Arnold W. Miller
Department of Mathematics
University of Wisconsin
Madison, WI 53706
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Dear Professor Fraïssé

I enjoyed very much reading your book, **Theory of Relations**. Thank you for writting it.

Probably someone has already answered the problem of Hagendorf[1] which you mention on page 136:

> Existence of a strictly decreasing ω_1-sequence of denumerable partial orderings.

The following result, which I proved jointly with Ken Kunen, answers this question in the affirmative.

Theorem 1 *There exists* $\langle P_X : X \in [\omega]^\omega \rangle$ *where each* P_X *is a countable poset and*

$$P_X \text{ embeds into } P_Y \iff X \subseteq^* Y$$

where $[\omega]^\omega$ *is the set of infinite subsets of* $\omega = \{0, 1, 2, \ldots\}$ *and* $X \subseteq^* Y$ *means inclusion mod finite, i.e.* $X \setminus Y$ *is finite.*

Since there are decreasing mod finite ω_1 sequences in $[\omega]^\omega$ we get that the same is true for countable posets under embedding.

Lemma 2 *There is a set* $\langle C_n : n \in \omega \rangle$ *of finite partial orders such for any* $n \in \omega$, C_n *cannot be embedded in the disjoint union of* $\{C_m : m \neq n\}$. *Furthermore no* C_n *contains a chain of length three.*

Proof: For $n \geq 3$ let C_{n-3} be the following ordering on $2n$ points

$$\{a_i, b_i : i < n\}$$

$$a_i < b_j \iff i = j \text{ or } j = i + 1 \text{ mod n}$$

[1] I have sent copies of this letter to Hagendorf, Kunen, Pouzet, and Veličković.

Otherwise incomparable. Hence the a's are all minimal and the b's maximal. I picture them as being wrapped around a cylindar or ring. The embedding claim is true for the same reason that a cyclic graph cannot be embedded into another one of different cycle length.
□

For any $Y \in [\omega]^\omega$ let $Q(Y)$ be the partial order which consists of the disjoint union of $\{C_m : m \in Y\}$ and in addition has a unique minimal element c.

Now we describe P_X for any $X \in [\omega]^\omega$. Let $\{X_n : n \in \omega\}$ be all $Y \in [\omega]^\omega$ such that $Y =^* X$. P_X is the disjoint union of $\{Q(X_n) : n \in \omega\}$.

It easy to show that if $Y \subseteq Z$ then $Q(Y)$ can be embedded into $Q(Z)$ and hence if $X \subseteq^* Y$ then P_X can be embedded into P_Y. On the other hand if P_X can be embedded into P_Y then for some n, $Q(X_0)$ can be embedded into $Q(Y_n)$ and thus $X_0 \subseteq Y_n$ and so $X \subseteq^* Y$. This proves the theorem.
□

B. Veličković (CalTech) asked whether it is possible to get a decreasing chain of countable posets of length $2^{\aleph_0}$. Assuming MA the answer to Veličković's question is yes, since such chains exist in $[\omega]^\omega$/finite. However in the Cohen real model (adding say $\kappa \geq \omega_2$ Cohen reals to a model of GCH) the continuum is large, but ω_2 does not embed into $[\omega]^\omega$/finite. This in fact, follows from an unpublished result in Kunen's Thesis. (The theorem in Kunen's thesis is that in the Cohen real model no well-order of ω_2 is in the $\sigma-$algebra generated by rectangles $\{A \times B : A, B \subseteq \omega_2\}$.)

His argument can be generalized to show:

Theorem 3 *It consistent relative to the consistency of ZFC that the continuum is arbitrarily large but there does not exist countable structures $\langle A_\alpha : \alpha < \omega_2\rangle$ such that for all $\alpha, \beta < \omega_2$*

$$\alpha < \beta \iff A_\alpha \text{ embeds into } A_\beta$$

Proof: Let M be a countable transitive model of ZFC+GCH and let P be $FIN(\kappa)$ the partial order of finite partial functions from κ into 2 where κ is any cardinal of M $\geq \omega_2^M$. Suppose for contradiction that in M[G] where G is P-generic over M there is such an ω_2 sequence.

Working in M, let $\langle A_\alpha : \alpha < \omega_2\rangle$ be a sequence of names for countable structures with ω as thier universe and $p \in P$ be such that

$$p \Vdash \forall \alpha, \beta\, [\alpha < \beta \iff A_\alpha \text{ embedds into } A_\beta]$$

Since P has c.c.c. we can assume that the names have countable support, i.e. we can find Γ_α countable subsets of κ such that $A_\alpha^G \in M[G|_{\Gamma_\alpha}]$.

By the delta systems lemma we can find $X \in [\omega_2]^{\omega_2}$ and root R such that for $\alpha, \beta \in X$ and distinct

$$\Gamma_\alpha \cap \Gamma_\beta = R$$

We can assume that the order type of every Γ_α for $\alpha \in X$ is the same (say α_0) and furthermore that the unique order preserving map between any two is the identity on R. (Since M satisfies CH.)

Let π_α be the partial order isomorphism from $FIN(\Gamma_\alpha)$ to $FIN(\alpha_0)$ induced by the unique order preserving map from Γ_α to α_0.

Let $\pi_\alpha(A_\alpha) = \tau_\alpha$. Since M satisfies CH we can assume that all τ_α are the same. It follows then that if π is the automorphism of $FIN(\kappa)$ induced by interchanging $\Gamma_\alpha \setminus R$ and $\Gamma_\beta \setminus R$ order preservingly and the identity outside of these, then $\pi(A_\alpha) = A_\beta$ and $\pi(A_\beta) = A_\alpha$

If we take $\alpha, \beta \in X$ such that $\alpha < \beta$ and $\Gamma_\alpha \setminus R$ and $\Gamma_\beta \setminus R$ are both disjoint from the domain of p (so $\pi(p) = p$) then since

$$p \Vdash A_\alpha \text{ embedds into } A_\beta$$

so

$$\pi(p) \Vdash \pi(A_\alpha) \text{ embedds into } \pi(A_\beta)$$

hence

$$p \Vdash A_\beta \text{ embedds into } A_\alpha$$

contradicting $\alpha < \beta$.
$\square$

sincerely,

Arnold W. Miller