\beta \subseteq^* \cup F_\beta$, for any $B \in (F_\alpha \setminus F_\beta)$, $B \subseteq^* \cup F_\beta$ but this implies that $\mathcal{B}$ is redundant.
QED

Although many ideals can be generated by an irredundant family, this is not the case for the prime ideals.

Proposition 3 *A non-principal prime ideal $\mathcal{I}$ on ω cannot be irredundantly generated.*

Proof
Suppose $\mathcal{B}$ is an irredundant generator of $\mathcal{I}$. Let $\{A_n : n < \omega\} \subseteq \mathcal{B}$ be distinct. By adding at most one thing to each A_n we may suppose $\cup_{n<\omega} A_n$ is ω. Let

$$B = \bigcup_n (A_{2n} \setminus \cup_{i<2n} A_i) \text{ and } C = \bigcup_n (A_{2n+1} \setminus \cup_{i<2n+1} A_i)$$

and note these are complementary sets. If $B \in \mathcal{I}$ then for some finite $F \subseteq \mathcal{B}$ we have $B \subseteq^* \cup F$. But this means $A_{2n} \subseteq^* \bigcup F \cup \bigcup_{i<2n} A_i$ which contradicts irredundancy for n large enough so that $A_{2n} \notin F$. Similarly if $C \in \mathcal{I}$.
QED

We will now show that $\mathfrak{s}_{mm}$ can be smaller than the continuum, in fact this holds in the side by side countable support Sacks model.

Proposition 4 *In the side by side countable support Sacks model there is a maximal irredundant generator of size ω_1. In this model the continuum can be made arbitrarily large but $\mathfrak{s}_{mm} = \mathfrak{u} = \mathfrak{d} = \omega_1$.*

Proof
We are forcing with the countable support product of κ-many Sacks posets for any κ over a model of CH.

To get an irredundant generator which remains maximal after forcing, we need only work with the ω-product of Sacks forcing $\mathbb{P} = \mathbb{S}^\omega$.

By Laver's combinatorial generalization of the Halpern-Lauchli Theorem [7] for any $\mathbb{P}$-name τ for a subset of ω and $p \in \mathbb{P}$ we may obtain $q \leq p$ and $Z \in [\omega]^\omega$ such that either

$$q \Vdash \text{“}Z \subseteq \tau \text{ or } q \Vdash Z \cap \tau = \emptyset.\text{”}$$

As Laver points out one may use this to build a descending mod finite sequence $Z_\alpha \in [\omega]^\omega$ for $\alpha < \omega_1$ in the ground model with the property that they generate a Ramsey ultrafilter in the extension.

Lemma 5 *Given $(Y_n \in [\omega]^\omega : n < \omega)$ pairwise disjoint in the ground model, τ a $\mathbb{P}$-name for a subset of ω, and $p \in \mathbb{P}$, there are $W_n \in [Y_n]^\omega$ and $q \leq p$ such that*

$$q \Vdash \text{“}\forall n \ \ (W_n \subseteq \tau \text{ or } W_n \cap \tau = \emptyset).\text{”}.$$

Proof
Let $(f_n : \omega \to Y_n)_n$ be a sequence of bijections in the ground model and define $\tau_n = f_n^{-1}(\tau)$. Let G be generic with $p \in G$. By properness of $\mathbb{P}$ in the generic extension for some $\alpha < \omega_1$

$$\forall n \ \ (Z_\alpha \subseteq^* \tau_n^G \text{ or } Z_\alpha \cap \tau_n^G =^* \emptyset)$$

Using the well-known fact that this forcing is ω^ω-bounding (in Shelah's terminology) or weakly distributive (in Namba's terminology) we may find $f \in \omega^\omega$ in the ground model and take $W_n = f_n(Z_\alpha) \setminus f(n)$ so that

$$W_n \subseteq \tau^G \text{ or } W_n \cap \tau^G = \emptyset \text{ for all } n.$$

Finally we take $q \leq p$ to do the required forcing.
QED

Working in the ground model let (p_α, τ_α) for $\alpha < \omega_1$ list with uncountable repetitions all pairs (p, τ) for $p \in \mathbb{P}$ and τ a canonical $\mathbb{P}$-name for a subset of ω. Construct an increasing family of irredundant countable families $\mathcal{I}_\alpha$ for $\alpha < \omega_1$.

At stage α given $\mathcal{I}_\alpha = \{A_n : n < \omega\}$.

Put $B_n = (A_n \setminus \cup_{i<n} A_i)$ and construct $Y_n \in [B_n]^\omega$ so that Y_n are infinite pairwise disjoint, $B_n \setminus Y_n$ is infinite, and $Y_n \cap A_k$ is finite for $k \neq n$.

By applying the Lemma we may obtain $(W_n \in [Y_n]^\omega : n < \omega)$ and $q \leq p_\alpha$ such that

$$q \Vdash \text{“}\forall n \ \ (W_n \subseteq \tau_\alpha \text{ or } W_n \cap \tau_\alpha = \emptyset).\text{”}.$$

Take $W = \cup_{n<\omega}(B_n \setminus W_n)$ and let $\mathcal{I}_{\alpha+1} = \mathcal{I}_\alpha \cup \{W\}$. It is not hard to see that this family is indeed irredundant. We claim that τ is forced by q to never be added to our irredundant family. Let G be generic with $q \in G$.

If for some n, $W_n \subseteq \tau_\alpha^G$, then $W \cup \tau_\alpha^G$ covers B_n and hence τ^G, W, A_i for $i < n$ cover A_n.

If for all n $W_n \cap \tau_\alpha^G = \emptyset$, then $\tau^G \subseteq W$ since the B_n partition ω, and so the pair is redundant.

Hence $\mathcal{I} = \bigcup_{\alpha<\omega_1} \mathcal{I}_\alpha$ will be a maximal irredundant family in the ground model which remains a maximal irredundant family in the generic extension.
QED

Recall the following definition by Vojtáš [15]

Definition 6 *We say* $(A, B, \to)$ *is an* invariant *if,*

1. $\to \subseteq A \times B$.
2. *For every* $a \in A$ *there is* $b \in B$ *such that* $a \to b$.
3. *There is no* $b \in B$ *such that* $a \to b$ *for all* $a \in A$.

We say that $D \subseteq B$ is dominating if for every $a \in A$ there is a $d \in D$ such that $a \to d$, so 3) means that B is dominating and 4) that no singleton is dominating. Given an invariant $(A, B, \to)$ we define it's *evaluation* by $\langle A, B, \to \rangle = min \{|D| : D \subseteq B \text{ and } D \text{ is dominating}\}$. An invariant $(A, B, \to)$ is called Borel if A, B and $\to$ are Borel subsets of a polish space. Most of the usual (but not all) invariants are actually Borel invariants. In [13] for any Borel invariant $(A, B, \to)$, a guessing principle $\Diamond (A, B, \to)$ is defined and it is proved that it implies $\langle A, B, \to \rangle \leq \omega_1$ and it holds in most of the natural models where this inequality holds. For our applications in this note, we need to work in a slightly more general framework than the one in [13].

Definition 7 *We say an invariant* $(A, B, \to)$ *is an* $L(\mathbb{R})$-invariant *if* A, B *and* $\to$ *are subsets of Polish spaces and all three of them belong to* $L(\mathbb{R})$.

Following [13] we define the following guessing principle for any $L(\mathbb{R})$-invariant $(A, B, \to)$.

Definition 8 $\Diamond_{L(\mathbb{R})} (A, B, \to)$

For every $C : 2^{<\omega_1} \to A$ *such that* $C \restriction 2^\alpha \in L(\mathbb{R})$ *for all* $\alpha < \omega_1$ *there is a* $g : \omega_1 \to B$ *such that for every* $R \in 2^{\omega_1}$ *the set* $\{\alpha \mid C(R \restriction \alpha) \to g(\alpha)\}$ *is stationary.*

Exactly as in the Borel case, $\Diamond_{L(\mathbb{R})} (A, B, \to)$ implies $\langle A, B, \to \rangle \leq \omega_1$. Given two $L(\mathbb{R})$-invariants $\mathbb{A} = (A_-, A_+, {}_{\mathbb{A}}\!\to)$ and $\mathbb{B} = (B_-, B_+, {}_{\mathbb{B}}\!\to)$ we define the *sequential composition* $\mathbb{A}; \mathbb{B} = (A_- \times Bor(B_-^{A_+}), A_+ \times B_+, \to)$ where $Bor(B_-^{A_+})$ denotes the set of codes of all Borel functions from A_+ to B_- and $(a_-, f) \to (a_+, b_+)$ if $a_{-\mathbb{A}} \to a_+$ and $f(a_-)_{\mathbb{B}} \to b_+$. It is easy to see that $\mathbb{A}; \mathbb{B}$ is an $L(\mathbb{R})$-invariant and in [3] it is proved that $\langle \mathbb{A}; \mathbb{B} \rangle = max \{\langle \mathbb{A} \rangle, \langle \mathbb{B} \rangle\}$.

As usual we will write $\mathfrak{d}$ instead of $(\omega^\omega, \omega^\omega, \leq^*)$ and $\mathfrak{r}_\sigma$ instead of the invariant $(([\omega]^\omega)^\omega, [\omega]^\omega, \text{is } \sigma\text{-reaped})$.

Proposition 9 $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma;\mathfrak{d})$ *implies* $\mathfrak{s}_{mm}=\omega_1$.

Proof

We need to define a function F into $[\omega]^\omega\times(\omega^\omega)^{[\omega]^\omega}$ such that for all $\alpha\in\omega_1$, $F\restriction\alpha$ is in $L(\mathbb{R})$. For each $\alpha<\omega_1$, let $e_\alpha:\omega\to\alpha$ be an enumeration of α in $L(\mathbb{R})$. By a suitable coding, we can assume that the domain of F is the set

$$\bigcup_{\alpha\in\omega_1}[\omega]^\omega\times([\omega]^\omega)^\alpha$$

Given $(A,\vec{\mathcal{I}})\in[\omega]^\omega\times(\omega^\omega)^\alpha$ proceed as follows. If $\vec{\mathcal{I}}$ is not an irredundant family, define $F(A,\vec{\mathcal{I}})=(\omega,e)$, where $e(X)$ for $X\in[\omega]^\omega$ is the enumeration of X. Otherwise, define $B_n^{\vec{\mathcal{I}}}=I_{e_\alpha(n)}\setminus\bigcup_{i<n}I_{e_\alpha(i)}$. For each n, let $Z_n^{\vec{\mathcal{I}}}\subseteq B_n^{\mathcal{I}}$ be an infinite subset such that for all $\beta\neq e_\alpha(n)$, $Z_n^{\mathcal{I}}\cap I_\beta$ is finite [1], and let $\varphi_{\vec{\mathcal{I}},n}$ be a recursive enumeration of $Z_n^{\vec{\mathcal{I}}}$. Then define $A_n=\varphi_{\vec{\mathcal{I}},n}^{-1}[Z_n^{\vec{\mathcal{I}}}\cap A]$. Now define a function $f_{A,\vec{\mathcal{I}}}:[\omega]^\omega\to\omega^\omega$ as follows: if $X\in[\omega]^\omega$ reaps A_n for all n, then define

$$f_{A,\vec{\mathcal{I}}}(X)(n)=\min\{k\in\omega:X\setminus k\subseteq A_n\vee(X\setminus k)\cap A_n=\emptyset\}$$

Otherwise define $f_{A,\vec{\mathcal{I}}}(X)$ to be the identity function. Finally, the value of F in $(A,\vec{\mathcal{I}})$ is given by $F(A,\vec{\mathcal{I}})=(\langle A_n:n\in\omega\rangle,f_{A,\vec{\mathcal{I}}})$. Let $g:\omega_1\to[\omega]^\omega$ be a $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma;\mathfrak{d})$-guessing sequence for F. We can assume that for all α the set A_α in $g(\alpha)=(A_\alpha,h_\alpha)$ is coinfinite. Recursively define an irredundant family as follows:

1) Start with a partition of ω into infinitely many infinite sets $\vec{\mathcal{I}}_\omega=\langle I_n:n\in\omega\rangle$.

2) Suppose we have defined $\vec{\mathcal{I}}_\alpha=\langle I_\beta:\beta<\alpha\rangle$. Now define I_α as follows:

$$I_\alpha=\bigcup_{n\in\omega}B_n^{\vec{\mathcal{I}}_\alpha}\setminus\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha\setminus h_\alpha(n)]$$

Let $\vec{\mathcal{I}}_{\alpha+1}$ be the family $\langle I_\beta:\beta\leq\alpha\rangle$. Finally, let $\mathcal{I}=\langle I_\alpha:\alpha\in\omega_1\rangle$ be the family obtained by the above recursion. Let's see that $\mathcal{I}$ is a witness for $\mathfrak{s}_{mm}$.

Claim 1. $\mathcal{I}$ is an irredundant family. We proceed by induction of $\alpha\in\omega_1$. Clearly $\mathcal{I}_\omega$ is irredundant. Assume $\vec{\mathcal{I}}_\alpha$ is irredundant. Then $\vec{\mathcal{I}}_{\alpha+1}$ is irredundant:

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[1] $Z_n^{\vec{\mathcal{I}}}\subseteq B_n^{\mathcal{I}}$ should be found in a recursive way and should depend only on $\vec{\mathcal{I}}$

a) For all $H \in [\alpha]^{<\omega}$, $I_\alpha \not\subseteq^* \bigcup H$. Let $n \in \omega$ be such that H is contained in $\{e_\alpha(0), \dots, e_\alpha(n)\}$, so $\bigcup H \subseteq \bigcup_{i \leq n} B_i^{\mathcal{I}_\alpha}$. By the definition of I_α, $I_\alpha \setminus \bigcup_{i \leq n} B_i^{\mathcal{I}_\alpha}$ is infinite.

b) For all $\beta \in \alpha \setminus H$, $I_\beta \not\subseteq^* I_\alpha \cup \bigcup H$. Let n be such that $\beta = e_\alpha(n)$. By the choice of $Z_n^{\vec{\mathcal{I}}_\alpha}$, we have that for any $\gamma \in \alpha \setminus \{\beta\}$, $Z_n^{\vec{\mathcal{I}}_\alpha} \cap I_\gamma$ is finite, so in particular, $Z_n^{\vec{\mathcal{I}}_\alpha} \cap \bigcup H$ is finite. Also by the construction of I_α, $B_n^{\vec{\mathcal{I}}_\alpha} \cap I_\alpha \cap \varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)]$ is finite. This both facts together give $\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)] \setminus I_\alpha \cup \bigcup H$ is infinite. Since $\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)] \setminus I_\alpha \cup \bigcup H \subseteq I_\beta \setminus I_\alpha \cup \bigcup H$, we are done.

Claim 2. $\mathcal{I}$ is maximal. Pick any $X \in [\omega]^\omega$. If g guesses $(X, \langle I_\alpha : \alpha \in \omega_1 \rangle)$ in γ, then we have that A_γ σ-reaps $\langle X_n : n \in \omega \rangle$ and h_γ almost dominates the function $l = f_{X, \mathcal{I}_\gamma}(A_\gamma)$. There are two cases:

i) There are infinitely many $n \in \omega$ such that $A_\gamma \subseteq^* X_n$. Pick n such that $l(n) \leq h_\gamma(n)$. Then $A_\gamma \setminus h_\gamma(n) \subseteq X_n$, so $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq X \cap B_n^{\vec{\mathcal{I}}_\gamma}$. Then by the definition of I_γ, $B_n^{\vec{\mathcal{I}}_\gamma} \subseteq I_\gamma \cup \varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq I_\gamma \cup X$, which implies $I_{e_\gamma(n)} \subseteq X \cup I_\gamma \cup \bigcup_{i<n} I_{e_\gamma(n)}$.

ii) For almost all $n \in \omega$ $A_\gamma \subseteq^* \omega \setminus X_n$. Then for almost all n, $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq Z_n^{\vec{\mathcal{I}}_\gamma} \setminus X$, so for almost all n, $X \cap Z_n^{\vec{\mathcal{I}}_\gamma} \subseteq I_\gamma$, and for finitely many n, $A_\gamma \subseteq^* X_n$, so $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq^* Z_n^{\vec{\mathcal{I}}} \cap X \subseteq B_n \cap X$, which implies $B_n \setminus X \subseteq^* B_n \setminus \varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq I_\gamma$. Putting all this together we have that $X \subseteq^* I_\gamma \cup \bigcup_{i \leq k} B_i$, for some $k \in \omega$.

QED

The following result was proved by Hiroaki Minami [10] for Borel invariants, however, the proof for $L(\mathbb{R})$-invariants is the same.

Proposition 10 *Let* $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_1 \right\rangle$ *a finite support iteration of ccc forcings and* $(A, B, \to)$ *be an* $L(\mathbb{R})$ *-invariant with the following property: For all* $\alpha < \omega_1$ *there is* $b \in B \cap V[G_{\alpha+1}]$ *such that* $a \to b$ *for all* $a \in A \cap V[G_\alpha]$. *Then* $\mathbb{P}_{\omega_1} \Vdash$ "$\diamondsuit_{L(\mathbb{R})}(A, B, \to)$ ".

With the previous proposition we can conclude the following:

Corollary 11 *There is a finite support iteration of ccc forcings of length ω_1 such that $\mathbb{P}_{\omega_1} \Vdash$ "$\mathfrak{s}_{mm} = \omega_1$".*

Proof

Define $\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha$ for $\alpha < \omega_1$ as follows. Let $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{M}\left(\dot{\mathcal{U}}_\alpha\right) * \dot{\mathbb{H}}$" where $\dot{\mathcal{U}}_\alpha$ is the name of any ultrafilter, $\mathbb{M}\left(\dot{\mathcal{U}}_\alpha\right)$ is its Mathias forcing and $\mathbb{H}$ is Hechler forcing. Using the previous proposition, it is easy to show that $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ holds in $\mathbb{P}_{\omega_1}$ and then $\mathfrak{s}_{mm}$ is equal to ω_1 in the extension.

QED

In [13] it is shown that for any Borel invariant $(A, B, \to)$, most countable support iterations of proper forcings that force $\langle A, B, \to\rangle \leq \omega_1$ will also force $\diamondsuit(A, B, \to)$. This is also true for $L(\mathbb{R})$-invariants and will be proved in [5].

Proposition 12 *Let $\langle \mathbb{Q}_\alpha \mid \alpha \in \omega_2\rangle$ be a sequence of Borel proper partial orders where each $\mathbb{Q}_\alpha$ is forcing equivalent to $\wp(2)^+ \times \mathbb{Q}_\alpha$ and let $\mathbb{P}_{\omega_2}$ be the countable support iteration of this sequence. If $(A, B, \to)$ is an $L(\mathbb{R})$-invariant and $\mathbb{P}_{\omega_2} \Vdash$ "$\langle A, B, \to\rangle \leq \omega_1$" then $\mathbb{P}_{\omega_2} \Vdash$ "$\diamondsuit_{L(\mathbb{R})}(A, B, \to)$".*

With the previous propositions we can conclude the following,

Corollary 13 *There is a model where $\mathfrak{s}_{mm} < non(\mathcal{M})$, hence $\mathfrak{s}_{mm} < \mathfrak{i}$.*

Proof

Do a countable support iteration of fat tree forcings (see [16], section 4.4.3) over a model of CH. It can be proved that this forcings preserve Ramsey ultrafilters, so in the final model we have $\mathfrak{r}_\sigma = \omega_1$. These forcings are also ω^ω-bounding, so in the generic extension we have $\mathfrak{d} = \omega_1$. This implies that $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ holds in the generic extension, and then $\mathfrak{s}_{mm} = \omega_1$ in this model. Since this forcings add eventually different reals then $non(\mathcal{M}) = \omega_2$ holds. Since $\mathfrak{i} \geq cof(\mathcal{M})$ ([1]), we are done.

QED

By the previous results it might be conjecture that $\mathfrak{s}_{mm} = max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$ (note this equality holds in all the Cohen, random, Hechler, Sacks, Laver, Mathias and Miller models) but we will now show this is not the case. Let κ be a measurable cardinal and $\mathcal{U}$ be a κ complete ultrafilter. Given a partial order $\mathbb{P}$ we denote by $\mathbb{P}^\kappa/\mathcal{U}$ its ultrapower and for every $f : \kappa \to \mathbb{P}$ we denote by $[f]$ its equivalence class mod $\mathcal{U}$. For some background about forcing with ultrapowers see [4]. It follows by the Łoš theorem that if $\mathbb{P}$ is ccc then $\mathbb{P}^\mu/\mathcal{U}$ is also ccc and $\mathbb{P}$ regularly embeds in its ultrapower. The next lemma shows that big irredundant families are destroyed when taking ultrapowers and the proof is very similar to the lemma 4 of [4], so we skip the proof.

Lemma 14 *Let $\mathbb{P}$ be ccc and $\dot{\mathcal{A}}$ be a $\mathbb{P}$-name for an irredundant family of size at least κ. Then $\mathbb{P}^\kappa/\mathcal{U}$ forces $\dot{\mathcal{A}}$ is not maximal.*

With the previous lemma it is possible to show the following,

Proposition 15 *Assume $V \models \mathsf{GCH}$, κ is a measurable cardinal and $\mathcal{U}$ is a κ-complete ultrafilter. Then there is $\mathbb{P}_{\kappa^+\kappa^{++}}$ a ccc partial order such that $\mathbb{P}_{\kappa^+\kappa^{++}} \Vdash$ "$\kappa^+ = \mathfrak{d} = \mathfrak{r}_\sigma = \mathfrak{u} < \mathfrak{s}_{mm} = \mathfrak{c} = \kappa^{++}$".*

Proof

The partial order $\mathbb{P}_{\kappa^+\kappa^{++}}$ is the forcing constructed in [4] theorem 1 that forces $\mathfrak{u} < \mathfrak{a}$. We construct a matrix iteration $\langle \mathbb{P}_{\alpha\beta} \mid \alpha \leq \kappa^+, \beta \leq \kappa^{++} \rangle$ where $\langle \mathbb{P}_{\alpha 0} \mid \alpha \leq \kappa^+ \rangle$ is a finite support iteration of Laver forcings based on some ultrafilters, $\mathbb{P}_{\alpha\beta+1} = \mathbb{P}^\kappa_{\alpha\beta}/\mathcal{U}$ and some " amalgamated limit" is taken at limit stages. We refer to [4] for details. In that paper it is shown that

$$\mathbb{P}_{\kappa^+\kappa^{++}} \Vdash \text{"}\kappa^+ = \mathfrak{d} = \mathfrak{r}_\sigma = \mathfrak{u} < \mathfrak{a} = \mathfrak{c} = \kappa^{++}\text{"}$$

and following the proof of $\mathbb{P}_{\kappa^+\kappa^{++}} \Vdash$ "$\mathfrak{a} = \kappa^{++}$" and using the previous lemma it is possible to show that $\mathbb{P}_{\kappa^+\kappa^{++}} \Vdash$ "$\mathfrak{s}_{mm} = \kappa^{++}$".
QED

However we do not know the answer to the following questions,

Question 16 *Is $\mathfrak{u} \leq \mathfrak{s}_{mm}$?*

Question 17 *Is $\mathfrak{s}_{mm} \leq \mathfrak{i}$?*

We would like to remark that in [14] Shelah built a model of $\mathfrak{i} < \mathfrak{u}$ so in that model one of the questions has a negative answer, but we do not know which one.

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