Hechler and Laver Trees

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Abstract

A Laver tree is a tree in which each node splits infinitely often. A Hechler tree is a tree in which each node splits cofinitely often. We show that every analytic set is either disjoint from the branches of a Heckler tree or contains the branches of a Laver tree. As a corollary we deduce Silver Theorem that all analytic sets are Ramsey. We show that in Godel's constructible universe that our result is false for co-analytic sets (equivalently it fails for analytic sets if we switch Hechler and Laver). We show that under Martin's axiom that our result holds for $\mathbf{\Sigma^1_2}$ sets. Finally we define two games related to this property.

Definition 1 *A subtree $H \subseteq \omega^{<\omega}$ is* Hechler *iff* $\forall s \in H\ \forall^\infty n \quad sn \in H$. *A subtree $L \subseteq \omega^{<\omega}$ is* Laver *iff* $\forall s \in L\ \exists^\infty n \quad sn \in L$.

These definitions are motivated by well-known forcing notions of Laver [4] and Hechler [3]. In the classical Hechler forcing the cofinite sets on the n^{th} level of the tree would all be the same.

Definition 2 *For any subtree $T \subseteq \omega^{<\omega}$ define*

$$[T] = \{x \in \omega^\omega \ : \ \forall n \ \ x \restriction n \in T\}$$

Theorem 3 *For any $\mathbf{\Sigma}^1_1$ set $A \subseteq \omega^\omega$ either there exists a Hechler tree H with $[H] \cap A = \emptyset$ or there exists a Laver tree L with $[L] \subseteq A$.*

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Proof
Since analytic sets are projections of closed sets there exists a tree T on $\omega^{<\omega} \times \omega^{<\omega}$ such that

$$A = p[T] =^{def} \{x \in \omega^\omega \ : \ \exists y \in \omega^\omega \ \forall n \ (x \restriction n, y \restriction n) \in T\}.$$

Assume that for every Hechler H that A meets $[H]$ and we will show there is a Laver L with $[L] \subseteq A$.

For $s, t \in \omega^{<\omega}$ define

$$A_{s,t} = \{x \in \omega^\omega \ ; \ s \subseteq x \exists y \supseteq t \ (x, y) \in [T]\}.$$

Definition 4 *We say that H is Hechler with root s if for all $t \in H$ either $s \subseteq t$ or $t \subseteq s$ and beneath s there is cofinite splitting.*

Lemma 5 *Suppose for every Hechler H with root s that $A_{s,t} \cap [H] \neq \emptyset$. Then there are infinitely many n such that for every Hechler H with root sn that $A_{sn,t} \cap [H] \neq \emptyset$.*

Proof
Otherwise for all but finitely many n (say $n > N$) there exists a Hechler H_n with root sn which misses $A_{sn,t}$. But then the Hechler tree: $H = \bigcup_{n>N} H_n$ misses $A_{s,t}$ and has root s.
QED

Lemma 6 *Suppose for every Hechler H with root s that $A_{s,t} \cap [H] \neq \emptyset$. Then there exists an infinite well-founded tree $T \subseteq \{r \ : \ s \subseteq r\}$ with root s and terminal nodes $B \subseteq T$ such that*

(1) The nonterminal nodes of T are ω-splitting, i.e., if $r \in T \setminus B$, then there are infinitely many n with $rn \in T$, and

(2) For every $r \in B$ there exists n such that for every Hechler tree H with root r, $A_{r,tn} \cap [H] \neq \emptyset$.

Proof

For each ordinal α define a set $B_\alpha \subseteq \{r \ : \ s \subseteq r\}$ as follows.

(a) $r \in B_0$ iff there exists n such that for every Hechler tree H with root r, $A_{r,tn} \cap [H] \neq \emptyset$.

(b) $B_{\alpha+1} = B_\alpha \cup \{r \ : \ \exists^\infty n \ \ rn \in B_\alpha\}$

(c) $B_\lambda = \bigcup_{\alpha<\lambda}$ for λ a limit ordinal.

Define function rank(r) on $r \supseteq s$ as follows, rank$(r) = \alpha$ if α is the least ordinal with $r \in B_\alpha$ and rank$(r) = \infty$ if there is no such ordinal.

Case 1. rank(s) is an ordinal. In this case it is easy to build T and B as required.

Case 2. rank$(s) = \infty$. We show that this is impossible. Note that if rank$(r) = \infty$ then for all but finitely many n we must have that rank$(rn) = \infty$. Hence we may construct a Hechler tree H with root s such that rank$(r) = \infty$ for all $r \in H$ below the root. For each $n < \omega$ for each $r \in \omega^{n+|s|} \cap H$ there exists a Hechler H_r with root r such that

$$[H_r] \cap A_{s,tn} = \emptyset.$$

Define

$$K_n = \bigcup \{H_r \ : \ r \in \omega^{n+|s|} \cap H\}$$

Note that K_n is a Hechler tree with root s whose $n + |s|$ level is the same as H. It also true that $K_n \cap A_{s,tn} = \emptyset$. Because they are so wide $K = \bigcap_{n<\omega} K_n$ is a Hechler tree with root s such that $[K] \cap \bigcup_n A_{s,tn} = \emptyset$. This contradicts the hypothesis of the Lemma since $\bigcup_n A_{s,tn} = A_{s,t}$.

Finally, if T is trivial, i.e., $T = B = \{s\}$ just apply Lemma 5 to make T infinite.
QED

Proof of Theorem 3:

Suppose for every Hechler tree H with trivial root that $A \cap [H] \neq \emptyset$. Apply Lemma 6 to obtain a non-trivial well-found tree T_0 with terminal nodes B_0 and witnesses of length one.

Suppose we are given a well-founded tree T_n with trivial root and terminal nodes B_n such that for all $s \in T_n \setminus B_n$ there are infinitely many immediate extensions of s in T_n and for each $s \in B_n$ there is a t_s of length $n+1$ such that for every Hechler tree H with root s, $A_{s,t_s} \cap [H] \neq \emptyset$. Apply Lemma 6 to each node (s, t_s) with $s \in B_n$. Union all these trees together to get T_{n+1} which end extends T_n. It follows that L is a Laver where

$$L = \bigcup_{n<\omega} T_n$$

Note that although the length of the witnesses grow much slower than the s-part, nevertheless, they union up to show that $L \subseteq A$.
QED

Definition 7 *For $\mathcal{F}$ a filter extending the cofinite filter on ω define $\mathbb{H}_\mathcal{F}$ to be the Hechler trees mod $\mathcal{F}$, i.e., instead of demanding that for each $s \in H$ that $sn \in H$ for cofinitely many n, we demand that*

$$\{n \ : \ sn \in H\} \in \mathcal{F}.$$

Analogously define $\mathbb{L}_\mathcal{F}$ the Laver trees mod $\mathcal{F}$ by for each $s \in L$

$$\{n \ : \ sn \in L\} \in \mathcal{F}^+$$

where $\mathcal{F}^+$ are the positive $\mathcal{F}$ sets, i.e, sets whose complement is not in $\mathcal{F}$.

Theorem 8 *For any filter $\mathcal{F}$ and any $\mathbf{\Sigma}^1_1$ set $A \subseteq \omega^\omega$ either there exists a Hechler tree $H \in \mathbb{H}_\mathcal{F}$ with $[H] \cap A = \emptyset$ or there exists a Laver tree $L \in \mathbb{L}_\mathcal{F}$ with $[L] \subseteq A$.*

Proof
The proof of this goes over mutatis mutandis, the proof of Theorem 3.
QED

Any Hechler tree H may be pruned so that every node in it is strictly increasing, i.e. , if $\langle x_0, x_1, \ldots, x_n\rangle \in H$ then $x_0 < x_1 < \ldots x_n$. By the range of H we mean all infinite subsets of ω which are the image of some branch $f \in H$, i.e.,

$$\text{range}(H) = \{\{f(n) \ : \ n < \omega\} \ : \ f \in [H]\}$$

Proposition 9 *Suppose $H \in \mathbb{H}_\mathcal{F}$. Then there exists $X \in [\omega]^\omega$ such that $[X]^\omega \subseteq$ range(H).*

Proof
We may suppose that the nodes of H are strictly increasing. Construct a strictly sequence $x_0, x_1, \ldots, x_n$ such that for every k and subsequence

$$0 \leq i_1 < i_2 < \cdots < i_k \leq n$$

we have that $\langle x_{i_1}, \ldots, x_{i_k}\rangle \in H$. To obtain x_{n+1} we need only intersect finitely many elements of the filter $\mathcal{F}$.
QED

Corollary 10 *(Silver [5]) Analytic sets have the Ramsey Property. This means that for any $\mathbf{\Sigma}^1_1$ set $A \subseteq [\omega]^\omega$ there exists an $X \in [\omega]^\omega$ with either $[X]^\omega \subseteq A$ or $[X]^\omega \cap A = \emptyset$.*

Proof
Let $\mathcal{F}$ be a nonprincipal ultrafilter. Note that $\mathbb{H}_\mathcal{F} = \mathbb{L}_\mathcal{F}$ for ultrafilters. Define $B \subseteq \omega^\omega$ by $f \in B$ iff f is strictly increasing with range in A. Then B is $\mathbf{\Sigma}^1_1$ and so by Theorem 8 there is a Hechler tree $H \in \mathbb{H}_\mathcal{F}$ with $[H] \subseteq B$ or $[H] \cap B = \emptyset$. By Proposition 9 there is an infinite X as required.
QED

This gives a proof of Silver's Theorem which avoids the accept-reject arguments of Galvin-Prikry [2] and Ellentuck [1].

Theorem 11 *(V=L) There exists a Π^1_1 set $A \subseteq \omega^\omega$ such that $[H] \cap A \neq \emptyset$ for every Hechler tree H but A contains no Laver $[L]$.*

Proof
Using the definable well-ordering of the reals in L construct $B \subseteq \omega^\omega$ a Σ^1_2 set with the following two properties:

(1) B is an $<^*$ scale, i.e., $B = \{g_\alpha \in \omega^\omega \;:\; \alpha < \omega_1\}$ where $\alpha < \beta$ implies $g_\alpha <^* g_\beta$ and for every $f \in \omega^\omega$ there exist α such that $f <^* g_\alpha$.

(2) B has the property that for any $\sigma : \omega^{<\omega} \to \omega$ there exists $g \in B$ such that for all $x \in 2^\omega$ if $f = 2g + x$ then $\forall n \;\; f(n) > \sigma(f \upharpoonright n)$,

Let $C \subseteq \omega^\omega \times 2^\omega$ be Π^1_1 so that $g \in B$ iff $\exists x \; (g, x) \in C$.

Given $f \in \omega^\omega$ define $Q(f) = (g, x)$ where $g \in \omega^\omega$ and $x \in 2^\omega$ are determined by $f = 2g + x$. Define the Π^1_1 set A by

$$A = \{f \in \omega^\omega \;:\; Q(f) \in C\}.$$

Note that for any Hechler H we can find $\sigma : \omega^{<\omega} \to \omega$ such that

$$H_\sigma =^{def} \{f \in \omega \;:\; \forall n \;\; f(n) > \sigma(f \upharpoonright n)\ \} \subseteq H$$

It follows from (2) that A meets every $[H]$. On the other hand A cannot contain the branches $[L]$ of a Laver tree. This is because of the scale (1). Take a 3 splitting subtree of $T \subseteq L$, i.e., for every $s \in T$ there are exactly 3 immediate extensions of s in T. For each $g \in \omega^\omega$ define

$$C_g = \{f \in \omega^\omega \;:\; \exists x \in 2^\omega \; f = 2g + x\}$$

and note that $A \subseteq \bigcup_{g \in B} C_g$. If $[T] \subseteq A$ then by the scale property of B there would have to be a countable set $Q \subseteq B$ with such that

$$[T] \subseteq \bigcup_{g \in Q} C_g$$

But the C_g are the branches of a binary splitting tree and since T is 3-splitting, it is easy to construct $f \in [T]$ such that $f \notin C_g$ for every $g \in Q$.
QED

Theorem 12 *Assume MA +¬CH. If* $A \subseteq \omega^\omega$ *is* $\mathbf{\Sigma^1_2}$*, then either there is a Hechler tree* H *with* $[H] \cap A = \emptyset$ *or there is a Laver tree* L *with* $[L] \subseteq A$*. In fact, this is true for any set* A *which is the union of* ω_1 *many Borel sets.*

Proof
Suppose $A = \bigcup_{\alpha < \omega_1} B_\alpha$ where each B_α is Borel and the union is increasing. Since no B_α contains the branches of a Laver tree we have H_α a Hechler tree with $[H_\alpha] \cap B_\alpha = \emptyset$. Without loss we may assume that

$$[H_\alpha] = \{ f \in \omega^\omega \ : \ \forall n \ f(n) > \sigma_\alpha(f \restriction n) \ \}$$

where $\sigma_\alpha : \omega^{<\omega} \to \omega$. By Martin's axiom we may find $\sigma : \omega^{<\omega} \to \omega$ which eventually dominates each σ_α. By a counting argument we can find a single $\sigma : \omega^{<\omega} \to \omega$ which everywhere dominates ω_1 of the σ_α. But this means that H_σ is a Hechler tree disjoint from A since the B_α's are an increasing union.
QED

Finally we make some remarks about games.

Game 1. Given $A \subseteq \omega^\omega$. Player I and II alternatingly play

$$n_0, \ m_0 > n_0, \ \ n_1, \ \ m_1 > n_1, \ \ \ldots$$

with Player I playing $n_k \in \omega$ and Player II responding with $m_k > n_k$. The play of the game is won by Player II iff $(m_i : i < \omega) \in A$.

Proposition 13 *(a) Player II has a winning strategy in Game 1 iff there exists a Laver tree* L *with* $[L] \subseteq A$*. (b) Player I has a winning strategy in Game 1 iff there exists a Hechler tree* H *with* $[H] \cap A = \emptyset$*.*

Proof
Given the trees it easy to get the strategies. For the other direction:

(a) Use player II's winning strategy to construct a Laver tree as required.

(b) If $\sigma : \omega^{<\omega} \to \omega$ is Player I's winning strategy, then for any sequence $(m_i : i < \omega)$ such that $m_{i+1} > \sigma(m_0, \ldots, m_i)$ for every i we have that $(m_i : i < \omega) \notin A$. But this gives a Hechler tree H with $[H]$ disjoint from A.
QED

Game 2. Given $A \subseteq \omega^\omega$. Player I and II alternatingly play

$$X_0, \ \ m_0 \in X_1, \ \ X_1, \ \ m_1 \in X_1, \ldots$$

with Player I playing $X_k \in [\omega]^\omega$ and Player II responding with $m_k \in X_k$. Player II wins the play of the game iff $(m_i : i < \omega) \in A$.

Proposition 14 *(a) Player II has a winning strategy in Game 2 iff there exists a Hechler tree H with $[H] \subseteq A$. (b) Player I has a winning strategy in Game 2 iff there exists a Laver tree L with $[L] \cap A = \emptyset$.*

Proof
From right-to-left in both cases is easy. For the other direction:

(a) Let σ be a winning strategy for Player II. Consider

$$\{m_0 \ : \ \exists X_0 \ \ \sigma(X) = m_0\}$$

This set must be cofinite, since otherwise consider σ's response to its complement. Similarly given any sequence $X_0, X_1, \ldots, X_{n-1}$ the set

$$\{m_n \ : \ \exists X_n \ \ \sigma(X_0, \ldots, X_n) = m_n\}$$

must be cofinite. Construct X_s for $s \in \omega^{<\omega}$ and get a Hechler tree H all of whose branches are plays of the winning strategy and hence are in A.

(b) The sequence of X_s played by winning strategy of Player I determine a Laver tree L.
QED

Some of the results in this note follow from Zapletal [6].

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