Uniquely Universal Sets

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Abstract[1]

We say that $X \times Y$ satisfies the Uniquely Universal property (UU) iff there exists an open set $U \subseteq X \times Y$ such that for every open set $W \subseteq Y$ there is a unique cross section of U with $U_x = W$. Michael Hrušák raised the question of when does $X \times Y$ satisfy UU and noted that if Y is compact, then X must have an isolated point. We consider the problem when the parameter space X is either the Cantor space 2^ω or the Baire space ω^ω. We prove the following:

1. If Y is a locally compact zero dimensional Polish space which is not compact, then $2^\omega \times Y$ has UU.

2. If Y is Polish, then $\omega^\omega \times Y$ has UU iff Y is not compact.

3. If Y is a σ-compact subset of a Polish space which is not compact, then $\omega^\omega \times Y$ has UU.

For any space Y with a countable basis there exists an open set $U \subseteq 2^\omega \times Y$ which is universal for open subsets of Y, i.e., $W \subseteq Y$ is open iff there exists $x \in 2^\omega$ with

$$U_x =^{\text{def}} \{y \in Y \ : \ (x, y) \in U\} = W.$$

To see this let $\{B_n \ : \ n < \omega\}$ be a basis for Y. Define

$$(x, y) \in U \text{ iff } \exists n \ (x(n) = 1 \text{ and } y \in B_n).$$

More generally if X contains a homeomorphic copy of 2^ω then $X \times Y$ will have a universal open set.

In 1995 Michael Hrušák mentioned the following problem to us. Most of the results in this note were proved in June and July of 2001.

[1]MSC2010: 03E15

Keywords: Universal set, Unique parameterization, Polish spaces, Cantor space, Baire space.

Last revised January 12, 2012.

Hrušák 's problem.

> Let X, Y be topological spaces, call X the parameter space, and Y the base space. When does there exists $U \subseteq X \times Y$ which is uniquely universal for the open subsets of Y? This means the U is open and for every open set $W \subseteq Y$ there is a unique $x \in X$ such that $U_x = W$.

Let us say that $X \times Y$ satisfies UU (uniquely universal property) if there exists such an open set $U \subseteq X \times Y$ which uniquely parameterizes the open subsets of Y. Note that the complement of U is a closed set which uniquely parameterizes the closed subsets of Y.

Proposition 1 *(Hrušák) $2^\omega \times 2^\omega$ does not satisfy UU.*

proof:

The problem is the empty set. Suppose U is uniquely universal for the closed subsets of 2^ω. Then there is an x_0 such that $U_{x_0} = \emptyset$ but all other cross sections are nonempty. Take $x_n \to x_0$ but distinct from it. Since all other cross sections are non-empty we can choose $y_n \in U_{x_n}$. But then y_n has a convergent subsequence, say to y_0, but then $y_0 \in U_{x_0}$.
QED

More generally:

Proposition 2 *(Hrušák) Suppose $X \times Y$ has UU and Y is compact. Then X must have an isolated point.*

proof:

Suppose $U \subseteq X \times Y$ witnesses UU for closed subsets of Y and $U_{x_0} = \emptyset$. For every $y \in Y$ there exists $U_y \times V_y$ open containing (x_0, y) and missing U. By compactness of Y finitely many V_y cover Y. The intersection of the corresponding U_y isolates x_0.
QED

Hence, for example, $2^\omega \times (\omega + 1)$, $\omega^\omega \times (\omega + 1)$, and $\omega^\omega \times 2^\omega$ cannot have UU.

Proposition 3 *Let $2^\omega \oplus 1$ be obtained by attaching an isolated point to 2^ω. Then $(2^\omega \oplus 1) \times 2^\omega$ has UU.*

proof:

Define $T \subseteq 2^{<\omega}$ to be a nice tree iff

(a) $s \subseteq t \in T$ implies $s \in T$ and

(b) if $s \in T$, then either $s\hat{}\langle 0\rangle$ or $s\hat{}\langle 1\rangle$ in T.

Let $\mathrm{NT} \subseteq \mathcal{P}(2^{<\omega})$ be the set of nice trees. Define the universal set U by

$$U = \{(T, x) \in \mathrm{NT} \times 2^\omega \ : \ \forall n \ \ x\restriction n \in T\}.$$

Note that the empty tree T is nice and parameterizes the empty set. Also NT is a closed subset of $\mathcal{P}(2^{<\omega})$ with exactly one isolated point (the empty tree), and hence it is homeomorphic to $2^\omega \oplus 1$.
QED

Question 4 *Does $(2^\omega \oplus 1) \times [0,1]$ have UU?*

Remark 5 *$2^\omega \times \omega$ has the UU property. Just let $(x, n) \in U$ iff $x(n) = 1$.*

Question 6 *Does either $\mathbb{R} \times \omega$ or $[0,1] \times \omega$ have UU? Or more generally, is there any example of UU for a connected parameter space?*

Recall that a topological space is Polish iff it is completely metrizable and has a countable dense subset. A set is G_δ iff it is the countable intersection of open sets. The countable product of Polish spaces is Polish. A G_δ subset of a Polish space is Polish (Alexandrov). A space is zero-dimensional iff it has a basis of clopen sets. All compact zero-dimensional Polish spaces without isolated points are homeomorphic to 2^ω (Brouwer). A zero-dimensional Polish space is homeomorphic to ω^ω iff compact subsets have no interior (Alexandrov-Urysohn). For proofs of these facts see Kechris [5] p.13-39.

Proposition 7 *Suppose Y is a zero dimensional Polish space. If Y is locally compact but not compact, then $2^\omega \times Y$ has UU. So, for example, $2^\omega \times (\omega \times 2^\omega)$ has UU.*

proof:

Let $\mathcal{B}$ be a countable base for Y consisting of clopen compact sets. Define $G \subseteq \mathcal{B}$ is good iff

$$G = \{b \in \mathcal{B} \ : \ b \subseteq \bigcup G\}$$

Let $\mathcal{G} \subseteq \mathcal{P}(\mathcal{B})$ be the family of good subsets of $\mathcal{B}$. We give the $\mathcal{P}(\mathcal{B})$ the topology from identifying it with $2^{\mathcal{B}}$. Since $\mathcal{B}$ is an infinite countable set $\mathcal{P}(\mathcal{B})$ is homeomorphic to 2^ω. A sequence G_n for $n < \omega$ converges to G iff for each finite $F \subseteq \mathcal{B}$ we have that $G_n \cap F = G \cap F$ for all but finitely many n. Hence $\mathcal{G}$ is a closed subset of $\mathcal{P}(\mathcal{B})$ since by compactness $b \subseteq \bigcup G$ iff $b \subseteq \bigcup F$ for some finite $F \subseteq G$.

There is a one-to-one correspondence between good families and open subsets of Y: Given any open set $U \subseteq Y$ define

$$G_U = \{b \in \mathcal{B} \ : \ b \subseteq U\}$$

and given any good G define $U_G = \bigcup G$. (Note that the empty set is good.)

We claim that no $G_0 \in \mathcal{G}$ is an isolated point. Suppose for contradiction it is. Then there must be a basic open set N with $\{G_0\} = \mathcal{G} \cap N$. A basic neighborhood has the following form

$$N = N(F_0, F_1) = \{G \subseteq \mathcal{B} \ : \ F_0 \subseteq G \text{ and } F_1 \cap G = \emptyset\}$$

where $F_0, F_1 \subseteq \mathcal{B}$ are finite.

For each $b \in F_1$ since G_0 is good, b is not a subset of $\bigcup G_0$, and since $\bigcup F_0 \subseteq \bigcup G_0$, we can choose a point $z_b \in b \setminus \bigcup F_0$. Since Y is not compact, $Y \setminus (\bigcup (F_0 \cup F_1))$ is nonempty. Fix $z \in Y \setminus (\bigcup (F_0 \cup F_1))$.

Now let $U_1 = Y \setminus \{z_b \ : \ b \in F_1\}$ and let $U_2 = U_1 \setminus \{z\}$. Then G_{U_1}, G_{U_2} are distinct elements of $N \cap \mathcal{G}$.

Hence $\mathcal{G}$ is a compact zero-dimensional metric space without isolated points and therefore it is homeomorphic to 2^ω.

To get a uniquely universal open set $U \subseteq \mathcal{G} \times Y$ define:

$$(G, y) \in U \text{ iff } \exists b \in G \ \ y \in b.$$

QED

Example 26 is a countable Polish space Z such that $2^\omega \times Z$ has UU, but Z is not locally compact.

Lemma 8 *Suppose $f : X \to Y$ is a continuous bijection and $Y \times Z$ has UU. Then $X \times Z$ has UU.*

proof:

Given $V \subseteq Y \times Z$ witnessing UU, let

$$U = \{(x, y) \ : \ (f(x), y) \in V\}.$$

QED

Many uncountable standard Borel sets[2] are the bijective continuous image of the Baire space ω^ω. According to the footnote on page 447 of Kuratowski [6] Sierpinski proved in a 1929 paper that any standard Borel set in which every point is a condensation point is the bijective continuous image of ω^ω. We weren't able to find Sierpinski's paper but we give a proof of his result in Lemma 21.

We first need a special case for which we give a proof.

Lemma 9 *There is a continuous bijection $f : \omega^\omega \to 2^\omega$.*

proof:

Let $\pi : \omega \to \omega + 1$ be a bijection. It is automatically continuous. It induces a continuous bijection $\pi : \omega^\omega \to (\omega+1)^\omega$. But $(\omega+1)^\omega$ is a compact Polish space without isolated points, hence it is homeomorphic to 2^ω.
QED

Remark. If $C \subseteq 2^\omega \times \omega^\omega$ is the graph of f^{-1}, then C is a closed set which uniquely parameterizes the family of singletons of ω^ω.

Corollary 10 *If $2^\omega \times Y$ has UU, then $\omega^\omega \times Y$ has UU.*

Question 11 *Is the converse of Corollary 10 false? That is: Does there exist Y such that $\omega^\omega \times Y$ has UU but $2^\omega \times Y$ does not have UU?*

Lemma 12 *Suppose X is a zero-dimensional Polish space without isolated points. Then there exists a continuous bijection $f : \omega^\omega \to X$.*

[2] A standard Borel set is a Borel subset of a Polish space.

proof:

Construct a subtree $T \subseteq \omega^{<\omega}$ and $(C_s \subseteq X \ : \ s \in T)$ nonempty clopen sets such that:

1. $C_{\langle\rangle} = X$,
2. if $s \in T$ is a terminal node, then C_s is compact, and
3. if $s \in T$ is not terminal, then $s\hat{}\langle n\rangle \in T$ for every $n \in \omega$ and C_s is partitioned by $(C_{s\hat{}\langle n\rangle} \ : \ n < \omega)$ into nonempty clopen sets each of diameter[3] less than $\frac{1}{|s|+1}$.

For each terminal node $s \in T$ choose a continuous bijection $f_s : [s] \to C_s$ given by Lemma 9. Define $f : \omega^\omega \to X$ by $f(x) = f_{x\restriction n}(x)$ if there exists n such that $x\restriction n$ is a terminal node of T and otherwise determine $f(x)$ by the formula:

$$\{f(x)\} = \bigcap_{n<\omega} C_{x\restriction n}$$

Checking that f is a continuous bijection is left to the reader.
QED

Remark. An easy modification of the above argument shows that any zero-dimensional Polish space is homeomorphic to a closed subspace of ω^ω. It also gives the classical result that if no clopen sets are compact, then X is homeomorphic to ω^ω. A different proof of Lemma 12 is given in Moschovakis [8] p. 12.

Definition 13 *We use $cl(X)$ to denote the closure of X.*

Proposition 14 *Suppose Y is Polish but not compact. Then $\omega^\omega \times Y$ has the UU. So for example, $\omega^\omega \times \omega^\omega$ and $\omega^\omega \times \mathbb{R}$ both have UU.*

proof:

We assume that the metric on Y is complete and bounded. Let $\mathcal{B}$ be a countable basis for Y of nonempty open sets which has the property that no finite subset of $\mathcal{B}$ covers Y.

For $s, t \in \mathcal{B}$ define $t \lhd s$ iff $cl(t) \subseteq s$ and $\text{diam}(t) \leq \frac{1}{2}\,\text{diam}(s)$.

[3]This is with respect to a fixed complete metric on X.

Lemma 15 *Suppose $G \subseteq \mathcal{B}$ has the following properties:*

(1) for all $t, s \in \mathcal{B}$ if $t \subseteq s \in G$, then $t \in G$ and

(2) $\forall s \in \mathcal{B}$ if $(\forall t \lhd s \;\; t \in G)$, then $s \in G$,

then for any $s \in \mathcal{B}$ if $s \subseteq \bigcup G$, then $s \in G$.

proof:

Suppose (1) and (2) hold but for some $s \subseteq \bigcup G$ we have $s \notin G$.

Note that there cannot be a sequence $(s_n : n \in \omega)$ starting with $s_0 = s$, and with $s_{n+1} \lhd s_n$ and $s_n \notin G$ for each n. This is because if $\{x\} = \bigcap_{n \in \omega} s_n$, then $x \in s \subseteq \bigcup G$ and so for some $t \in G$ we have $x \in t$. But then for some sufficiently large n we have that $s_n \subseteq t$ putting $s_n \in G$ by (1).

Hence there must be some $t \unlhd s$ with $t \notin G$ but for all $r \lhd t$ we have $r \in G$. This is a contradiction to (2).

QED

Let $\mathcal{G} \subseteq \mathcal{P}(\mathcal{B})$ be the set of all G which satisfy the hypothesis of Lemma 15. Then we have a G_δ subset $\mathcal{G}$ of $2^{\mathcal{B}}$, which uniquely parameterizes the open sets. Hence the set U witnesses the unique universal property:

$$U = \{(G, x) \in \mathcal{G} \times Y \;:\; x \in \bigcup G\}.$$

To finish the proof it is enough to see that no point in $\mathcal{G}$ is isolated. If $\mathcal{G}$ has an isolated point, there must be $s_1, \ldots, s_n$ and $t_1, \ldots, t_m$ from $\mathcal{B}$ such that

$$N = \{G \in \mathcal{G} \;:\; s_1 \in G, \ldots, s_n \in G,\; t_1 \notin G, \ldots, t_m \notin G\}$$

is a singleton. Let $W = s_1 \cup \cdots \cup s_n$. Since N contains the point

$$G = \{s \in \mathcal{B} \;:\; s \subseteq W\}$$

it must be that $N = \{G\}$. For each j choose $x_j \in t_j \setminus W$. Since the union of the s_i and t_j does not cover Y, we can choose

$$z \in Y \setminus (s_1 \cup \cdots \cup s_n \cup t_1 \cup \cdots \cup t_m).$$

Take $r \in \mathcal{B}$ with $z \in r$ but $x_j \notin r$ for each $j = 1, \ldots, m$. Let

$$G' = \{s \in \mathcal{B} \;:\; s \subseteq r \cup W\}.$$

Then $G' \in N$ but $G' \neq G$ which shows that N is not a singleton.

It follows from Lemma 8 and 12 that $\omega^\omega \times Y$ has the UU.
QED

Proposition 2 and 14 show that:

Corollary 16 *For Y Polish:*

$$\omega^\omega \times Y \textit{ has UU iff } Y \textit{ is not compact.}$$

Question 17 *Does $2^\omega \times \omega^\omega$ have UU?*

Question 18 *Does $2^\omega \times \mathbb{R}$ have UU?*

Our next result follows from Proposition 22 but has a simpler proof so we give it first.

Proposition 19 *If X is a countable metric space which is not compact, then $\omega^\omega \times X$ has UU. So, for example, $\omega^\omega \times \mathbb{Q}$ has UU.*

proof:

We produce a uniquely universal set for the open subsets of X.

First note that there exists a countable basis $\mathcal{B}$ for X with the property that it is closed under finite unions and $X \setminus B$ is infinite for every $B \in \mathcal{B}$. To see this fix $\{x_n : n < \omega\} \subseteq X$ an infinite set without a limit point, i.e., an infinite closed discrete set. Given a countable basis $\mathcal{B}$ replace it with finite unions of sets from

$$\{B \setminus \{x_m : m > n\} \ : \ n \in \omega \text{ and } B \in \mathcal{B}\}.$$

We may assume also that $\mathcal{B}$ includes the empty set.

Next let

$$\mathbb{P} = \{(B, F) : B \in \mathcal{B}, \ \ F \in [X]^{<\omega}, \text{ and } B \cap F = \emptyset\}.$$

Then $\mathbb{P}$ is a partial order determined by $p \leq q$ iff $B_q \subseteq B_p$ and $F_q \subseteq F_p$. For $p \in \mathbb{P}$ we write $p = (B_p, F_p)$. For $p, q \in \mathbb{P}$ we write $p \perp q$ to stand for p and q are incompatible, i.e., there does not exist $r \in \mathbb{P}$ with $r \leq p$ and $r \leq q$.

We will code open subsets of X by good filters on $\mathbb{P}$. Define the family $\mathcal{G}$ of good filters on $\mathbb{P}$ to be the set of all $G \subseteq \mathbb{P}$ such that

1. $p \leq q$ and $p \in G$ implies $q \in G$,
2. $\forall p, q \in G$ exists $r \in G$ with $r \leq p$ and $r \leq q$,
3. $\forall x \in X\ \exists p \in G\ \ x \in B_p \cup F_p$, and
4. $\forall p \in \mathbb{P}$ either $p \in G$ or $\exists q \in G\ \ p \perp q$.

Since the poset $\mathbb{P}$ is countable we can identify $\mathcal{P}(\mathbb{P})$ with $\mathcal{P}(\omega)$ and hence 2^ω. We give $\mathcal{G} \subseteq \mathcal{P}(\mathbb{P})$ the subspace topology. Note that $\mathcal{G}$ is G_δ in this topology. Note also that the sets

$$[p] = \{G \in \mathcal{G}\ :\ p \in G\}$$

form a basis for $\mathcal{G}$ (use conditions (2) and (4) to deal with finitely many p_i in G and finitely many q_j not in G).

Note that since $X \setminus B_p$ is always infinite, for any $p \in \mathbb{P}$ there exists $r, q \leq p$ with $r \perp q$. Namely, for some $x \in X \setminus (B_p \cup F_p)$ put x into $B_r \cap F_q$. It follows that no element of $\mathcal{G}$ is isolated. So $\mathcal{G}$ is a zero-dimensional Polish space without isolated points. Hence by Lemma 12 there is a continuous bijection $f : \omega^\omega \to \mathcal{G}$.

For $G \in \mathcal{G}$, let

$$U_G = \bigcup\{B_p\ :\ p \in G\}.$$

For any $U \subseteq X$ open, define

$$G_U = \{p \in \mathbb{P}\ :\ B_p \subseteq U \text{ and } F_p \cap U = \emptyset\}.$$

The maps $G \to U_G$ and $U \to G_U$ show that there is a one-to-one correspondence between $\mathcal{G}$ and the open subsets of X.

Finally define $\mathcal{U} \subseteq \mathcal{G} \times X$ by

$$(G, x) \in \mathcal{U} \text{ iff } \exists p \in G\ \ x \in B_p.$$

This witnesses the UU property for $\mathcal{G} \times X$ and so by Lemma 8, we have UU for $\omega^\omega \times X$.
QED

Question 20 *Does $2^\omega \times \mathbb{Q}$ have UU?*

Our next result Proposition 22 implies Proposition 19 but needs the following Lemma:

Lemma 21 *(Sierpinski) Suppose B is a Borel set in a Polish space for which every point is a condensation point. Then there exists a continuous bijection from ω^ω to B.*

proof:

We use that every Borel set is the bijective image of a closed subset of ω^ω. This is due to Lusin-Souslin see Kechris [5] p.83 or Kuratowski-Mostowski [7] p.426.

Using the fact that every uncountable Borel set contains a perfect subset it is easy to construct K_n for $n < \omega$ satisfying:

1. $K_n \subseteq B$ are pairwise disjoint,
2. K_n are homeomorphic copies of 2^ω which are nowhere dense in B, and
3. every nonempty open subset of B contains infinitely many K_n.

Let $B_0 = B \setminus \bigcup_{n<\omega} K_n$. We may assume B_0 is nonempty, otherwise just split K_0 into two pieces. Since it is a Borel set, there exists $C \subseteq \omega^\omega$ closed and a continuous bijection $f : C \to B_0$. Define

$$\Gamma = \{s \in \omega^{<\omega} \;:\; [s] \cap C = \emptyset \text{ and } [s^*] \cap C \neq \emptyset\}$$

where s^* is the unique $t \subseteq s$ with $|t| = |s| - 1$. Without loss we may assume that C is nowhere dense and hence Γ infinite. Let $\Gamma = \{s_n \;:\; n < \omega\}$ be a one-one enumeration. Note that $\{C\} \cup \{[s_n] \;:\; n < \omega\}$ is a partition of ω^ω.

Inductively choose $l_n > l_{n-1}$ with K_{l_n} a subset of the ball of radius $\frac{1}{n+1}$ around $f(x_n)$ for some $x_n \in C \cap [s_n^*]$. For each $n < \omega$ let $f_n : [s_n] \to K_{l_n}$ be a continuous bijection.

Then $g = f \cup \bigcup_{n<\omega} f_n$ is a continuous bijection from ω^ω to $B_0 \cup \bigcup_{n<\omega} K_{l_n}$.

To see that it is continuous suppose for contradiction that $u_n \to u$ as $n \to \infty$ and $|g(u_n) - g(u)| > \epsilon > 0$ all n. Since C is closed if infinitely many u_n are in C, so is u and we contradict continuity of f. If $u \in [s_n]$, then we contradict the continuity of f_n, So, we may assume that all u_n are not in C but u is in

C. By the continuity of f we may find $s \subseteq u$ with $f([s] \cap C)$ inside a ball of radius $\frac{\epsilon}{3}$ around $f(u)$. Find n with $\frac{1}{n+1} < \frac{\epsilon}{3}$ for which there is m such that $u_m \in [s_n]$ and $s_n \supseteq s$. But then $g(u_m) = f_n(u_m) \in K_{l_n}$ and K_{l_n} is in a ball of radius $\frac{1}{n+1}$ around some $f(x_n)$ with $x_n \in [s_n^*] \cap C$. This is a contradiction:

$$d(g(u), g(u_m)) \leq d(f(u), f(x_n)) + d(f(x_n), f_n(u_m)) \leq \frac{2}{3}\epsilon.$$

Next let $I = \omega \setminus \{l_n \,:\, n < \omega\}$. Then there exists continuous bijection

$$h : I \times \omega^\omega \to \bigcup_{i \in I} K_i.$$

Finally $g \cup h$ is a continuous bijection from $\omega^\omega \oplus (I \times \omega^\omega)$ to $B_0 \cup \bigcup_{n<\omega} K_n = B$. Since $\omega^\omega \oplus (I \times \omega^\omega)$ is a homeomorphic copy of ω^ω we are done.
QED

Proposition 22 *$\omega^\omega \times Y$ has UU for any σ-compact but not compact subspace Y of a Polish space. So for example, $\omega^\omega \times (\mathbb{Q} \times 2^\omega)$ has UU.*

proof:

Let $Y = \bigcup_{n<\omega} K_n$ where each K_n is compact. Since Y is not compact it contains an infinite closed discrete set D. Choose a countable basis $\mathcal{B}$ for Y such that for any $b \in \mathcal{B}$ the closure of b contains at most finitely many points of D. Define $G \subseteq \mathcal{B}$ to be good iff for every $b \in \mathcal{B}$ if $cl(b) \subseteq \bigcup G$ then $b \in G$. Let $\mathcal{G} \subseteq \mathcal{P}(\mathcal{B})$ be the family of good sets.

Note that $\mathcal{G}$ is a $\mathbf{\Pi}^0_3$ set:

$$G \in \mathcal{G} \text{ iff } \forall b \in \mathcal{B} \;\; (\forall n \;\; cl(b) \cap K_n \subseteq \bigcup G) \to b \in G$$

Note that $cl(b) \cap K_n \subseteq \bigcup G$ iff there is a finite $F \subseteq G$ with $cl(b) \cap K_n \subseteq \bigcup F$.

To finish the proof it is necessary to see that basic open sets in $\mathcal{G}$ are uncountable. Given $b_i, c_j \in \mathcal{B}$ for $i < n$ and $j < m$ suppose that

$$N = \{G \in \mathcal{G} \,:\, b_0 \in G, \ldots, b_{n-1} \in G, \;\; c_0 \notin G, \ldots, c_{m-1} \notin G\}$$

is nonempty. Since it is nonempty we can choose points

$$u_j \in cl(c_j) \setminus \bigcup_{i<n} cl(b_i)$$

for $j < m$. Note that the set

$$Z = D \setminus cl(\bigcup_{i<n, j<m} b_i \cup c_j)$$

is an infinite discrete closed set. But then given any $Q \subseteq Z$ we can find an open set U_Q with $\bigcup_{i<n} cl(b_i) \subseteq U_Q$, $u_j \notin U_Q$ for $j < m$, and $U_Q \cap Z = Q$. Let

$$G_Q = \{b \in \mathcal{B} \ : \ cl(b) \subseteq U_Q\}$$

Since each $G_Q \in N$ we have that N is uncountable. By Lemma 21 and 8, we are done.
QED

Question 23 *For what Borel spaces Y does $\omega^\omega \times Y$ have UU?*
For example, does $\omega^\omega \times \mathbb{Q}^\omega$ have UU?

Proposition 24 *For every Y a $\mathbf{\Sigma^1_1}$ set there exists a $\mathbf{\Sigma^1_1}$ set X such that $X \times Y$ has UU.*

proof:
Suppose $Y \subseteq Z$ where Z is Polish and $\mathcal{B}$ is a countable base for Z. Define $\mathcal{G} \subseteq \mathcal{P}(\mathcal{B})$ by

$$G \in \mathcal{G} \text{ iff } \forall b \in \mathcal{B} \ \ [(b \cap Y \subseteq \bigcup G) \to b \in G]$$

Note that $b \cap Y \subseteq \bigcup G$ is $\mathbf{\Pi^1_1}$ and so $\mathcal{G}$ is $\mathbf{\Sigma^1_1}$.
QED

We use the next Lemma for Example 26.

Lemma 25 *For any space Y*
$(\omega \times 2^\omega) \times Y$ *has UU* *iff* $2^\omega \times Y$ *has UU.*

proof:
Suppose $C \subseteq (\omega \times 2^\omega) \times Y$ is a closed set uniquely universal for the closed subsets of Y.

Since the whole space Y occurs as a cross section of C without loss we may assume that $Y = C_{(0,\vec{0})}$ where $\vec{0}$ is the constant zero function.

For each $n > 0$ let

$$K_n = \{(0^n {}^\frown \langle \star, x_0, x_1, \ldots \rangle, y) \in 2^\omega \times Y \ : \ ((n,x),y) \in C\}$$

By $0^n {}^\frown \langle \star, x_0, x_1, \ldots \rangle$ we mean a sequence of n zeros followed by the special symbol $\star$ and then the (binary) digits of x. Note that the K_n converge to $\vec{0}$. Let

$$K_0 = \{(x,y) \ : \ ((0,x),y) \in C\}$$

Let $B = \bigcup_{n<\omega} K_n \subseteq S \times Y$ where

$$S = 2^\omega \cup \bigcup_{n>0} \{0^n {}^\frown \langle \star, x_0, x_1, \ldots \rangle \ : \ x \in 2^\omega\}.$$

Note that S is homeomorphic to 2^ω and there is a one-to-one correspondence between the cross sections of B and cross sections of C. Note that B is closed in $S \times Y$: If $(x_n, y_n) \in B$ for $n < \omega$ is a sequence converging to $(x,y) \in S \times Y$ and x is not the zero vector it is easy to see that $(x,y) \in B$. On the other hand if x is the zero vector, then since $B_{\vec{0}} = Y$, it is automatically true that $(x,y) \in B$.

Hence UU holds for $2^\omega \times Y$.

For the opposite direction just note that $(\omega+1) \times 2^\omega$ is homeomorphic to 2^ω and there is a continuous bijection from $\omega \times 2^\omega$ onto $(\omega+1) \times 2^\omega$. QED

Next we describe our counterexample to a converse of Proposition 7. Let $Z = (\omega \times \omega) \cup \{\infty\}$. Let each $D_n = \{n\} \times \omega$ be an infinite closed discrete set and let the sequence of D_n "converge" to ∞, i.e., each neighborhood of ∞ contains all but finitely many D_n. Equivalently Z is homeomorphic to:

$$Z' = \{x \in \omega^\omega \ : \ |\{n \ : \ x(n) \neq 0\}| \leq 1\}.$$

The point ∞ is the constant zero map, while D_n are the points in Z' with $x(n) \neq 0$. Note that Z' is a closed subset of ω^ω, hence it is Polish. This seems to be the simplest nonlocally compact Polish space.

Example 26 *Z is a countable nonlocally compact Polish space such that $2^\omega \times Z$ has the UU.*

proof:

$Z = \bigcup_{n<\omega} D_n \cup \{\infty\}$. Note that $X \subseteq Z$ is closed iff $\infty \in X$ or $X \subseteq \bigcup_{i\leq k} D_i$ for some $k < \omega$. By Lemma 25 it is enough to see that $(\omega \times 2^\omega) \times Z$ has the UU.

Let $P_n = \{n\} \times 2^\omega$.

Use P_0 to uniquely parameterize all subsets of Z which contain the point at infinity, see Remark 5.

Use P_1 to uniquely parameterize all $X \subseteq D_0$, including the empty set.

For $n = 1 + 2^{k-1}(2l-1)$ with $k, l > 0$ use P_n to uniquely parameterize all $X \subseteq \bigcup_{i\leq k} D_i$ such that D_k meets X and the minimal element of $D_k \cap X$ is the l-th element of D_k.

QED

Our next two results have to do with Question 17.

Proposition 27 *Existence of UU for $2^\omega \times \omega^\omega$ is equivalent to:*

There exist a $\mathcal{C} \subseteq \mathcal{P}(\omega^{<\omega})$ homeomorphic to 2^ω such that every $T \in \mathcal{C}$ is a subtree of $\omega^{<\omega}$ (possibly with terminal nodes) and such that for every closed $C \subseteq \omega^\omega$ there exists a unique $T \in \mathcal{C}$ with $C = [T]$.

proof:

Given $C \subseteq 2^\omega \times \omega^\omega$ witnessing UU for closed subsets of ω^ω. Let

$$[T] = \{(s,t) : ([s] \times [t]) \cap C \neq \emptyset\}.$$

Define $f : 2^\omega \to \mathcal{P}(\omega^{<\omega}) = 2^{\omega^{<\omega}}$ by $f(x)(s) = 1$ iff $(x{\restriction}n, s) \in T$ where $n = |s|$. Then f is continuous, since $f(x)(s)$ depends only on $x{\restriction}n$ where $n = |s|$.

If $T_x = f(x)$, then $[T_x] = C_x$. Hence f is one-to-one, so its image $\mathcal{C}$ is as described.

QED

Proposition 28 *Suppose $\omega^\omega \times Y$ has UU where Y is any topological space in which open sets are F_σ. Then there exists $U \subseteq 2^\omega \times Y$ an F_σ set such that all cross sections U_x are open and for every open $W \subseteq Y$ there is a unique $x \in 2^\omega$ with $U_x = W$.*

proof:

Let $\omega \oplus 1$ denote the discrete space with one isolated point adjoined and let $\omega+1$ denote the compact space consisting of a single convergent sequence. Then $(\omega \oplus 1)^\omega$ is homeomorphic to ω^ω and $(\omega+1)^\omega$ is homeomorphic to 2^ω.

Assume that $U \subseteq (\omega \oplus 1)^\omega \times Y$ is an open set witnessing UU. Then U is an F_σ set in $(\omega+1)^\omega \times Y$.

To see this note that a basic clopen set in $(\omega \oplus 1)^\omega$ could be defined by some $s \in (\omega \oplus 1)^{<\omega}$ by

$$[s] = \{x \in (\omega \oplus 1)^\omega \ : \ s \subseteq x\}.$$

But it is easy to check that $[s] \subseteq (\omega+1)^\omega$ is closed in the topology of $(\omega+1)^\omega$. Since U is open in $(\omega \oplus 1)^\omega \times Y$ there exists s_n and open sets $W_n \subseteq Y$ such that

$$U = \bigcup_{n<\omega} [s_n] \times W_n$$

Hence U is the countable union of F_σ sets and so it is F_σ in $(\omega+1)^{<\omega}$.
QED

Here $\oplus$ refers to the topological sum of disjoint copies or equivalently a clopen separated union.

Proposition 29 *Suppose $X_i \times Y_i$ has UU for $i \in I$. Then*

$$(\prod_{i \in I} X_i) \times (\bigoplus_{i \in I} Y_i) \textit{ has UU.}$$

So, for example, if $2^\omega \times Y$ has UU, then $2^\omega \times (\omega \times Y)$ has UU.

proof:

Define

$$((x_i)_{i \in I}, y) \in U \text{ iff } \exists i \in I \ \ (x_i, y) \in U_i$$

where the $U_i \subseteq X_i \times Y_i$ witness UU.
QED

Except for Proposition 2 we have given no negative results. The following two propositions are the best we could do in that direction.

Proposition 30 *Suppose $U \subseteq X \times Y$ is an open set universal for the open subsets of Y. If X is second countable, then so is Y.*

proof:

U is the union of open rectangles of the form $B \times C$ with B open in X and C open in Y. Clearly we may assume that B is from a fixed countable basis for X. Since $\bigcup_i B \times C_i = B \times \bigcup_i C_i$ we may write U as a countable union:

$$U = \bigcup_{n<\omega} B_n \times C_n$$

where the B_n are basic open sets in X and the C_n are open subsets of Y. But this implies that $\{C_n : n < \omega\}$ is a basis for Y since for each $x \in X$

$$U_x = \bigcup \{C_n : x \in B_n\}$$

QED

Proposition 31 *There exists a partition $X \cup Y = 2^\omega$ into Bernstein sets X and Y such that for every Polish space Z neither $Z \times X$ nor $Z \times Y$ has UU.*

proof:

Note that up to homeomorphism there are only continuum many Polish spaces. If there is a UU set for $Z \times X$, then there is an open $U \subseteq Z \times 2^\omega$ such that $U \cap (Z \times X)$ is UU. Note that the cross sections of U must be distinct open subsets of 2^ω. Hence it suffices to prove the following:

Given $\mathcal{U}_\alpha$ for $\alpha < \mathfrak{c}$ such that each $\mathcal{U}_\alpha$ is a family of open subsets of 2^ω either

(a) there exists distinct $U, V \in \mathcal{U}_\alpha$ with $U \cap X = V \cap X$ or

(b) there exists $U \subseteq 2^\omega$ open such that $U \cap X \neq V \cap X$ for all $V \in \mathcal{U}_\alpha$.

And the same for Y in place of X.

Let P_α for $\alpha < \mathfrak{c}$ list all perfect subsets of 2^ω and let $\{z_\alpha : \alpha < \mathfrak{c}\} = 2^\omega$. Construct $X_\alpha, Y_\alpha \subseteq 2^\omega$ with

1. $X_\alpha \cap Y_\alpha = \emptyset$

2. $\alpha < \beta$ implies $X_\alpha \subseteq X_\beta$ and $Y_\alpha \subseteq Y_\beta$

3. $|X_\alpha \cup Y_\alpha| = |\alpha| + \omega$

4. If there exists distinct $U, V \in \mathcal{U}_\alpha$ such that $U \Delta V$ is a countable set disjoint from X_α, then there exists such a pair with $U \Delta V \subseteq Y_{\alpha+1}$

5. If there exists distinct $U, V \in \mathcal{U}_\alpha$ such that $U \Delta V$ is a countable set disjoint from $Y_{\alpha+1}$, then there exists such a pair with $U \Delta V \subseteq X_{\alpha+1}$

6. P_α meets both $X_{\alpha+1}$ and $Y_{\alpha+1}$

7. $z_\alpha \in (X_{\alpha+1} \cup Y_{\alpha+1})$

First we do (4) then (5) and then take care of (6) and (7).

Let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$ and $Y = \bigcup_{\alpha<\mathfrak{c}} Y_\alpha$.

Fix α and let us verify (a) or (b) holds. Take any point $p \in X \setminus X_{\alpha+1}$. If (b) fails there must be $U, V \in \mathcal{U}_\alpha$ with $X = X \cap U$ and $X \setminus \{p\} = X \cap V$. Then $(U \Delta V) \cap X_{\alpha+1} = \emptyset$. Since X is Bernstein and $(U \Delta V) \cap X$ has only one point in it, it must be that $U \Delta V$ is countable. Then by our construction we have chosen distinct $U, V \in \mathcal{U}_\alpha$ with $U \Delta V \subseteq Y$ therefore $U \cap X = V \cap X$, so (a) holds.

A similar argument goes through for Y.

QED

Finally, and conveniently close to the bibliography, we note some papers in the literature which are related to the property UU. Friedberg [3] proved that there is one-to-one recursively enumerable listing of the recursively enumerable sets. This is the same as saying that there is a (light-face) Σ^0_1 subset $U \subseteq \omega \times \omega$ which is uniquely universal for the Σ^0_1 subsets of ω. Brodhead and Cenzer [1] prove the analogous result for (light-face) Σ^0_1 subsets of 2^ω.

Becker [2] considers unique parameterizations of the family of countable sets by Borel or analytic sets. Gao, Jackson, Laczkovich, and Mauldin [4] consider several other problems of unique parameterization.

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