Cohen forcing preserves being a γ-set but not the Borel-Hurewicz property

Arnold W. Miller
Sept 2012

Marion Scheepers proved that random real forcing preserves being a γ-set. Boas Tsaban asked if the same is true for Cohen real forcing.

Proposition 1 *Suppose in the ground model M that $X \subseteq 2^\omega$ is a γ-set and $\mathbb{P} = 2^{<\omega}$ is Cohen real forcing. Then for any G $\mathbb{P}$-generic over M*

$$M[G] \models X \textit{ is a } \gamma\textit{-set.}$$

Proof
Work in M. Suppose

$$p_0 \Vdash \overset{\circ}{\mathcal{U}} \text{ is an } \omega\text{-cover of } X \text{ with clopen sets and is downward closed.}$$

To simplify our notation assume that p_0 is the trivial condition or replace $\mathbb{P}$ by the conditions stronger than p_0. For each $p \in \mathbb{P}$ define

$$\mathcal{U}_p = \{C \ : \ \mu(C) < \frac{1}{2^{|p|+1}} \text{ and } \exists q \leq p \quad q \Vdash C \in \overset{\circ}{\mathcal{U}}\}.$$

It is easy to check that each $\mathcal{U}_p$ is an ω-cover of X. Hence we may find $(C_p \in \mathcal{U}_p \ : \ p \in \mathbb{P})$ a γ-cover of X. Let $f : \mathbb{P} \to \mathbb{P}$ be such that $f(p) \leq p$ and $f(p) \Vdash C_p \in \overset{\circ}{\mathcal{U}}$.

Let G be $\mathbb{P}$-generic over M and define

$$\mathcal{V} = \{C_p \ : \ f(p) \in G\}.$$

Then $\mathcal{V} \subseteq \mathcal{U}$ is a γ-cover of X. Note that there must be infinitely many p with $f(p) \in G$ since no p can force that there are only finitely many. The measure condition on $\mathcal{U}_p$ guarantees that $\mathcal{V}$ is infinite.
QED

Another question asked by Tsaban is whether it possible that adding a Cohen real can destroy the Hurewicz property in the case of a totally imperfect set. Scheepers and Tall showed that adding a Cohen real destroys the property that the ground model's Cantor set is Hurewicz.

Proposition 2 *Suppose that $M \models X \subseteq 2^\omega$ is a Sierpinski set. If $\mathbb{P}$ is Cohen real forcing, then for any G $\mathbb{P}$-generic over M*

$$M[G] \models X \text{ does not have the Hurewicz property.}$$

Proof

It is well-known that forcing with $\mathbb{P}$ is equivalent to forcing with any non-trivial countable poset. Here is the poset we use:

$p \in \mathbb{P}$ iff $p = (\vec{C_k} : k < n)$ for some n where each $\vec{C_k} = (C_{k,i} : i < n_k)$ is a finite sequence of clopen sets in 2^ω with $\mu(\bigcup_{i<n_k} C_{k,i}) < \frac{1}{2^k}$. Then $p \leq q$ iff $n_p \geq n_q$ and $\vec{C_k}^p$ extends $\vec{C_k}^q$ for $k < n_q$.

Now let G be $\mathbb{P}$-generic over M and in $M[G]$ define $(\mathcal{U}_n : n < \omega)$ by

$$\mathcal{U}_k = \{(C_{k,i})^p \ : \ \exists p \in G \ \ k < (n)^p \text{ and } i < (n_k)^p\}.$$

It easy to check that each $\mathcal{U}_k$ is a cover of $2^\omega \cap M$ and hence of X. We claim that in $M[G]$ there does not exists $g : \omega \to \omega$ with the property that for every $x \in X$ there exists N such that $x \in \bigcup_{i<g(n)} C_{n,i}$ for all $n > N$.

Work in M. Suppose for contradiction that there exists p_0 such that

$$p_0 \Vdash \forall x \in X \ \exists N \ \forall n \geq N \ \ \check{x} \in \bigcup_{i<\mathring{g}(n)} \mathring{C}_{n,i} \ .$$

For each $p \leq p_0$ and N define

$$X(p,N) = \{x \in X \ : \ \forall n \geq N \ \ p \Vdash \check{x} \in \bigcup_{i<\mathring{g}(n)} \mathring{C}_{n,i}\}.$$

Note that

$$X = \bigcup \{X(p,N) \ : \ p \leq p_0 \text{ and } N < \omega\}.$$

Since $\mathbb{P}$ is countable there must exist $p \leq p_0$ and N for which $X(p,N)$ is uncountable and since X is Sierpinski, $X(p,N)$ has positive outer measure. Note that if $q \leq p$ and $N' \geq N$, then $X(q,N') \supseteq X(p,N)$. Hence by extending p and increasing N if necessary we may suppose that

1. $\mu^*(X(p,N)) > \frac{1}{2^N}$,

2. $p \Vdash \mathring{g}\ (N) = \check{L}$, and

3. $N < n_p$ and the length of $(\vec{C}_N)^p$ is at least L.

Since $\mu(\bigcup_{i<L}(C_{N,i})^p) < \frac{1}{2^N} < \mu^*(X(p,N))$, we can choose $x \in X(p,N)$ with x not in $\bigcup_{i<L}(C_{N,i})^p)$. But this contradicts

$$p \Vdash \check{x} \in \bigcup_{i<\overset{\circ}{g}(N)} \overset{\circ}{C}_{N,i} \text{ and } p \Vdash \overset{\circ}{g}(N) = \check{L}$$

QED

Note that Sierpinski sets have the Hurewicz property with respect to Borel covers also. Zdomskyy and Tsaban point out that Proposition 2 directly contradicts Theorem 40 of Scheepers and Tall [1].

Alan Dow asked "Can adding one Cohen real lower the value of $\mathfrak{b}$?"

Proposition 2 shows that this is possible. Start with a model M where $\mathfrak{b} = \omega_2 = \mathfrak{c}$ or any larger regular cardinal. Then force with the measure algebra on 2^{ω_1}. In the model $M[H]$ the smallest unbounded set is still ω_2 since the reals added by random real forcing are bounded by ground model reals. Let G be $\mathbb{P}$ generic over $M[H]$. Proposition 2 gives us a sequence of covers $\mathcal{U}_n = \{C_{n,m} : m < \omega\}$ of the generic Sierpinski set X_H determined by H. For each $x \in X_H$ let $f_x(n)$ be the least m with $x \in C_{n,m}$. Then in $M[H][G]$ the set $\{f_x : x \in X_H\}$ is unbounded in ω^ω. Hence $\mathfrak{b} = \omega_1$.

References

[1] Scheepers, Marion; Tall, Franklin D.; Lindelof indestructibility, topological games and selection principles. Fund. Math. 210 (2010), no. 1, 1-46.

The following remark is due to Janusz Pawlikowski (email June 2013)

1. any set that is null and Hurewicz is covered by a null F_σ set,

2. given models $M \subseteq N$: if no real from N is eventually different over M (e.g., if the reals from M are nonmeager in N), then any null F_σ set coded in N is covered by a null G_δ set coded in M, so, if a nonnull set from M becomes in N null and Hurewicz, then N adds an eventually different real over M,

3. in particular, Cohen cannot force a Sierpinski set to keep the Hurewicz property.