“Descriptive Set Theory”

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Descriptive set theory is the study of definable sets of real numbers. More generally, we could consider subsets of any Polish space, i.e., a separable complete metric space.

In fact, it is often most convenient to work in the Baire space, ω^ω. The symbol ω denotes set of nonnegative integers or equivalently the first infinite ordinal number, $\omega = \{0, 1, 2, 3, \ldots\}$. Elements of ω^ω can be thought of either as infinite sequences of elements of ω or as functions $f : \omega \to \omega$. The metric on ω^ω is defined by

$$d(f, g) = \begin{cases} \frac{1}{n+1} & \text{where } n \text{ is the least such that } f(n) \neq g(n) \\ 0 & \text{if } f = g \end{cases}$$

Baire showed that under this topology the Baire space ω^ω is homeomorphic to the irrational numbers with their usual topology.

The first person to consider definable sets of real number was probably Borel. Borel reasoned that basic open sets should be considered definable and if we allow countably many bits of information, then the family of definable sets should be closed under taking countable unions and countable intersections. This is the family of Borel sets.

The classical hierarchy on Borel sets is defined as follows.

1. $\utilde{\mathbf{\Sigma}}^0_1$ is the family of open sets.
2. $\utilde{\mathbf{\Pi}}^0_1$ is the family of closed sets.
3. $\utilde{\mathbf{\Sigma}}^0_2$ is the family of sets which are the countable unions of $\utilde{\mathbf{\Pi}}^0_1$ sets.
4. $\utilde{\mathbf{\Pi}}^0_2$ is the family of sets which are the countable intersections of $\utilde{\mathbf{\Sigma}}^0_1$ sets.
5. In general, for each countable ordinal α, $\utilde{\mathbf{\Sigma}}^0_\alpha$ is the family of sets which are the countable unions of sets each of which is $\utilde{\mathbf{\Pi}}^0_\beta$ for some $\beta < \alpha$ and $\utilde{\mathbf{\Pi}}^0_\alpha$ is the family of sets which are the countable intersection of sets each of which is $\utilde{\mathbf{\Sigma}}^0_\beta$ for some $\beta < \alpha$.
6. $\utilde{\mathbf{\Delta}}^0_\alpha = \utilde{\mathbf{\Sigma}}^0_\alpha \cap \utilde{\mathbf{\Pi}}^0_\alpha$

By DeMorgan’s laws it is easy to see that the $\utilde{\mathbf{\Pi}}^0_\alpha$ sets are precisely the sets whose complement is $\utilde{\mathbf{\Sigma}}^0_\alpha$. Lebesgue proved that each of these classes are nontrivial, i.e., for any countable ordinal α there are sets in $\utilde{\mathbf{\Sigma}}^0_\alpha$ which are not in $\utilde{\mathbf{\Pi}}^0_\alpha$ and hence vice-versa. We can think of the Borel sets as being closed under quantification over ω, so the next higher classes would be those in the projective hierarchy, i.e., closure under quantification over ω^ω.

Suppose X is a Polish space and $A \subseteq X$. Then A is $\utilde{\mathbf{\Sigma}}^1_1$ in X iff there exists a Borel set $B \subseteq \omega^\omega \times X$ such that

$$y \in A \text{ iff } \exists y \in \omega^\omega \ (x, y) \in B$$

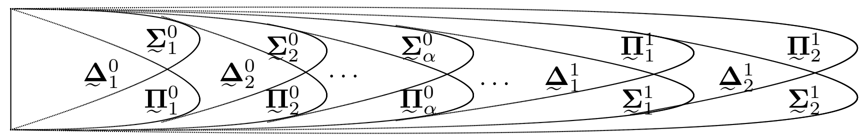

Figure 1: The Borel hierarchy and a little beyond

Thus the $\boldsymbol{\Sigma}^1_1$ sets or analytic sets are precisely those which are the projection of a Borel set. The family of $\boldsymbol{\Pi}^1_1$ are exactly those which are the complements of $\boldsymbol{\Sigma}^1_1$. Figure 1 shows how some of these classes of sets are arranged.

Analytic sets have been studied in the context of general topological spaces by C.A.Rogers, L.E.Jayne, A.H.Stone, and many others, see [12]. Continuing up the projective hierarchy, we take $\boldsymbol{\Sigma}^1_2$ to be the family of projections of $\boldsymbol{\Pi}^1_1$ sets and $\boldsymbol{\Pi}^1_2$ to be the complements of $\boldsymbol{\Sigma}^1_2$, etc. It is a classical result that the class of Borel sets coincides exactly with the $\boldsymbol{\Delta}^1_1$-sets, i.e. those sets which are both $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$.

Early descriptive set theorists, Sierpiński, Luzin, Suslin, Kuratowski, Borel, Baire, and Lebesgue, were concerned with questions about the regularity of projective sets of reals. For example, it was shown that any uncountable $\boldsymbol{\Sigma}^1_1$ set must contain a perfect subset, i.e., a homeomorphic copy of the Cantor space, $2^\omega = \{0,1\}^\omega$. They showed that $\boldsymbol{\Sigma}^1_1$ sets are Lebesgue measurable and have the Property of Baire. They also proved that any $\boldsymbol{\Sigma}^1_2$ is the ω_1 union of Borel sets. See [6] [7] [8] [11].

In the 1930's Kurt Gödel as a consequence of his work on the consistency of continuum hypothesis, showed that it was consistent with the usual axioms of set theory that there is a $\boldsymbol{\Delta}^1_2$ set which neither contains nor is disjoint from a perfect set. Such a set cannot have the property of Baire or be Lebesgue measurable.

In 1960's Robert Solovay using the forcing technique of Paul Cohen, was able to show (relative to the consistency of an inaccessible cardinal) that it is consistent that every projective set of reals is Lebesgue measurable, has the property of Baire, and has the perfect subset property.

The strongest known regularity property is called determinacy, it arose from the study of infinite two person games in the 1950's (see [5]). It implies the perfect set property, Lebesgue measurability and the Baire property, as well as, many other natural properties of the projective sets. It was a celebrated theorem of D.A.Martin that Borel sets are determined. In fact, one of the reasons this proof was difficult to find, was proved earlier by Harvey Friedman who showed that a proof of Borel determinacy must necessarily use uncountable many uncountable cardinals. The axiom of determinacy for projective sets was established by D.A.Martin and John Steel using large cardinal axioms, specifically infinitely many Woodin cardinals. The large cardinal assumption was also shown to be necessary. This confirmed a conjecture of Solovay who was the first to connect large cardinal theory with the axiom of determinacy. Kanomori's book [3] contains many of the results on large cardinal theory in set theory.

In 1970's, Jack Silver proved the following theorem about Borel equivalence relations, or actually $\underset{\sim}{\mathbf{\Pi}}^1_1$ equivalence relations. Namely, every $\underset{\sim}{\mathbf{\Pi}}^1_1$ equivalence relation with uncountably many equivalence classes must contain a perfect set of inequivalent members. John Burgess established a similar theorem for $\underset{\sim}{\mathbf{\Sigma}}^1_1$ equivalence relations, namely any such equivalence relation with more than ω_1 equivalence classes must have a perfect set of inequivalent elements. Leo Harrington using a topology invented by Robin Gandy gave a simpler proof of Silver's Theorem. This technique achieved great success at proving a number of other results using effective descriptive set theory. For example, Louveau's theorem, the Borel-Dilworth Theorem, and Glimm-Effros-Kechris-Harrington Dichotomy Theorem were all proved using this technique. See [2] [4] [5] [9] [10].

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