

CLASSICAL SOLUTIONS TO SEMI-GEOSTROPHIC SYSTEM WITH VARIABLE CORIOLIS PARAMETER

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ABSTRACT. We prove short time existence and uniqueness of smooth solutions (in $C^{k+2,\alpha}$ with $k \geq 2$) to the 2-D semi-geostrophic system and the semi-geostrophic shallow water system with variable Coriolis parameter f and periodic boundary conditions, under the natural convexity condition on the initial data. The dual space used in analysis of the semi-geostrophic system with constant f is not available for the variable Coriolis parameter case, and we develop a time-stepping procedure in Lagrangian coordinates in the physical space to overcome this difficulty.

1. INTRODUCTION

The semi-geostrophic system (abbreviated as SG) is a model of large-scale atmospheric/oceanic flows, where “large-scale” means that the flow is rotation dominated. All previous works on analysis of the SG system have been restricted to the case of constant Coriolis force, where the ability to write the equations in “dual” coordinates enables the equations to be solved in that space and then mapped back to physical space. Examples are the results of Benamou and Brenier [2], Cullen and Gangbo [7], Cullen and Feldman [6], Ambrosio, Colombo, De Philippis and Figalli [1]. All these solve SG subject to a convexity condition introduced by Cullen and Purser [8]. The convexity condition allows the mapping between the physical and dual spaces to be interpreted as an optimal map for a Monge-Kantorovich mass transport problem, which makes possible the use of methods of Monge-Kantorovich theory in the study of SG with constant Coriolis force.

The background and applicability of this model is reviewed by Cullen [3]. In the atmosphere, this model is applicable on scales larger than about 1000km, which is comparable to the radius of the Earth. Thus the variations of the vertical component of the Coriolis force have to be taken into account, as these are a fundamental part of atmospheric dynamics on this scale. Thus SG with variable rotation (i.e. Coriolis parameter) is more physically realistic.

Attempts to extend the theory to the case of variable rotation were made by Cullen *et al.* [5] and Cullen [4]. These included formal arguments why the equations should be solvable. In particular, they derived a solvability condition in the form of the positive definiteness of a stability matrix which generalises the convexity condition used in the constant rotation case. They also showed that geostrophic balance could be defined by the condition that the energy was stationary under a certain class of Lagrangian displacements. These properties suggest that a rigorous existence proof should be possible.

In this paper we prove short time existence and uniqueness of smooth solutions to SG with variable Coriolis parameter subject to the strict positive definiteness of the stability matrix. Since dual variables are not available, the result has to be proved

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working directly in the physical coordinates. Somewhat surprisingly, Monge-Ampere type equations appear in this process, even though we do not use Monge-Kantorovich mass transport as in the case of constant Coriolis parameter. We consider two versions of the SG equations. The simplest to analyse is the two-dimensional incompressible SG flow. However, this is not a physically relevant model. We therefore also analyse the SG shallow water equations, which are an accurate approximation to the full shallow water equations on large scales.

2. FORMULATION OF THE PROBLEMS AND MAIN RESULTS

We consider the SG equations with non-constant Coriolis force on a 2-D flat torus:

$$(2.1) \quad (u_{1,g}, u_{2,g}) = f^{-1}(-\frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_1}),$$

$$(2.2) \quad D_t u_{1,g} + \frac{\partial p}{\partial x_1} - f u_2 = 0,$$

$$(2.3) \quad D_t u_{2,g} + \frac{\partial p}{\partial x_2} + f u_1 = 0,$$

$$(2.4) \quad \nabla \cdot \mathbf{u} = 0,$$

with initial data $p|_{t=0} = p_0(x)$. Here $\mathbf{u} = (u_1, u_2)$ is the physical velocity and $D_t = \partial_t + \mathbf{u} \cdot \nabla$, the material derivative. $\mathbf{u}_g = (u_{1,g}, u_{2,g})$ is the geostrophic wind velocity, p is the pressure, and $f = f(x)$ is the Coriolis parameter, which is a given smooth strictly positive function.

In this paper we consider the two-dimensional periodic case. That is, all the functions appearing above are assumed to be defined on $\mathbb{R}^2$ and periodic with respect to $\mathbb{Z}^2$, hence can be thought of as defined on a 2-D torus.

Physically interesting solutions of the SG system must satisfy the convexity principle introduced by Cullen and Purser [8]. In the case when $f \equiv 1$, the convexity condition means that the modified pressure function $P(x_1, x_2) = p(x_1, x_2) + \frac{1}{2}(x_1^2 + x_2^2)$ is convex. We will introduce the analogue of this convexity condition when f is not a constant, see (2.6) below, and prove short time existence and uniqueness of solutions when this condition is satisfied by the initial data. Before we state the main results of this paper, we first introduce some notation:

In the following, we identify $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$.

We will denote $C^{k,\alpha}(\mathbb{T}^2)$ to be the space of $C^{k,\alpha}$ functions on $\mathbb{R}^2$ and periodic with respect to $\mathbb{Z}^2$, which is equipped with the norm

$$\|p\|_{k,\alpha} = \sum_{0 \leq |\beta| \leq k} \|D^\beta p\|_0 + \sum_{|\beta|=k} [D^\beta u]_\alpha,$$

where

$$\|v\|_0 = \max_{x \in \mathbb{R}^2} |v(x)|, \quad [v]_\alpha = \sup_{x,y \in \mathbb{R}^2} \frac{|v(x) - v(y)|}{|x - y|^\alpha}.$$

Sometimes we will write $C^{k,\alpha}$ instead of $C^{k,\alpha}(\mathbb{T}^2)$ for simplicity. Similarly define $L^2(\mathbb{T}^2)$ which consists of periodic functions which are in $L^2_{loc}(\mathbb{R}^2)$. All these spaces can be equivalently understood as corresponding spaces on the 2-D torus $\mathbb{T}^2$.

We will also need the following remark:

Remark 2.1. Let A be an $n \times n$ matrix (not necessarily symmetric). We say $A > 0$ if there exists a constant $c_0 > 0$, so that $a_{ij}\xi_i\xi_j \geq c_0|\xi|^2$, for all $\xi \in \mathbb{R}^n$. $A \geq 0$ means $a_{ij}\xi_i\xi_j \geq 0$ for all $\xi \in \mathbb{R}^n$. Note that $A > 0$ is equivalent to $A + A^T > 0$, and $A \geq 0$ is equivalent to $A + A^T \geq 0$. We say $A > B$ (resp. $A \geq B$) as long as $A - B > 0$ (resp. $A - B \geq 0$) in the sense just defined.

Now we state the main result of this paper:

Theorem 2.1. Let $k \geq 2$ be an integer. Let $f \in C^{k,\alpha}(\mathbb{T}^2)$ with $f(x) > 0$ on $\mathbb{T}^2$. Let $p_0 \in C^{k+2,\alpha}(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} p_0(x)dx = 0$. Suppose also the following convexity-type condition is satisfied

$$(2.5) \quad I + f^{-1}D(f^{-1}Dp_0) \geq c_0I \text{ on } \mathbb{T}^2 \text{ for some } c_0 > 0.$$

Then there exists $T_0 > 0$, depending on $\|p_0\|_{k+2,\alpha}$, c_0 , f and k , such that there exists a solution $(p, u_g, v_g, \mathbf{u})$ to (2.1)-(2.4) with initial data p_0 on $[0, T_0] \times \mathbb{T}^2$ which satisfies

$$(2.6) \quad I + f^{-1}D(f^{-1}Dp) > 0 \text{ on } [0, T_0] \times \mathbb{T}^2,$$

$$(2.7) \quad \int_{\mathbb{T}^2} p(t, x)dx = 0 \text{ for all } t \in [0, T_0],$$

and the following regularity

$$(2.8) \quad \partial_t^m p \in L^\infty(0, T_0; C^{k+2-m,\alpha}(\mathbb{T}^2)) \text{ for } 0 \leq m \leq k+1, \mathbf{u} \in L^\infty(0, T_0; C^{k,\alpha}(\mathbb{T}^2)).$$

Moreover, any solution $(p, \mathbf{u}_g, \mathbf{u})$ to (2.1)-(2.4) with initial data p_0 , defined on $[0, T] \times \mathbb{T}^2$ for some $T > 0$, which satisfies (2.6), (2.7) and has regularity $p \in L^\infty(0, T; C^3(\mathbb{T}^2))$, $\partial_t p \in L^\infty(0, T; C^2(\mathbb{T}^2))$ is unique.

Similar results hold for the semi-geostrophic shallow water system (8.1)-(8.4).

Theorem 2.2. Let $k \geq 2$ be an integer. Let $f \in C^{k,\alpha}(\mathbb{T}^2)$ with $f(x) > 0$ on $\mathbb{T}^2$. Let $h_0 \in C^{k+2,\alpha}(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} h_0(x)dx = 1$. Suppose also the following convexity and positivity conditions are satisfied for initial data:

$$(2.9) \quad I + f^{-1}D(f^{-1}Dh_0) \geq c_0I \text{ and } h_0 \geq c_1 \text{ on } \mathbb{T}^2 \text{ for some } c_0, c_1 > 0.$$

Then there exists $T_0 > 0$, depending on $\|h_0\|_{k+2,\alpha}$, c_0, c_1 , f and k , such that there exists a solution $(h, u_g, v_g, \mathbf{u})$ to (8.1)-(8.4) with initial data h_0 on $[0, T_0] \times \mathbb{T}^2$ which satisfies

$$(2.10) \quad I + f^{-1}D(f^{-1}Dh) > 0, h > 0 \text{ on } [0, T_0] \times \mathbb{T}^2,$$

$$(2.11) \quad \int_{\mathbb{T}^2} h(t, x)dx = 1 \text{ for all } t \in [0, T_0],$$

and the following regularity

$$(2.12) \quad \partial_t^m h \in L^\infty(0, T_0; C^{k+2-m,\alpha}(\mathbb{T}^2)) \text{ for } 0 \leq m \leq k+1, \mathbf{u} \in L^\infty(0, T_0; C^{k,\alpha}(\mathbb{T}^2)).$$

Moreover, any solution $(h, u_g, v_g, \mathbf{u})$ to (8.1)-(8.4) with initial data h_0 , defined on $[0, T] \times \mathbb{T}^2$ for some $T > 0$ which satisfies (2.10), (2.11) and has regularity $h \in L^\infty(0, T; C^3(\mathbb{T}^2))$, $\partial_t h \in L^\infty(0, T; C^2(\mathbb{T}^2))$ is unique.

All previous works on existence of solutions for the SG system concern the case when the Coriolis parameter f is constant (and then by rescaling we can set $f \equiv 1$), and make

use of the dual space. Namely, we introduce the “potential” $P = p(x_1, x_2) + \frac{1}{2}(x_1^2 + x_2^2)$, then the system (2.1)-(2.4) can be put in the form.

$$(2.13) \quad D_t(\nabla P) = J(\nabla P - id),$$

$$(2.14) \quad \nabla \cdot \mathbf{u} = 0,$$

with initial conditions

$$(2.15) \quad P|_{t=0} = p_0 + \frac{1}{2}(x_1^2 + x_2^2),$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Cullen-Purser convexity condition is that P is convex, which coincides with condition (2.6) for $f \equiv 1$. For each $t > 0$, introduce the measure $\nu_t = \nabla P(t, \cdot)_\#(\mathcal{L}_{\mathbb{T}^2})$ (i.e. the push-forward of the Lebesgue measure $\mathcal{L}_{\mathbb{T}^2}$ on the torus by the map $\nabla P(t, \cdot)$), and let ν be a measure on $[0, \infty) \times \mathbb{T}^2$ defined by $d\nu = d\nu_t dt$. Then the measure ν will satisfy the equation

$$(2.16) \quad \partial_t \nu + \nabla \cdot (U\nu) = 0,$$

with

$$(2.17) \quad U = J(id - \nabla P^*),$$

with also the initial condition

$$(2.18) \quad \nu|_{t=0} = \nu_0 := \nabla P_0_\#(\mathcal{L}_{\mathbb{T}^2}).$$

Here P^* is the Legendre transform of the convex function P . Notice the vector field (2.17) is divergence-free.

To prove existence of solutions to the SG system (2.1)-(2.4) when $f \equiv 1$, or equivalently (2.13)-(2.14), it is easier to first consider the existence of solutions to the system (2.16)-(2.18). In general, such solutions have low regularity which makes it difficult to transform them back to the physical space. For general initial data with $\nu_0 \in L^p$, it is shown in [6] that a solution in physical space exists in a Lagrangian sense. In [9, 10], a weaker form of Lagrangian solution in physical space was obtained in the case when ν_0 is a general measure. If the solutions (ν, P) in dual space have enough regularity, then such solutions can be transformed back to physical variables and give Eulerian solutions to the original equations (2.13)-(2.14). In the case when the density of the initial measure ν_0 is between two positive constants: $0 < \lambda \leq \nu_0 \leq \Lambda$ on $\mathbb{T}^2$, Ambrosio et al [1] obtained a solution to (2.16)-(2.18) with $P^* \in W^{2,1}(\mathbb{T}^2)$, and this regularity turns out to be sufficient to transform back and give a weak solution to (2.1)-(2.4) in the sense of distributions. For smooth solutions, Loeper [12] obtained short time existence of smooth solutions to (2.16)-(2.18) when ν_0 is smooth and positive. And because of smoothness, there is no difficulty to rewrite the equation in terms of the original physical variables.

The approach described above does not work for the system (2.1)-(2.4) when f is not a constant, because a dual space is not available in such case. Therefore, we work directly with the system (2.1)-(2.4).

As a first attempt, we may try the following argument. Note that (2.2), (2.3) is a linear algebraic system for the physical velocity $\mathbf{u}$. Denote

$$(2.19) \quad Q = I + f^{-1}D(f^{-1}Dp).$$

Assume that $Q > 0$, then we can solve for $\mathbf{u}$ and use the definition of $\mathbf{u}_g$ by (2.1) to obtain

$$(2.20) \quad \mathbf{u} = Q^{-1}(f^{-1}J\nabla p - f^{-2}\partial_t\nabla p).$$

Substituting (2.20) into (2.4), we obtain an elliptic equation for $\partial_t p$:

$$(2.21) \quad \nabla \cdot [Q^{-1}f^{-2}\partial_t\nabla p] = \nabla \cdot [Q^{-1}f^{-1}J\nabla p].$$

Then we may try to solve (2.1)-(2.4) by a fixed point argument. For a given p , we solve the elliptic equation:

$$(2.22) \quad \nabla \cdot [Q^{-1}f^{-2}\nabla w] = \nabla \cdot [Q^{-1}f^{-1}J\nabla p].$$

We expect the solution w to give $\partial_t p$. Hence we define $\hat{p}(t, x) = p_0(x) + \int_0^t w(s, x)ds$. This procedure gives a map $p \rightarrow \hat{p}$. If this map has a fixed point p and it is smooth, then it gives a solution to the system (2.1)-(2.4). This approach runs into a serious difficulty because of the loss of a derivative. Indeed, if we assume p to be in $C^{k+2, \alpha}$ in spatial variables, then the coefficients of the elliptic equation written in divergence form in (2.22) are in $C^{k, \alpha}$, and the right-hand is the divergence of a $C^{k, \alpha}$ function. Then, from the standard elliptic estimates, the solution w will be $C^{k+1, \alpha}$ in x -variables. Next we integrate w in time, and the resulting $\hat{p}$ has regularity $C^{k+1, \alpha}$ in spatial variables. Thus we lose one derivative (in space) by performing this procedure. For this reason, we take a different approach.

We will construct solutions using a time-stepping procedure in the Lagrangian coordinates in physical space.

The system (2.1)-(2.4) can be written in Lagrangian coordinates as follows. We use (2.1) to write (2.2)-(2.3) as

$$(2.23) \quad D_t \mathbf{u}_g - fJ\mathbf{u}_g + fJ\mathbf{u} = 0.$$

Denote by $\phi(t, x)$ the flow map generated by $\mathbf{u}$. Then $\phi(t, x)$ satisfies

$$(2.24) \quad \begin{aligned} \partial_t \phi(t, x) &= \mathbf{u}(t, \phi(t, x)) \text{ in } \mathbb{R} \times \mathbb{R}^2, \\ \phi(0, x) &= x \text{ on } \mathbb{R}^2. \end{aligned}$$

From the standard ODE theory, $\phi(t, x+h) = \phi(t, x) + h$ for any $h \in \mathbb{Z}^2$. Equation (2.4) implies that for each t the map $\phi(t, \cdot)$ is Lebesgue measure preserving: $\phi(t, \cdot)_\# \mathcal{L}_{\mathbb{R}^2}^2 = \mathcal{L}_{\mathbb{R}^2}^2$, where the left-hand side denotes the push-forward of the Lebesgue measure $\mathcal{L}_{\mathbb{R}^2}^2$ by the map $\phi(t, \cdot)$. Express the geostrophic wind velocity in Lagrangian variables:

$$(2.25) \quad \mathbf{v}_g(t, x_0) = \mathbf{u}_g(t, \phi(t, x_0)) \text{ for } (t, x_0) \in \mathbb{R}_+ \times \mathbb{R}^2,$$

where $x_0 = (x_{01}, x_{02})$ is the spatial coordinate at time $t = 0$, and $\mathbf{u}_g = (u_{1,g}, u_{2,g})$ has been defined in (2.1). Then $\mathbf{v}_g$ is periodic with respect to $\mathbb{Z}^2$ since $\mathbf{u}_g$ is assumed to be periodic in its spatial arguments. Now, the equation (2.23) can be written as

$$(2.26) \quad \partial_t \mathbf{v}_g(t, x_0) - f(\phi(t, x_0))J\mathbf{v}_g(t, x_0) + f(\phi(t, x_0))J\partial_t \phi(t, x_0) = 0.$$

We also note that

$$(2.27) \quad \mathbf{v}_g(t, x_0) = f^{-1}(\phi_t(x_0))J\nabla p_t(\phi_t(x_0))$$

by (2.1), (2.25).

We thus have rewritten the system (2.1)–(2.4) in the following Lagrangian form: for $T > 0$, find a function $p \in C^1([0, T] \times \mathbb{T}^2)$ and a family of maps $\phi \in C^1([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$ such that:

$$(2.28) \quad \begin{aligned} &\text{Equation (2.26) holds on } (0, T) \times \mathbb{T}^2 \text{ with } \mathbf{v}_g \text{ defined by (2.27);} \\ &\phi(t, \cdot)_{\#} \mathcal{L}_{\mathbb{R}^2}^2 = \mathcal{L}_{\mathbb{R}^2}^2 \text{ for all } t \in (0, T), \phi(t, x + h) = \phi(t, x) + h \text{ for any } h \in \mathbb{Z}^2; \\ &p(0, x) = p_0(x), \quad \phi(0, x) = x \text{ on } \mathbb{R}^2, \end{aligned}$$

where $p_0(x)$ is a given periodic function. For sufficiently small $T > 0$, we will find a smooth solution of (2.28) such that $\phi(t, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism for each $t \in [0, T]$. This determines a solution of (2.1)–(2.4) by defining $\mathbf{u}(t, x) = (\partial_t \phi)(t, \phi_t^{-1}(x))$, where $\phi_t(\cdot) := \phi(t, \cdot)$.

The advantage of using Lagrangian coordinates for the iteration is the following. Note that in Eulerian coordinates, equation (2.23), written in terms of $(p, \mathbf{u})$ by using (2.1) to express $\mathbf{u}_g$, involves second spatial derivatives of p , see equation (2.20), where $Q = Q(D^2 p, Dp, x)$ by (2.19). In Lagrangian coordinates, writing equation (2.26) in terms of (p, ϕ) by using (2.27), we obtain a second order equation which does not involve second spatial derivatives of p . This allows us to define the time-stepping procedure in Lagrangian coordinates so that ϕ will be expressed in terms of Dp by solving the equations on individual timesteps. Then, noting that the measure-preserving property of ϕ implies $\det D\phi = 1$, we obtain a second-order equation of Monge-Ampere type for p , instead of the equation (2.22), in which the coefficients and the right-hand side depend on the third spatial derivatives of p , where we recall that $Q = Q(D^2 p, Dp, x)$. This allows us to avoid the loss of a derivative in the iteration.

In the rest of this section, we briefly describe the plan of the paper.

In Section 3 we define a time-stepping approximation of the system (2.28). For a time step size $\delta t > 0$ and $n = 0, 1, \dots, N$, a periodic function p_n on $\mathbb{R}^2$ is an approximation of $p(n\delta t, \cdot)$, and a measure-preserving map $F_{n+1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $F(\cdot + h) = F(\cdot) + h$ for any $h \in \mathbb{Z}^2$ is an approximation of the flow map connecting time steps $n\delta t$ and $(n+1)\delta t$. Then p_0 is given by the initial data. On the n -th step of iteration, assuming that p_n is known, we define equations for F_{n+1} and p_{n+1} . The equation for p_{n+1} is of Monge-Ampere type.

In Section 4, we show that for any p_n, p_{n+1} close enough to p_0 in the $C^{2,\alpha}$ norm, and δt small depending on p_0 , it is always possible to define a map F_{n+1} which is a $C^{1,\alpha}$ diffeomorphism and satisfies equation (3.7), and such a map is unique among all maps close enough to the identity.

In Sections 5, 6 we show that if the step size δt is sufficiently small, then for any p_n which is close to p_0 in the $C^{3,\alpha}$ -norm, we can find p_{n+1} close to p_0 in $C^{2,\alpha/2}$, such that the map defined in Section 4 is measure preserving. This is done by iteratively solving an equation of Monge-Ampere type using the Implicit Function Theorem. Also we establish an estimate which shows that if $p_0 \in C^{k+2,\alpha}$, $k \geq 1$, then p_{n+1} will remain bounded in $C^{k+2,\alpha}$ and will be $C\delta t$ -close to p_n in $C^{k+1,\alpha}$. This allows to define a time-stepping solution on time interval $[0, T]$, independent of (small) δt .

In Section 7, we take the limit of the time-stepping solutions as $\delta t \rightarrow 0$, and show that the limit is a smooth solution of system (2.1)–(2.4) in Lagrangian coordinates. Because of smoothness, there is no difficulty in transforming the solution to Eulerian coordinates.

In Section 8, we extend the above discussion to the SG shallow water case.

In Section 9, we prove uniqueness of solutions, both for the SG and the SG shallow water equations, under the assumptions of Theorem 2.1 and 2.2 respectively.

3. TIME-STEPPING IN LAGRANGIAN COORDINATES

In this section we define a time-stepping approximation of the system (2.28). We first give a heuristic motivation for the equations defined below.

In the following argument it will be more convenient to work with periodic functions on $\mathbb{R}^2$, instead of functions on $\mathbb{T}^2$.

Discretize the time at t_0 with step size δt . Then the time difference equation corresponding to (2.26) is

$$(3.1) \quad \begin{aligned} \mathbf{v}_g(t_0 + \delta t, x_0) - \mathbf{v}_g(t_0, x_0) - f(\phi(t_0, x_0))J\mathbf{v}_g(t_0, x_0)\delta t \\ + f(\phi(t_0, x_0))J(\phi(t_0 + \delta t, x_0) - \phi(t_0, x_0)) = 0. \end{aligned}$$

On the other hand, we have

$$(3.2) \quad R_{f(\phi(t_0, x_0))\delta t}\mathbf{v}_g(t_0, x_0) = \mathbf{v}_g(t_0, x_0) + f(\phi(t_0, x_0))J\mathbf{v}_g(t_0, x_0)\delta t + O(\delta t^2|\mathbf{v}_g|).$$

where

$$(3.3) \quad R_a = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}$$

is the matrix defining a rotation by angle a . In the later construction of approximate solutions, $\mathbf{v}_g$ will remain bounded in some smooth norm for a short time, So we can replace (3.1) with

$$(3.4) \quad \mathbf{v}_g(t_0 + \delta t, x_0) - R_{f\delta t}\mathbf{v}_g(t_0, x_0) + f(\phi(t_0, x_0))J(\phi(t_0 + \delta t, x_0) - \phi(t_0, x_0)) = 0,$$

where $R_{f\delta t} = R_{f(\phi(t_0, x_0))\delta t}$. Write the flow map from time t_0 to $t_0 + \delta t$ as F , then $\phi(t_0 + \delta t, x_0) = F \circ \phi(t_0, x_0)$. Write $x = \phi(t_0 + \delta t, x_0)$, then $\phi(t_0, x_0) = F^{-1}(x)$. With this notation, and recalling (2.25), equation (3.4) becomes

$$(3.5) \quad \mathbf{u}_g(t_0 + \delta t, x) - R_{f\delta t}\mathbf{u}_g(t_0, F^{-1}(x)) + f(F^{-1}(x))J(x - F^{-1}(x)) = 0,$$

where in $R_{f\delta t}$, the function f is evaluated at $F^{-1}(x)$. Recalling that $\mathbf{u}_g = f^{-1}J\nabla p$ and noting that $R_{f\delta t}J = JR_{f\delta t}$, we obtain

$$(3.6) \quad \begin{aligned} f^{-1}(x)J\nabla p(t_0 + \delta t, x) \\ - f^{-1}(F^{-1}(x))JR_{f\delta t}\nabla p(t_0, F^{-1}(x)) + f(F^{-1}(x))J(x - F^{-1}(x)) = 0. \end{aligned}$$

where $R_{f\delta t} = R_{f(F^{-1}(x))\delta t}$.

Let $t_0 = n\delta t$. Write $p_{n+1} = p((n+1)\delta t)$, $p_n = p(n\delta t)$, and F_{n+1} for the flow map connecting time step $n\delta t$ and $(n+1)\delta t$. Then we obtain from (3.6):

$$(3.7) \quad x + f^{-1}(x)f^{-1}(F_{n+1}^{-1}(x))\nabla p_{n+1}(x) = F_{n+1}^{-1}(x) + (f^{-2}R_{f\delta t}\nabla p_n)(F_{n+1}^{-1}(x)).$$

In the second term of the right-hand side, functions f and p_n are evaluated at $F_{n+1}^{-1}(x)$.

We require the map F_{n+1}^{-1} to be a measure preserving diffeomorphism of $\mathbb{T}^2$.

Next we will set up the iteration scheme based on the ideas described above. Let p_0 be the initial data. Then we define $p_1, p_2, \dots, p_N$ inductively as follows. Let $n \in \{0, 1, 2, \dots, N\}$, and assume a function p_n is given. We look for a function p_{n+1} and a

measure preserving map F_{n+1} such that (3.7) holds. Since we want F_{n+1} to be measure preserving, we take the gradient of both sides of (3.7), and collect terms involving DF_{n+1}^{-1} :

$$(3.8) \quad \begin{aligned} & (I + f^{-1}\nabla p_{n+1} \otimes \nabla(f^{-1})(x) + f^{-2}D^2p_{n+1})(x) + A_{n+1}(x) \\ &= [(I + f^{-1}\nabla p_n \otimes \nabla(f^{-1}) + f^{-2}D^2p_n)(F_{n+1}^{-1}(x)) + B_{n+1}(x)]DF_{n+1}^{-1}(x), \end{aligned}$$

where

$$(3.9) \quad \begin{aligned} A_{n+1}(x) &= [f^{-1}(F_{n+1}^{-1}(x)) - f^{-1}(x)]\nabla p_{n+1}(x) \otimes \nabla(f^{-1})(x) \\ &\quad + f^{-1}(x)(f^{-1}(F_{n+1}^{-1}(x)) - f^{-1}(x))D^2p_{n+1}(x), \end{aligned}$$

$$(3.10) \quad \begin{aligned} B_{n+1}(x) &= [(f^{-1}\nabla p_n)(F_{n+1}^{-1}(x)) - (f^{-1}\nabla p_{n+1})(x)] \otimes \nabla(f^{-1})(F_{n+1}^{-1}(x)) \\ &\quad + D[f^{-2}(R_{f\delta t} - I)\nabla p_n](F_{n+1}^{-1}(x)). \end{aligned}$$

Taking the determinant of both sides of (3.8), we see that $\det DF_{n+1}^{-1} \equiv 1$ if and only if

$$(3.11) \quad \begin{aligned} & \det[(I + f^{-1}\nabla p_{n+1} \otimes \nabla(f^{-1}) + f^{-2}D^2p_{n+1})(x) + A_{n+1}(x)] \\ &= \det[(I + f^{-1}\nabla p_n \otimes \nabla(f^{-1}) + f^{-2}D^2p_n)(F_{n+1}^{-1}(x)) + B_{n+1}(x)]. \end{aligned}$$

The number N will be defined below so that $N\delta t$ is small and $p_1, \dots, p_N$ are close to p_0 in the norms specified below. After (3.7)–(3.11) are solved for $n = 1, \dots, N$, we define the approximate solution $(p_{\delta t}, \phi^{\delta t})$ of (2.28) with step size δt to be $p_{\delta t}(t) = p_n$ if $t \in [n\delta t, (n+1)\delta t)$, the approximate flow map $\phi^{\delta t}(t) = F_n \circ F_{n-1} \circ \dots \circ F_1$, for $t \in [n\delta t, (n+1)\delta t)$.

In the following, to simplify the notation, we write $q(x) = p_{n+1}(x)$, $p(x) = p_n(x)$, and $F = F_{n+1}$, $A(x) = A_{n+1}(x)$, $B(x) = B_{n+1}(x)$ for the functions and maps used in (3.7)–(3.11). In the present notation, (3.7) becomes

$$(3.12) \quad x + f^{-1}(x)f^{-1}(F^{-1}(x))\nabla q(x) = F^{-1}(x) + (f^{-2}R_{f\delta t}\nabla p)(F^{-1}(x)).$$

Here, in the last term, all functions are evaluated at $F^{-1}(x)$. Equation (3.11) in the present notation becomes the following:

$$(3.13) \quad \begin{aligned} & \det[(I + f^{-1}\nabla q \otimes \nabla(f^{-1}) + f^{-2}D^2q)(x) + A(x)] \\ &= \det[(I + f^{-1}\nabla p \otimes \nabla(f^{-1}) + f^{-2}D^2p)(F^{-1}(x)) + B(x)], \end{aligned}$$

where the expressions of $A(x)$, $B(x)$ are given by (3.9), (3.10) with $p_n = p$, $p_{n+1} = q$, $F_{n+1} = F$.

In next two sections, for a given p which is close to p_0 in $C^{2,\alpha}$ and small $\delta t > 0$, we find q and F which satisfy (3.12), (3.13). Here $\alpha \in (0, 1)$ is fixed from now on.

4. CONSTRUCTION OF MAPS

Let $p_0 \in C^{3,\alpha}(\mathbb{T}^2)$ satisfy $\int_{\mathbb{T}^2} p_0(x)dx = 0$ and (2.5). In this section we show that, given p, q which are close to p_0 and small $\delta t > 0$, the map F^{-1} satisfying (3.12) exists and is invertible. For this we use the Implicit Function Theorem.

We continue to work with periodic functions on $\mathbb{R}^2$, instead of working directly on $\mathbb{T}^2$. Then, for $k = 0, 1, \dots$ and $\alpha \in (0, 1)$, we denote by $C^{k,\alpha}(\mathbb{T}^2)$ the space of functions $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ which are in $C^{k,\alpha}(\mathbb{R}^2)$ and $\mathbb{Z}^2$ -periodic:

$$C^{k,\alpha}(\mathbb{T}^2) = \{\varphi \in C^{k,\alpha}(\mathbb{R}^2) \mid \varphi(x + h) = \varphi(x) \text{ for all } h \in \mathbb{Z}^2\}.$$

Then $C^{k,\alpha}(\mathbb{T}^2)$ is a closed subspace of $C^{k,\alpha}(\mathbb{R}^2)$, thus $C^{k,\alpha}(\mathbb{T}^2)$ with norm $\|\varphi\|_{k,\alpha} := \|\varphi\|_{k,\alpha,\mathbb{R}^2}$ is a Banach space.

Let $U_1 \subset C^{2,\alpha}$ be the open subset defined by:

$$(4.1) \quad U_1 = \{p \in C^{2,\alpha} : I + f^{-1}D(f^{-1}Dp) > \frac{c_0}{2}I\}.$$

Also, we denote by $C^{k,\alpha}(\mathbb{T}^2; \mathbb{R}^2)$ the space of mappings $w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which are in $C^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$ and $\mathbb{Z}^2$ -periodic:

$$(4.2) \quad C^{k,\alpha}(\mathbb{T}^2; \mathbb{R}^2) = \{w \in C^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2) \mid w(x+h) = w(x) \text{ for all } x \in \mathbb{R}^2, h \in \mathbb{Z}^2\}.$$

We also consider mappings $w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with the following periodicity property:

$$C_p^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2) = \{w \in C_{loc}^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2) \mid w(x+h) = w(x) + h \text{ for all } h \in \mathbb{Z}^2\}.$$

Note that $C_p^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$ is not a subspace. We also note that

$$(4.3) \quad C_p^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2) = id + C^{k,\alpha}(\mathbb{T}^2; \mathbb{R}^2),$$

where id is the identity map $id(x) = x$ in $\mathbb{R}^2$. Indeed, $w(x+h) = w(x) + h$ for all $x \in \mathbb{R}^2, h \in \mathbb{Z}^2$ if and only if $v := w(x) - x$ is $\mathbb{Z}^2$ -periodic. The reason to introduce $C_p^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$ is the following: if p, q are periodic, and $\|p - q\|_{2,\alpha}$ and δt are small, then the map $z = F^{-1}$ solving (3.12) and close to id satisfies $z \in C_p^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$, see Lemma 4.2 below.

We first rewrite equation (3.12) as follows:

For fixed $p, q \in C^2(\mathbb{T}^2)$, and $\delta t \in (-1, 1)$, consider the map

$$(4.4) \quad \begin{aligned} \hat{Q}_{p,q,\delta t} &= \hat{Q} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ defined by} \\ \hat{Q}(x, w) &= x + f^{-1}(x)f^{-1}(w)\nabla q(x) - w - (f^{-2}R_{f\delta t}\nabla p)(w). \end{aligned}$$

Solving (3.12) for F , with given $p, q, \delta t$, is equivalent to solving

$$(4.5) \quad \hat{Q}_{p,q,\delta t}(x, z) = 0$$

for z for each $x \in \mathbb{R}^2$, then $F^{-1}(x) = z(x)$. Also, we note that for any $p \in C^{2,\alpha}(\mathbb{T}^2)$

$$(4.6) \quad \hat{Q}_{p,p,0}(x, x) = 0 \text{ for all } x \in \mathbb{R}^2,$$

which is obtained directly from (4.4) using that $R_0 = I$ in (3.3). Thus we expect that $z(x) - x$ is small if $\|p - q\|_{2,\alpha}$ and δt are small.

In the next lemma we use the set U_1 defined by (4.1):

Lemma 4.1. *For any $p_0 \in U_1$ there exists $\varepsilon > 0$ such that for any $p, q \in C^{2,\alpha}(\mathbb{T}^2)$ satisfying $\|p - p_0\|_{2,\alpha,\mathbb{R}^2} \leq \varepsilon$, $\|q - p_0\|_{2,\alpha,\mathbb{R}^2} \leq \varepsilon$, and any $\delta t \in (-\varepsilon, \varepsilon)$*

(i) $D_w \hat{Q}_{p,q,\delta t}(x, w) < -\frac{c_0}{4}I$ if $|x - w| < \varepsilon$;

(ii) For any $x \in \mathbb{R}^2$, the map $\hat{Q}_{p,q,\delta t}(x, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is injective on $B_\varepsilon(x)$.

Proof. We note first that for $q = p = p_0$ and $\delta t = 0$, we get for any $x \in \mathbb{R}^2$:

$$\begin{aligned} D_w \hat{Q}_{p_0,p_0,0}(x, x) &= -I - (f^{-1}\nabla p_0 \otimes \nabla(f^{-1}) + f^{-2}D^2 p_0)(x) \\ &= -I - (f^{-1}D(f^{-1}Dp_0))(x) < -\frac{c_0}{2}I. \end{aligned}$$

Now consider any $p, q, \delta t$ satisfying conditions of the Lemma, and any $x, w \in \mathbb{R}^2$ with $|x - w| < \varepsilon$. Then, assuming that $\varepsilon \in (0, 1)$, we have $\|p\|_{2,\alpha}, \|q\|_{2,\alpha} \leq \|p_0\|_{2,\alpha} + 1$, thus, from (4.4), we get:

$$\begin{aligned} |D_w \hat{Q}_{p_0, p_0, 0}(x, x) - D_w \hat{Q}_{p, q, \delta t}(x, w)| &\leq |D_w \hat{Q}_{p_0, p_0, 0}(x, x) - D_w \hat{Q}_{p, p_0, \delta t}(x, w)| \\ &\quad + |D_w \hat{Q}_{p, p_0, \delta t}(x, w) - D_w \hat{Q}_{p, q, \delta t}(x, w)| \\ &\leq C(\|p - p_0\|_{2,0} + \|q - p_0\|_{1,0} + |x - w|^\alpha + |\delta t|) \\ &\leq C\varepsilon^\alpha, \end{aligned}$$

where C depends on $\|f^{-1}\|_{C^{1,\alpha}(\mathbb{T}^2)}$ and $\|p_0\|_{C^{2,\alpha}(\mathbb{T}^2)}$, and C may be different in different occurrences. Thus, choosing ε small depending only on $\|f^{-1}\|_{C^{1,\alpha}(\mathbb{T}^2)}$ and $\|p_0\|_{C^{2,\alpha}(\mathbb{T}^2)}$, we get assertion (i) of the Lemma.

Now we prove assertion (ii) of the Lemma. For $w, \hat{w} \in B_\varepsilon(x)$ with $w \neq \hat{w}$ we have $\tau w + (1 - \tau)\hat{w} \in B_\varepsilon(x)$ for any $\tau \in [0, 1]$, and then denoting $\mathbf{e} := w - \hat{w}$, we get $\mathbf{e} \neq 0$ and thus

$$(\hat{Q}(x, w) - \hat{Q}(x, \hat{w})) \cdot \mathbf{e} = \int_0^1 (D_w \hat{Q}(x, \tau w + (1 - \tau)\hat{w}) \mathbf{e}) \cdot \mathbf{e} d\tau \leq -\frac{c_0}{4} |\mathbf{e}|^2 < 0.$$

□

Next we show that solutions $z = F^{-1}$ of (3.12), which are close to the identity map, lie in the set $C_p^{1,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$. We first note the property

$$(4.7) \quad \hat{Q}(x + k, w + h) = \hat{Q}(x, w) + k - h \text{ for any } x, w \in \mathbb{R}^2, h, k \in \mathbb{Z}^2,$$

which follows from (4.4).

Lemma 4.2. *For any $p_0 \in U_1$ there exists $\varepsilon > 0$ such that if $p, q \in C^{2,\alpha}(\mathbb{T}^2)$, $z : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\delta t \in (-\varepsilon, \varepsilon)$ satisfy (3.12) with $F^{-1} := z$, and also satisfy $\|p - p_0\|_{2,\alpha,\mathbb{R}^2} \leq \varepsilon$, $\|q - p_0\|_{2,\alpha,\mathbb{R}^2} \leq \varepsilon$, $\|z - id\|_{L^\infty(\mathbb{R}^2)} < \varepsilon$, $|\delta t| < \varepsilon$, then $z \in C_p^{1,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$.*

Proof. From (4.7), if $\hat{Q}(x, z(x)) = 0$, then $\hat{Q}(x + h, z(x) + h) = \hat{Q}(x, z(x)) = 0$ for any $x \in \mathbb{R}^2, h \in \mathbb{Z}^2$. Combined with the property $|z(x) - x| < \varepsilon$ and the injectivity of $\hat{Q}(x, \cdot)$ on the $B_\varepsilon(x)$ shown in Lemma 4.1(ii), we obtain $z(x + h) = z(x) + h$.

Finally, the fact $z \in C_{loc}^{1,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$ follows from the Implicit Function Theorem applied to the equation $\hat{Q}(x, w) = 0$, using nondegeneracy of $D_w \hat{Q}(x, z(x))$ for any x which follows from Lemma 4.1(i) since $|z(x) - x| < \varepsilon$, and using the regularity $\hat{Q} \in C_{loc}^{1,\alpha}(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{R}^2)$ which follows from (4.4) for $p, q \in C^{2,\alpha}$. □

Now we show that (3.12) has a solution $F^{-1} = z \in C_p^{1,\alpha/2}(\mathbb{R}^2; \mathbb{R}^2)$. For that, we use the Implicit Function Theorem in the following spaces. Define a map

$$(4.8) \quad Q : C^{3,\alpha}(\mathbb{T}^2) \times C^{2,\alpha/2}(\mathbb{T}^2) \times C_p^{1,\alpha/2}(\mathbb{R}^2; \mathbb{R}^2) \times (-1, 1) \rightarrow C^{1,\alpha/2}(\mathbb{T}^2; \mathbb{R}^2) \quad \text{by}$$

$$(4.9) \quad Q(p, q, z, \delta t)(x) = \hat{Q}_{p,q,\delta t}(x, z(x)).$$

Thus, Q is given by the expression:

$$(4.10) \quad Q(p, q, z, \delta t)(x) = x + f^{-1}(x) f^{-1}(z(x)) \nabla q(x) - z(x) - (f^{-2} R_{f\delta t} \nabla p)(z(x)).$$

The fact that Q in (4.10) acts into $C^{1,\alpha/2}(\mathbb{T}^2; \mathbb{R}^2)$ is seen as follows: the regularity $Q(p, q, z, \delta t)(\cdot) \in C_{loc}^{1,\alpha/2}(\mathbb{R}^2; \mathbb{R}^2)$ follows directly from the choice of spaces in the domain

of Q and the explicit expression (4.10). The $\mathbb{Z}^2$ -periodicity of $Q(p, q, z, \delta t)(\cdot)$ follows from the property

$$(4.11) \quad Q(p, q, z + h, \delta t)(x + k) = Q(p, q, z, \delta t)(x) + h - k \text{ for any } x \in \mathbb{R}^2, h, k \in \mathbb{Z}^2,$$

where (4.11) follows from (4.10) using $\mathbb{Z}^2$ -periodicity of f , p , q and property (4.3) for $z \in C_p^{1, \alpha/2}(\mathbb{R}^2; \mathbb{R}^2)$.

Given $(p, q, \delta t)$, solving (3.12) for F^{-1} is equivalent to solving

$$(4.12) \quad Q(p, q, z, \delta t) = 0$$

for z , then $F^{-1} = z$ solves (3.12). From (4.6) and (4.9), we have for any $p \in C^{3, \alpha}(\mathbb{T}^2)$

$$(4.13) \quad Q(p, p, id, 0) = 0,$$

where id is the identity map in $\mathbb{R}^2$. Then we will solve (4.12) for $z(\cdot)$ when $p \in U_1 \cap C^{3, \alpha}$, and when $\|p - q\|_{2, \alpha}$ and δt are small.

Since the set of functions $C_p^{1, \alpha}(\mathbb{R}^2; \mathbb{R}^2)$ is not a space, it is convenient to replace $z(\cdot)$ by $w(x) = z(x) - x$ in Q , since then $w \in C^{1, \alpha}(\mathbb{T}^2; \mathbb{R}^2)$ by (4.3). Thus we define

$$(4.14) \quad \begin{aligned} Q_1 : C^{3, \alpha}(\mathbb{T}^2) \times C^{2, \alpha/2}(\mathbb{T}^2) \times C^{1, \alpha/2}(\mathbb{T}^2; \mathbb{R}^2) \times (-1, 1) &\rightarrow C^{1, \alpha/2}(\mathbb{T}^2; \mathbb{R}^2) \\ \text{by } Q_1(p, q, w, \delta t) &= Q(p, q, w + id, \delta t), \end{aligned}$$

that is

$$(4.15) \quad \begin{aligned} Q_1(p, q, w, \delta t)(x) &= f^{-1}(x) f^{-1}(x + w(x)) \nabla q(x) - w(x) \\ &\quad - (f^{-2} R_{f \delta t} \nabla p)(x + w(x)). \end{aligned}$$

Expressing equation (4.12) in terms of Q_1 , we see that solving (3.12) for F , for a given $p, q, \delta t$, is equivalent to solving

$$(4.16) \quad Q_1(p, q, w, \delta t) = 0$$

for w , then $F = (id + w)^{-1}$. From (4.13), for any $p \in C^{3, \alpha}(\mathbb{T}^2)$

$$(4.17) \quad Q_1(p, p, w_0, 0) = 0,$$

where w_0 is the zero map in $\mathbb{R}^2$, i.e. $w_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by $w_0(x) = 0$. Then we will solve (4.16) for $w(\cdot)$ with small $\|w\|_{2, \alpha}$, when $p \in U_1 \cap C^{3, \alpha}$, and when $\|p - q\|_{2, \alpha/2}$ and δt are small.

Now, in order to solve (4.16), we will apply the Implicit Function Theorem in spaces given in (4.14), near the background solution given in (4.17).

For that we first note that the higher regularity of p implies that the map Q_1 is smooth:

Lemma 4.3. *The map*

$$(4.18) \quad Q_1 : C^{3, \alpha}(\mathbb{T}^2) \times C^{2, \alpha/2}(\mathbb{T}^2) \times C^{1, \alpha/2}(\mathbb{T}^2; \mathbb{R}^2) \times (-1, 1) \rightarrow C^{1, \alpha/2}(\mathbb{T}^2; \mathbb{R}^2),$$

defined by (4.10), is continuously Frechet-differentiable.

Proof. Lemma follows directly from the expression (4.10) and Lemma 10.1, proved in the Appendix. $\square$

Now we prove existence of solution of (4.16) near $(p, p, w_0, 0)$. For $p_0 \in C^{3,\alpha}(\mathbb{T}^2)$ and $\varepsilon > 0$, denote:

$$\begin{aligned} V_\varepsilon(p_0) &:= \{(p, q) \in C^{3,\alpha}(\mathbb{T}^2) \times C^{2,\alpha/2}(\mathbb{T}^2) \mid \|p - p_0\|_{3,\alpha}, \|q - p_0\|_{2,\alpha/2} < \varepsilon\} \\ (4.19) \quad &\subset C^{3,\alpha}(\mathbb{T}^2) \times C^{2,\alpha/2}(\mathbb{T}^2); \\ W_\varepsilon &:= \{w \in C^{1,\alpha/2}(\mathbb{T}^2; \mathbb{R}^2) \mid \|w\|_{1,\alpha/2} < \varepsilon\} \subset C^{1,\alpha/2}(\mathbb{T}^2; \mathbb{R}^2). \end{aligned}$$

Lemma 4.4. *For any $p_0 \in U_1 \cap C^{3,\alpha}(\mathbb{T}^2)$ there exist $\varepsilon_1, \varepsilon_2 > 0$ such that for any $(p, q, \delta t) \in V_{\varepsilon_1}(p_0) \times (-\varepsilon_1, \varepsilon_1)$ there exists a unique $w \in W_{\varepsilon_2}$ such that $(p, q, w, \delta t)$ satisfy (4.16). The map $\mathcal{G} : V_{\varepsilon_1}(p_0) \times (-\varepsilon_1, \varepsilon_1) \rightarrow W_{\varepsilon_2}$, defined by $\mathcal{G}(p, q, \delta t) = w$, is continuously Frechet-differentiable.*

Proof. This follows directly from the Implicit Function Theorem in Banach spaces. Indeed, by Lemma 4.3, the map Q_1 is continuously Frechet-differentiable, and (4.17) holds. Using (4.15), we find that the linear map

$$(4.20) \quad D_w Q_1(p_0, p_0, w_0, 0) : C^{1,\alpha/2}(\mathbb{T}^2; \mathbb{R}^2) \rightarrow C^{1,\alpha/2}(\mathbb{T}^2; \mathbb{R}^2)$$

is given, for $h \in C^{1,\alpha/2}(\mathbb{T}^2; \mathbb{R}^2)$, by

$$\begin{aligned} (D_w Q_1(p_0, p_0, w_0, 0))h &= (-I - (f^{-1} \nabla p_0 \otimes \nabla(f^{-1}) + f^{-2} D^2 p_0))h \\ &= (-I - (f^{-1} D(f^{-1} D p_0)))h. \end{aligned}$$

Since the matrix $(-I - (f^{-1} D(f^{-1} D p_0)))(x)$ is nondegenerate for each $x \in \mathbb{R}^2$ (which holds because $p \in U_1$), and since $-I - (f^{-1} D(f^{-1} D p_0)) \in C^{1,\alpha}(\mathbb{T}^2; \mathbb{R}^2)$, it follows that the map (4.20) is a linear isomorphism. Now the lemma follows from the Implicit Function Theorem. $\square$

Remark 4.5. *From Lemma 4.4 and (4.17), it follows that*

$$\mathcal{G}(p, p, 0) = w_0 \text{ for all } (p, p) \in V_{\varepsilon_1}(p_0).$$

Then, since $\mathcal{G} : V_{\varepsilon_1}(p_0) \times (-\varepsilon_1, \varepsilon_1) \rightarrow W_{\varepsilon_2}$ is continuous, it follows that for any $\varepsilon'_2 \in (0, \varepsilon_2)$ there exists $\varepsilon'_1 \in (0, \varepsilon_1)$ such that $\mathcal{G}(p, q, \delta t) \in W_{\varepsilon'_2}$ if $(p, q, \delta t) \in V_{\varepsilon'_1}(p_0) \times (-\varepsilon'_1, \varepsilon'_1)$.

Lemma 4.6. *For any $p_0 \in U_1 \cap C^{3,\alpha}(\mathbb{T}^2)$ there exists $\varepsilon'_1 \in (0, \varepsilon_1]$ such that for each $(p, q, \delta t) \in V_{\varepsilon'_1}(p_0) \times (-\varepsilon'_1, \varepsilon'_1)$ the map $z = id + \mathcal{G}(p, q, \delta t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism.*

Proof. Let ε_1 and ε_2 be sufficiently small for the map $\mathcal{G}$ to be defined by Lemma 4.4.

Then, by (4.8) and (4.14)

$$(4.21) \quad \hat{Q}_{p,q,\delta t}(x, z(x)) = 0 \text{ for all } x \in \mathbb{R}^2.$$

We show that $z(\mathbb{R}^2) = \mathbb{R}^2$ for each $(p, q, \delta t) \in V_{\varepsilon_1}(p_0) \times (-\varepsilon_1, \varepsilon_1)$.

First we note that, after possibly reducing ε_1 , we have that $z(\mathbb{R}^2)$ is an open set. Indeed, using Remark 4.5, we find that for any $\varepsilon'_2 \in (0, \varepsilon_2)$ there exists $\varepsilon'_1 \in (0, \varepsilon_1)$ such that, if $(p, q, \delta t) \in V_{\varepsilon'_1}(p_0) \times (-\varepsilon'_1, \varepsilon'_1)$, we get

$$(4.22) \quad \|z - id\|_{L^\infty(\mathbb{R}^2)} = \|\mathcal{G}(p, q, \delta t)\|_{L^\infty(\mathbb{R}^2)} \leq \varepsilon'_2.$$

Then choosing ε'_2 smaller than $\varepsilon/2$ in Lemma 4.1(i) and choosing the corresponding ε'_1 , we get $D_w \hat{Q}_{p,q,\delta t}(x, z(x)) < -\frac{\varepsilon_0}{4} I$ for each $x \in \mathbb{R}^2$ if $(p, q, \delta t) \in V_{\varepsilon'_1}(p_0) \times (-\varepsilon'_1, \varepsilon'_1)$. Then, fixing $\hat{x} \in \mathbb{R}^2$, we obtain by the Implicit Function Theorem that there exists a

neighborhood $B_r(z(\hat{x}))$ of $z(\hat{x})$, where $r > 0$, and a $C^{1,\alpha/2}$ map $g : B_r(z(\hat{x})) \rightarrow \mathbb{R}^2$ with $g(z(\hat{x})) = \hat{x}$, such that

$$(4.23) \quad \hat{Q}_{p,q,\delta t}(g(v), v) = 0 \text{ for all } v \in \mathcal{N}.$$

Since

$$\|g(z(\hat{x})) - z(\hat{x})\| = \|\hat{x} - z(\hat{x})\| \leq \varepsilon'_2 < \varepsilon/2,$$

then reducing r , we get $\|g(v) - v\| < \varepsilon$ for each $v \in B_r(z(\hat{x}))$. Then, by (4.21), (4.22), (4.23) and Lemma 4.1(ii), it follows that $v = z(g(v))$, i.e. $B_r(z(\hat{x})) \subset z(\mathbb{R}^2)$. Thus, the set $z(\mathbb{R}^2)$ is open. Also, from now on we set $\hat{\varepsilon}_1$ to be equal to ε'_1 chosen above.

Next, we show that the set $z(\mathbb{R}^2)$ is closed. If $z(x_i) \rightarrow \hat{v} \in \mathbb{R}^2$ for some points $x_i \in \mathbb{R}^2$, then from (4.22) it follows that there exists a positive N such that $x_i \in B_{2\varepsilon'_2}(\hat{v})$ for all $i > N$. Thus there exists a convergent subsequence $x_{i_j} \rightarrow \hat{x} \in \mathbb{R}^2$. Since $z(\cdot)$ is continuous, then $z(\hat{x}) = \hat{v}$, thus $z(\mathbb{R}^2)$ is closed.

Now, $z(\mathbb{R}^2)$ is an open, closed, and non-empty set, thus $z(\mathbb{R}^2) = \mathbb{R}^2$. Also, by Lemma 4.1(ii), $z(\cdot)$ is injective on $\mathbb{R}^2$. Thus the map $z^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is uniquely defined. Also, locally this map is determined by the Implicit Function Theorem as we discussed above: $z^{-1} = g$ locally, where $g(\cdot)$ is given by (4.23). Thus $z^{-1} \in C_{loc}^{1,\alpha/2}$. $\square$

5. SOLVING THE ITERATION EQUATIONS

Let $\hat{\varepsilon}_1, q, p, \delta t$ be as in Lemma 4.6. Then we can define the map $F(x) = (id + \mathcal{G}(p, q, \delta t))^{-1}$, so that $F^{-1}(x) = id + \mathcal{G}(p, q, \delta t)$. Then $(q, p, F^{-1}, \delta t)$ satisfy equation (3.12) by Lemma 4.4 and (4.15).

To make F^{-1} measure preserving, we solve equation (3.13), with $F^{-1} = id + \mathcal{G}(p, q, \delta t)$, for q . We will use the Implicit Function Theorem in the setting described below.

Lemma 5.1. *Let $p_0 \in U_1$. There exists $\varepsilon_3 \in (0, \hat{\varepsilon}_1)$, such that for any $(p, q) \in V_{\varepsilon_3}(p_0)$, $|\delta t| < \varepsilon_3$, one has $\|A\|_0, \|B\|_0 < \frac{c_0}{8}$. Here $A(x), B(x)$ are given in (3.9), (3.10) with $p = p_n$, $q = p_{n+1}$, and $F_{n+1}^{-1} = id + \mathcal{G}(p, q, \delta t)$.*

Proof. Let $0 < \varepsilon < \hat{\varepsilon}_1$ be small and assume $(p, q) \in V_\varepsilon(p_0)$. We first estimate the matrix A . From (3.9), for some constant C which depends only on f we get:

$$\|A\|_0 \leq C\|q\|_{2,\alpha/2}\|\mathcal{G}\|_0 \leq C(\|q_0\|_{2,\alpha/2} + 1)\|\mathcal{G}\|_0.$$

Next we can write the expression of B in (3.10) as

$$\begin{aligned} B = & [(f^{-1}\nabla(p - q))(F^{-1}(x)) + (f^{-1}\nabla q)(F^{-1}(x)) - (f^{-1}\nabla q)(x)] \otimes \nabla(f^{-1})(F^{-1}(x)) \\ & + D[f^{-2}(R_{f\delta t} - I)\nabla p](F^{-1}(x)). \end{aligned}$$

Since one has $\|q - p\|_{2,\alpha/2} \leq \varepsilon_3$, $|\delta t| \leq \varepsilon_3$, one can estimate

$$\begin{aligned} \|B\|_0 & \leq C[\|p - q\|_{C^1} + \|f\|_{C^1}\|q\|_{C^2}\|\mathcal{G}\|_0 + |\delta t|\|p\|_{C^2}], \\ & \leq C[\varepsilon_3 + (\|p_0\|_{C^2} + 1)(\|\mathcal{G}\|_0 + \varepsilon_3)]. \end{aligned}$$

Now from Remark 4.5, $\|\mathcal{G}(p, q, \delta t)\|_0$ can be made as small as we want as long as $(p, q) \in V_{\varepsilon_3}(p_0)$ and $|\delta t| < \varepsilon_3$ with ε_3 chosen small enough. It follows that, as long as ε is chosen small enough, we can make $\|A\|_0, \|B\|_0 < \frac{c_0}{8}$. $\square$

Denote

$$C_0^{k,\alpha} := \{\varphi \in C^{k,\alpha}(\mathbb{T}^2) : \int_{[0,1]^2} \varphi(x) dx = 0\}.$$

For the rest of this section, we fix $p_0 \in U_1 \cap C^{k+2,\alpha}$ for some $k \geq 2$.

Given p near p_0 and small δt , we solve the equation (3.13) (with F defined by $F^{-1} = id + \mathcal{G}(p, q, \delta t)$) for q using the Implicit Function Theorem. Consider the following open subset U_2 of $C_0^{2,\alpha/2}$

$$(5.1) \quad U_2 = \{w \in C_0^{2,\alpha/2} : I + f^{-1}D(f^{-1}Dw) > \frac{c_0}{2}, \|w - p_0\|_{2,\alpha/2} < \varepsilon_3\}.$$

where ε_3 is chosen in Lemma 5.1. Let $\tilde{U}_2 \subset C_0^{3,\alpha}$ be defined similarly, namely

$$(5.2) \quad \tilde{U}_2 = \{w \in C_0^{3,\alpha} : I + f^{-1}D(f^{-1}Dw) > \frac{c_0}{2}, \|w - p_0\|_{3,\alpha} < \varepsilon_3\}.$$

Lemma 5.1 implies that

$$(5.3) \quad (I + f^{-1}\nabla p \otimes \nabla(f^{-1}) + f^{-2}D^2p) \circ (id + \mathcal{G}(p, q, \delta t)) + B \geq \frac{c_0}{4}$$

for all $(q, p, \delta t) \in U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3)$,

where B is as in Lemma 5.1. Also, by (4.19),

$$(5.4) \quad U_2 \times \tilde{U}_2 \subset V_{\varepsilon_3}.$$

Then we can define the following map:

$$(5.5) \quad P : U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3) \rightarrow C_0^{0,\alpha/2} \quad \text{by}$$

$$P(q, p, \delta t) = \frac{\det[I + f^{-1}\nabla q \otimes \nabla(f^{-1}) + f^{-2}D^2q + A]}{\det[(I + f^{-1}\nabla p \otimes \nabla(f^{-1}) + f^{-2}D^2p) \circ (id + \mathcal{G}(p, q, \delta t)) + B]} - 1,$$

where A and B are as in Lemma 5.1.

Lemma 5.2. *Map (5.5) has the following properties:*

- (i) $\int_{[0,1]^2} (P(q, p, \delta t))(x) dx = 0$ for any $(q, p, \delta t) \in U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3)$. Thus $P(q, p, \delta t) \in C_0^{0,\alpha/2}$, which means that P acts in the following spaces:
- (5.6) $P : U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3) \rightarrow C_0^{0,\alpha/2}$
- (ii) $(q, p, \delta t) \in U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3)$ satisfies equations (3.12), (3.13) with $F^{-1} = id + \mathcal{G}(p, q, \delta t)$ if and only if $P(q, p, \delta t) = 0$ on $\mathbb{T}^2$.
- (iii) P is continuously Frechet differentiable in the spaces given in (5.5), or equivalently in (5.6).

Proof. First we show P maps into $C_0^{0,\alpha/2}$.

Fix $(q, h, \delta t) \in U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3)$. From (3.8) with $p_n = p$, $p_{n+1} = q$, $F_{n+1}^{-1} = id + \mathcal{G}(p, q, \delta t)$, one sees that the right hand side of (5.5) at $x \in \mathbb{R}^2$ is exactly $\det D_x(id + (\mathcal{G}(p, q, \delta t)(x))) - 1$. Denote $G := id + \mathcal{G}(h, q, \delta t)$. Then $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism by Lemma 4.6, and the right hand side of (5.5) is $\det DG(x) - 1$. Then we calculate, changing variables:

$$\int_{[0,1]^2} \det DG(x) dx = \int_{G([0,1]^2)} dy = \int_{[0,1]^2} dx = 1,$$

where the second equality follows from $\mathbb{Z}^2$ -periodicity of $G - id = \mathcal{G}(h, q, \delta t)$, and from the fact that $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism, see Lemma 10.5 (applied now with $h \equiv 1$). This completes the proof of assertion (i) of the Lemma.

Assertion (ii) of the Lemma follows from Lemma 4.4 and (5.5).

Now we prove assertion (iii) of the Lemma.

Using (5.3), Lemma 10.2 and Corollary 10.3 we see that it is sufficient to show that for any $(i, j) \in \{1, 2\}^2$, the following maps acting in the spaces $U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3) \rightarrow C^{0, \alpha/2}$ are continuously Frechet differentiable:

$$(5.7) \quad (p, q, \delta t) \mapsto \delta_{ij} + f^{-1} \partial_i p \cdot \partial_j (f^{-1}) + f^{-2} \partial_{ij} q + A_{ij},$$

$$(5.8) \quad (p, q, \delta t) \mapsto \delta_{ij} + [f^{-1} \partial_i p \cdot \partial_j (f^{-1}) + f^{-2} \partial_{ij} p](id + \mathcal{G}(p, q, \delta t)) + B_{ij}.$$

Here A_{ij} and B_{ij} are elements of the matrices $A(x)$ and $B(x)$ which given in (3.9), (3.10) with $p = p_n$, $q = p_{n+1}$, and $F_{n+1}^{-1} = id + \mathcal{G}(p, q, \delta t)$. We now show differentiability of maps (5.7), (5.8). From Lemma 10.4 with Lemma 4.4, the terms $f^{-1}(id + \mathcal{G}(p, q, \delta t))$ and $\nabla(f^{-1})(id + \mathcal{G}(p, q, \delta t))$ are Frechet differentiable, where we include the terms in expressions of A and B . Then by Lemma 10.2, one can see that the mapping (5.7) is Frechet differentiable. From Lemma 10.1(ii), the terms $D^2 p(id + \mathcal{G}(p, q, \delta t))$ are also differentiable. Then we obtain differentiability of the map (5.8). $\square$

Now we will show that the partial Frechet derivative $D_q P(p_0, p_0, 0) : C_0^{2, \alpha/2} \rightarrow C_0^{0, \alpha/2}$ is invertible.

First we can calculate from (4.15), (4.16) and Lemma 4.4:

$$(5.9) \quad D_q \mathcal{G}(p_0, p_0, 0) h_1 = [I + f^{-2} D^2 p_0 + f^{-1} \nabla p_0 \otimes \nabla(f^{-1})]^{-1} (f^{-2} \nabla h_1).$$

Then by explicit calculation, we find that $D_q P$ is

$$(5.10) \quad \begin{aligned} D_q P(p_0, p_0, 0) : C_0^{2, \alpha/2} &\rightarrow C_0^{0, \alpha/2} \\ h &\mapsto L(h), \end{aligned}$$

where

$$(5.11) \quad L(h) = \frac{\sum_{i,j=1}^2 M_{ij} [f^{-2} \partial_{ij} h - \partial_j (f^{-1} \nabla(f^{-1} \partial_i p_0)) \cdot (D_q \mathcal{G}(p_0, p_0, 0) h)]}{\det(I + f^{-1} D(f^{-1} D p_0))}.$$

Here $M = M_{ij}$ is the cofactor matrix of $I + f^{-1} D(f^{-1} D p_0)$, which is strictly positive definite due to (5.2). Notice we already computed $D_q \mathcal{G}(p_0, p_0, 0)$ in (5.9).

Remark 5.3. Note that the operator (5.11) acts in spaces given in (5.10), i.e. that $\int_{[0,1]^2} (L(h))(x) dx = 0$ for any $h \in C_0^{2, \alpha/2}$. This follows from (5.6), since $L = D_q P(p_0, p_0, 0)$.

Next we argue the linear operator L defined above is invertible and the inverse is a bounded linear operator. First we observe that L can be put in the form

$$L(h) = a_{ij} \partial_{ij} h + b_i \partial_i h.$$

with coefficients $a_{ij}, b_i \in C^{\alpha/2}(\mathbb{T}^2)$, with the norms depending only on $\|p_0\|_{C^{3, \alpha}}$ and $\frac{1}{c_0}$. Also it follows from (5.11) that L is uniformly elliptic, precisely

$$a_{ij} = \frac{M_{ij}}{f^2 \det(I + f^{-1} D(f^{-1} D p_0))}.$$

Thus ellipticity follows from (5.2) and regularity of p_0, f, f^{-1} .

Now invertibility of $D_q P(p_0, p_0, 0) : C_0^{2,\alpha/2} \rightarrow C_0^{0,\alpha/2}$ follows from the following lemma.

Lemma 5.4. *Let $L(h) = a_{ij}\partial_{ij}h + b_i\partial_i h$ be a uniformly elliptic operator on $\mathbb{T}^2$, with coefficients $a_{ij}, b_i \in C^\alpha(\mathbb{T}^2)$. Suppose that coefficients a_{ij}, b_i satisfy the following additional property: $L(h) \in C_0^\alpha(\mathbb{T}^2)$ for any $h \in C_0^{2,\alpha}(\mathbb{T}^2)$. Then $L : C_0^{2,\alpha/2} \rightarrow C_0^{0,\alpha/2}$ is an isomorphism.*

Proof. The injectivity follows from the strong maximum principle. Indeed, if $L(h) = 0$ for some $h \in C^{2,\alpha}$ then, by the strong maximum principle, h must be a constant. Since $h \in C_0^{2,\alpha}$, i.e. $\int_{[0,1]^2} h dx = 0$, this constant must be zero.

To show surjectivity, we use the method of continuity. We consider the following family of operators:

$$(5.12) \quad \begin{aligned} L_t : C_0^{2,\alpha} &\rightarrow C_0^{0,\alpha} \text{ with } t \in [0, 1] \\ L_t(h) &= (1-t)\Delta h + tL(h). \end{aligned}$$

When $t = 0$, $L_0 = \Delta$. The equation $\Delta h = k$ has a unique solution in $C_0^{2,\alpha}$ with any $k \in C_0^{0,\alpha}$. Uniqueness is again the result of the strong maximum principle. Existence can be obtained by minimizing the functional $I[v] = \int_{\mathbb{T}^2} \frac{1}{2} |\nabla v|^2 + kv$ over the space $H_0^1(\mathbb{T}^2) := \{v \in H_{loc}^1(\mathbb{R}^2) : v \text{ is } \mathbb{Z}^2\text{-periodic, and } \int_{\mathbb{T}^2} v = 0\}$.

By Theorem 5.20 in [11], to see that L_1 is surjective, we just have to show the estimate

$$(5.13) \quad \|h\|_{2,\alpha} \leq C \|L_t h\|_{0,\alpha} \quad \text{for all } t \in [0, 1] \text{ and } h \in C_0^{2,\alpha}.$$

By the Schauder estimates, we have

$$(5.14) \quad \|h\|_{2,\alpha} \leq C(\|h\|_0 + \|L_t h\|_{0,\alpha}).$$

Here C depends on the C^α norm of the coefficients and the ellipticity constant of operator L . Both are independent of t . So we just need to show

$$(5.15) \quad \|h\|_0 \leq C \|L_t h\|_{0,\alpha} \quad h \in C_0^{2,\alpha} \text{ and } \forall t \in [0, 1].$$

We use compactness and argue by contradiction.

If (5.15) were false, then for any $n \geq 1$, there exists $t_n \in [0, 1]$, $h_n \in C_0^{2,\alpha}$, such that $\|h_n\|_0 \geq n \|L_{t_n} h_n\|_{0,\alpha}$. After normalization, we can assume $\|h_n\|_0 \equiv 1$ and $\|L_{t_n} h_n\|_{0,\alpha} \rightarrow 0$. By the Schauder estimates, h_n is bounded in $C^{2,\alpha}$. So up to a subsequence, we can assume $t_n \rightarrow t_* \in [0, 1]$, $h_n \rightarrow h_*$ in C^2 , and $h_* \in C_0^{2,\alpha}$. Then we will have $L_{t_*} h_* = 0$. By the strong maximum principle, we have $h_* \equiv 0$. On the other hand, because $h_n \rightarrow h_*$ uniformly, we have $\|h_*\|_0 = 1$. This is a contradiction. $\square$

Hence we can conclude the following:

Proposition 5.5. *There exist $\varepsilon_4, \varepsilon_5 \in (0, \varepsilon_3]$ with $\varepsilon_5 \leq \varepsilon_4$, such that for any $p \in C_0^{3,\alpha}(\mathbb{T}^2)$ with $\|p - p_0\|_{3,\alpha} < \varepsilon_5$ and $\delta t \in (-\varepsilon_5, \varepsilon_5)$, there exists a unique $q \in C^{2,\alpha/2}(\mathbb{T}^2)$ which solves (3.13) with $F^{-1} = \text{id} + \mathcal{G}(p, q, \delta t)$ and satisfies $\|q - p_0\|_{2,\alpha/2} < \varepsilon_4$.*

Thus, denoting $q := \mathcal{H}(p, \delta t)$ and $U_3 = \{p \in C_0^{3,\alpha}(\mathbb{T}^2) : \|p - p_0\|_{3,\alpha} < \varepsilon_5\}$, we obtain a map

$$\mathcal{H} : U_3 \times (-\varepsilon_5, \varepsilon_5) \rightarrow U_2,$$

such that for any $(p, \delta t) \in U_3 \times (-\varepsilon_5, \varepsilon_5)$, defining $q = \mathcal{H}(p, \delta t)$ and $F^{-1} = id + \mathcal{G}(p, \mathcal{H}(p, \delta t), \delta t)$, we get a solution $(p, q, F^{-1}, \delta t)$ of (3.13).

Proof. This follows from Lemma 5.2(ii), and (5.10), (5.11) with Lemma 5.4 by the Implicit Function Theorem. $\square$

Remark 5.6. If p_0 is chosen in a compact subset of $C^{3,\alpha}(\mathbb{T}^2)$, one can see that such a choice of ε_5 is actually uniform. In particular, this choice of ε_5 is uniform on any bounded subset in $C^{4,\alpha}$.

We also prove the following lemma which will be used below:

Lemma 5.7. If ε_4 , and hence ε_5 , are sufficiently small, then $C(x) \geq \frac{c_0}{4}$ for any p and δt as in Proposition 5.5 and $q = \mathcal{H}(p, \delta t)$. Here $C(x)$ is:

$$(5.16) \quad \begin{aligned} C(x) = & I + \int_0^1 D[f^{-2}Dp]((1-\theta)F^{-1}(x) + \theta x)d\theta \\ & - f^{-1}(x) \int_0^1 \nabla(f^{-1})((1-\theta)F^{-1}(x) + \theta x)d\theta \otimes \nabla p, \\ & \text{with } F^{-1} = id + \mathcal{G}(p, q, \delta t). \end{aligned}$$

Proof. We start by observing that

$$\begin{aligned} C(x) - (I + f^{-1}D(f^{-1}Dp))(x) = & \int_0^1 [D(f^{-2}Dp)((1-\theta)F^{-1}(x) + \theta x) - D(f^{-2}Dp)(x)]d\theta \\ & - f^{-1}(x) \int_0^1 [\nabla(f^{-1})((1-\theta)F^{-1}(x) + \theta x) - \nabla(f^{-1})(x)]d\theta \otimes \nabla p. \end{aligned}$$

Hence, recalling $F^{-1} = id + \mathcal{G}(p, q, \delta t)$, we get:

$$\begin{aligned} |C(x) - (I + f^{-1}D(f^{-1}Dp))(x)| & \leq \|D^2(f^{-2}Dp)\|_0 \|\mathcal{G}(p, q, \delta t)\|_0 \\ & \quad + \|f^{-1}\|_0 \|D^2(f^{-1})\|_0 \|\mathcal{G}(p, q, \delta t)\|_0 \\ & \leq (10\|f^{-2}\|_{C^2}(\|p_0\|_{3,\alpha} + \varepsilon_4) + \|f^{-1}\|_0 \|\nabla(f^{-1})\|_0) \|\mathcal{G}(p, q, \delta t)\|_0. \end{aligned}$$

By Remark 4.5, we can make $\|\mathcal{G}(p, q, \delta t)\|_0$ as small as we wish as long as we choose ε_4 small. In particular, we can make the above expression less than $\frac{c_0}{4}$. Now, since $p \in \tilde{U}_2$, we know that $I + f^{-1}D(f^{-1}Dp) > \frac{c_0}{2}$, and it follows that $C(x) \geq \frac{c_0}{4}$. $\square$

From now on, we fix ε_4 and ε_5 such that Lemma 5.7 holds.

6. ESTIMATES OF SOLUTIONS ON TIME STEPS

Suppose the initial data satisfies $p_0 \in C_0^{k+2,\alpha}$ and $I + f^{-1}D(f^{-1}Dp) > c_0$.

Fix $\delta t \in (0, \varepsilon_5)$, and define $p_1, p_2, \dots$ as following. Assume that, for $n = 0, 1, \dots$, we have defined $p_n \in U_3$, where U_3 is from Proposition 5.5. Then we can define $p_{n+1} := \mathcal{H}(p_n, \delta t)$. Thus we have $p_{n+1} \in U_2$, and by Lemma 4.6 and (5.4) with ε_3 determined by Lemma 5.1, we can define the flow map F_{n+1} which is a diffeomorphism and solves

$$(6.1) \quad x + f^{-1}(x)f^{-1}(F_{n+1}^{-1}(x))\nabla p_{n+1}(x) = F_{n+1}^{-1}(x) + (f^{-2}R_{f\delta t}\nabla p_n)(F_{n+1}^{-1}(x));$$

$$(6.2) \quad \det DF_{n+1} = 1,$$

where (6.2) follows from (3.13) written in the form (3.11). By the definition of U_2 , we have $|A_{n+1}(x)| \leq \frac{c_0}{4}$. Hence $I + f^{-1}D(f^{-1}Dp_{n+1}) + A_{n+1} > \frac{c_0}{4}$.

In order to continue the process, we need to show that $p_{n+1} \in U_3$. We will show that this is true if $n\delta t$ is sufficiently small, i.e. if $n\delta t \leq T$, where $T > 0$ does not depend on δt . In order to show this, we establish some estimates for the approximate solutions p_n .

Lemma 6.1. *Let $p_{n+1} = \mathcal{H}(p_n, \delta t)$ with $\|p_n - p_0\|_{3,\alpha} < \varepsilon_5$, $|\delta t| < \varepsilon_5$. Then*

$$(6.3) \quad \|p_{n+1}\|_{k+2,\alpha} \leq C_0(\|p_n\|_{k+2,\alpha}).$$

Proof. First we show that

$$(6.4) \quad \|p_{n+1}\|_{k+2,\alpha/2} \leq C_{13}(\|p_n\|_{k+2,\alpha/2}).$$

This follows from differentiating (3.13). Indeed, we have from our assumption and the definition of the map $\mathcal{H}$ that $\|p_{n+1}\|_{2,\alpha/2} \leq \|p_0\|_{2,\alpha/2} + 1$. Also it follows from Lemma 4.4 that $\|\mathcal{G}(p_n, p_{n+1}, \delta t)\|_{1,\alpha/2} \leq 1$. Now by differentiating (3.13) we see that Dp_{n+1} solves an elliptic equation with the main coefficients given by M_{ij} , the entries of the cofactor matrix of $I + f^{-1}\nabla p_{n+1} \otimes \nabla(f^{-1}) + f^{-2}D^2p_{n+1}(x) + A_{n+1}(x)$. The resulting operator is uniformly elliptic, with ellipticity constant depending on c_0 and $\|p_0\|_{2,\alpha/2}$, because, by Lemma 5.1, $I + f^{-1}D(f^{-1}Dp_{n+1}) + A_{n+1} > \frac{c_0}{4}$. Calculations based on (3.9), (3.10) show that all the coefficients of this equation are in $C^{0,\alpha/2}$, with norm bounded by $\|p_n\|_{3,\alpha/2}$ and $\|p_{n+1}\|_{2,\alpha/2}$. So one can apply Schauder estimates to conclude that Dp_{n+1} is bounded in $C^{2,\alpha/2}$, or that p_{n+1} is bounded in $C^{3,\alpha/2}$. Now one looks at (3.8) to conclude that $\|\mathcal{G}(p, q, \delta t)\|_{2,\alpha/2}$ can be bounded by $\|p_{n+1}\|_{3,\alpha/2}$, $\|p_n\|_{3,\alpha/2}$. Then differentiate (3.13) twice to see that D^2p_{n+1} solves a uniformly elliptic equation with coefficients bounded in $C^{0,\alpha/2}$ by $\|p_{n+1}\|_{3,\alpha/2}$ and $\|p_n\|_{4,\alpha/2}$. One can further differentiate (3.13) and use Schauder estimates again and again to get (6.4).

Then (6.4) gives a bound for $\|p_{n+1}\|_{2,\alpha}$, since $k \geq 1$. Therefore by looking at (3.8), one sees that $\|\mathcal{G}(p_n, p_{n+1}, \delta t)\|_{1,\alpha}$ can be bounded by $\|p_{n+1}\|_{2,\alpha}$ and $\|p_n\|_{2,\alpha}$. So the same argument as in the previous paragraph gives the desired conclusion. $\square$

Lemma 6.2. *Under conditions of Lemma 6.1,*

$$(6.5) \quad \|p_{n+1} - p_n\|_{k+1,\alpha} \leq C_1\delta t,$$

where the constant $C_1 = C_1(\|p_{n+1}\|_{k+2,\alpha}, \|p_n\|_{k+2,\alpha})$.

Proof. In this argument, all the constants C depend only on $\|p_{n+1}\|_{k+2,\alpha}$, $\|p_n\|_{k+2,\alpha}$ and may change line from line. Write $q(x) = p_{n+1}(x)$, $p(x) = p_n(x)$. First we observe that $\|\mathcal{G}(p, q, \delta t)\|_{k+1,\alpha} = \|F^{-1} - id\|_{k+1,\alpha} \leq C$. Here C has the dependence as stated in the Lemma. This estimate follows from differentiating (3.8) and a bootstrap argument. Indeed, first from Lemma 4.4, we know that $\|\mathcal{G}(p, q, \delta t)\|_{1,\alpha/2} \leq \varepsilon_2 \leq 1$. Also it follows from (5.3) that $(I + f^{-1}\nabla p \otimes \nabla(f^{-1}) + f^{-2}D^2p)(F^{-1}) + B \geq \frac{c_0}{4}$, therefore we can invert and obtain

$$(6.6) \quad DF^{-1} = [(I + f^{-1}\nabla p \otimes \nabla(f^{-1}) + f^{-2}D^2p)(F^{-1}) + B]^{-1} \cdot [I + f^{-1}\nabla q \otimes \nabla(f^{-1}) + f^{-2}D^2q + A].$$

Since we already have $p, q \in C^{2,\alpha}$, the formulae in (3.10), (3.11) for A and B with (6.6) gives $D_x\mathcal{G}(p, q, \delta t)(x) \in C^{0,\alpha}$, with a $C^{0,\alpha}$ bound having the stated dependence. This shows that $\mathcal{G}(p, q, \delta t) \in C^{1,\alpha}$. Now since $k \geq 2$, we know that actually $p, q \in C^{3,\alpha}$.

This implies the right hand side of (6.6) is in $C^{1,\alpha}$. Therefore we obtain from (6.6) that $\mathcal{G}(p, q, \delta t)$ is in $C^{2,\alpha}$. If it happens that $p, q \in C^{4,\alpha}$, then we know the right hand side of (6.6) is in $C^{2,\alpha}$, and hence it gives $\mathcal{G}(p, q, \delta t) \in C^{3,\alpha}$. One can repeat this argument and it gives in general that if $p, q \in C^{k+2,\alpha}$, then $\mathcal{G}(p, q, \delta t) \in C^{k+1,\alpha}$, with an estimate on the $C^{k+1,\alpha}$ norm which has the dependence stated in the Lemma.

We subtract from both sides of (3.13) the quantity $\det(I + f^{-1}D(f^{-1}Dp))$, and write the resulting equation as a linear equation for $q - p$. Then the the left hand side of the resulting equation can be written as

$$(6.7) \quad a_{ij}[f^{-2}(x)\partial_{ij}(q - p) + f^{-1}(x)\partial_i(q - p)\partial_j(f^{-1}) + A_{ij}],$$

where

$$a_{ij} = \int_0^1 M_{ij}[(1 - \theta)(M_1) + \theta(M_2)]d\theta.$$

Here M_{ij} denotes the entries of the cofactor matrix, and $M_1 = I + f^{-1}D(f^{-1}Dq) + A$, $M_2 = I + f^{-1}D(f^{-1}Dp)$, A_{ij} is the element of the matrix $A(x)$ from (3.9). The right hand side becomes

$$(6.8) \quad \begin{aligned} & b_{ij}[(f^{-1}\partial_i p \partial_j(f^{-1})(F^{-1}(x)) - f^{-1}\partial_i p \partial_j(f^{-1})(x)) + (f^{-2}\partial_{ij}p(F^{-1}(x)) - f^{-2}\partial_{ij}p(x)) + B_{ij}] \\ &= b_{ij}[\int_0^1 \partial_l(f^{-1}\partial_i p \partial_j(f^{-1}) + f^{-2}\partial_{ij}p)((1 - \theta)x + \theta F^{-1}(x))d\theta(F_l^{-1} - x_l) + B_{ij}], \end{aligned}$$

where

$$b_{ij} = \int_0^1 M_{ij}[(1 - \theta)(M'_1) + \theta(M_2)]d\theta.$$

Here $M'_1 = I + f^{-1}D(f^{-1}Dp)(F^{-1}(x)) + B$. Now observe we can write

$$(6.9) \quad A(x) = \int_0^1 \partial_k(f^{-1})((1 - \theta)F^{-1}(x) + \theta x)d\theta(F_l^{-1} - x_l) \cdot (\nabla q \otimes \nabla(f^{-1}) + f^{-1}D^2q).$$

As for $B(x)$, the first term can be rewritten as

$$(6.10) \quad \begin{aligned} & [(f^{-1}\nabla p)(F^{-1}(x)) - f^{-1}\nabla p(x)] \otimes \nabla(f^{-1})(F^{-1}(x)) \\ &= [(f^{-1}\nabla p)(F^{-1}(x)) - f^{-1}\nabla p(x)] \otimes \nabla(f^{-1})(F^{-1}(x)) + f^{-1}\nabla(p - q) \otimes \nabla(f^{-1})(F^{-1}(x)) \\ &= \int_0^1 \partial_l(f^{-1}\nabla p)(\theta F^{-1}(x) + (1 - \theta)x)d\theta \otimes \nabla(f^{-1})(F^{-1}(x))(F_l^{-1} - x_l) \\ &+ f^{-1}\nabla(p - q) \otimes \nabla(f^{-1})(F^{-1}(x)). \end{aligned}$$

The second term of $B(x)$ can be rewritten as

$$(6.11) \quad D[f^{-2}(R_{f\delta t} - I)\nabla p](F^{-1}(x)) = g\delta t,$$

with $\|g\|_{k,\alpha} \leq C$.

To summarize, the difference $q - p$ satisfies an equation of the following form

$$(6.12) \quad a_{ij}\partial_{ij}(q - p) + b_i\partial_i(q - p) + c_l(F_l^{-1} - x_l) = g\delta t,$$

with $\|a_{ij}\|_{k,\alpha}, \|b_i\|_{k,\alpha}, \|c_l\|_{k-1,\alpha} \leq C$,

Next we represent $F^{-1}(x) - x$ in terms of $\nabla(q - p)$, with an error term controlled by δt . For this we need to go back to (3.12). Subtracting $x + f^{-2}(x)\nabla p(x)$ from both sides of (3.12), we obtain

$$(6.13) \quad \begin{aligned} f^{-1}(x)(f^{-1}(F^{-1}(x))\nabla q(x) - f^{-1}(x)\nabla p(x)) &= F^{-1}(x) - x \\ &\quad + (f^{-2}\nabla p)(F^{-1}(x)) - f^{-2}\nabla p(x) + [f^{-2}(R_{f\delta t} - I)]\nabla p(F^{-1}(x)). \end{aligned}$$

After rearranging terms, we get

$$(6.14) \quad f^{-1}(x)f^{-1}(F^{-1}(x))\nabla(q - p) - [f^{-2}(R_{f\delta t} - I)\nabla p](F^{-1}(x)) = C(x)(F^{-1}(x) - x),$$

where

$$(6.15) \quad \begin{aligned} C(x) &= I + \int_0^1 D[f^{-2}Dp]((1 - \theta)F^{-1}(x) + \theta x)d\theta \\ &\quad - f^{-1}(x) \int_0^1 \nabla(f^{-1})((1 - \theta)F^{-1}(x) + \theta x)d\theta \otimes \nabla p. \end{aligned}$$

From Proposition 5.5, we know $C(x) \geq \frac{c_0}{4}$. Now in (6.14) we have $f^{-1}(F^{-1}(x)) \in C^{k+1,\alpha}$, with norms controlled by $\|p\|_{k+2,\alpha}$, $\|q\|_{k+2,\alpha}$. Also the term $f^{-2}(R_{f\delta t} - I)\nabla p(F^{-1}(x)) = m\delta t$ with $\|m\|_{k,\alpha}$ bounded by $C(\|p\|_{k+2,\alpha}, \|q\|_{k+2,\alpha})$. Therefore in the equation (6.12), $c_l(F_l^{-1} - x_l)$ can be dispensed with and the result follows from the Schauder estimates. $\square$

Lemma 6.3. *Let $(p_n, p_{n+1}, \delta t)$ be as in Lemma 6.1, and*

$$(6.16) \quad \|F^{-1} - id\|_{k,\alpha} \leq C_2\delta t.$$

Here $C_2 = C_2(\|p_{n+1}\|_{k+2,\alpha}, \|p_n\|_{k+2,\alpha})$.

Proof. We use (6.14), (6.15) to get

$$(6.17) \quad F^{-1}(x) - x = C(x)^{-1}[f^{-1}(x)f^{-1}(F^{-1}(x))\nabla(q - p) - f^{-2}(R_{f\delta t} - I)\nabla p(F^{-1}(x))].$$

We use the inequality $\|fg\|_{k,\alpha} \leq C_k\|f\|_{k,\alpha}\|g\|_{k,\alpha}$. Notice that $C(x) \in C^{k,\alpha}$, hence $C^{-1} \in C^{k,\alpha}$, with $C^{k,\alpha}$ norm controlled by $\|p_i\|_{k+2,\alpha}$, $i = n, n+1$, and $\frac{1}{c_0}$, by Proposition 5.5. The $C^{k,\alpha}$ norm of the square bracket is controlled by $C\delta t$ by Lemma 6.2. $\square$

For immediate use, we prove the following lemma,

Lemma 6.4. *Let $\alpha \in [0, 1)$. Let $k \geq 1$, $G \in C^{k,\alpha}(\mathbb{T}^2)$, and let $F^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a map which satisfies $F^{-1}(x + h) = F^{-1}(x) + h$ for any $h \in \mathbb{Z}^2$ and $\|F^{-1} - id\|_{k,\alpha} \leq C^*\delta t$, then*

$$(6.18) \quad \|G \circ F^{-1}\|_{k,\alpha} \leq \|G\|_{k,\alpha} + C\delta t,$$

where C depends on $\|G\|_{k,\alpha}$, C^* , α and k .

Proof. In the following argument, the constant C has dependence as in the Lemma, and may change from expression to expression.

We assume $\alpha \in (0, 1)$, since the case $\alpha = 0$ is simpler, and follows from the argument below.

It is obvious that

$$(6.19) \quad \|G \circ F^{-1}\|_0 \leq \|G\|_0.$$

Thus we need to estimate terms $D^\beta(G \circ F^{-1})$ for multi-indices β satisfying $1 \leq |\beta| \leq k$.

Using that $D_i(G \circ F^{-1}) = \sum_{j=1}^2 ((D_j G) \circ F^{-1}) D_i F_j^{-1}$, we obtain

$$(6.20) \quad D_i(G \circ F^{-1}) = (D_i G) \circ F^{-1} + \sum_{j=1}^2 ((D_j G) \circ F^{-1}) D_i(F^{-1} - id)_j.$$

Next we show that for any multi-index β with $|\beta| \geq 1$,

$$(6.21) \quad D^\beta(G \circ F^{-1}) = (D^\beta G) \circ F^{-1} + \sum_{1 \leq |\gamma| \leq |\beta|} \sum_{j=1}^2 a_{\beta, \gamma, j} D^\gamma(F^{-1} - id)_j,$$

where $a_{\beta, \gamma, j}$ is a polynomial expression of $\{(D^{\gamma'} G) \circ F^{-1}\}_{1 \leq |\gamma'| \leq |\beta|}$, $\{D^{\gamma'} F^{-1}\}_{1 \leq |\gamma'| \leq |\beta|-1}$.

Indeed, we prove (6.21) by induction over $m = |\beta|$. Case $m = 1$ follows from (6.20). Next, assume that $m \geq 1$ and that (6.21) is proved for all $|\beta| \leq m$. Let $|\beta'| = m + 1$, then $D^{\beta'} = D_i d^\beta$ for some $i \in \{1, 2\}$ and $|\beta| = m$. Now, taking D_i -derivative of (6.21) for β , and applying (6.20) with $D^\beta G$ instead of G to handle the derivative of the first term in the right-hand side of (6.21), we obtain (6.21) for β' .

Now from (6.21), recalling the structure of $a_{\beta, \gamma, j}$, we see that if $1 \leq |\beta| \leq k$, then

$$(6.22) \quad \|D^\beta(G \circ F^{-1})\|_0 \leq \|D^\beta G\|_0 + C\delta t,$$

where dependence of C is as in the lemma.

In order to estimate $[D^\beta(G \circ F^{-1})]_\alpha$ for $|\beta| = k$, we first estimate this seminorm for the first term in the right-hand side of (6.21):

$$\begin{aligned} [(D^\beta G) \circ F^{-1}]_\alpha &= \sup_{x, y} \frac{|(D^\beta G) \circ F^{-1}(x) - (D^\beta G) \circ F^{-1}(y)|}{|x - y|^\alpha} \\ &\leq [D^\beta G]_\alpha \sup_{x, y} \frac{(|(F^{-1}(x) - x) - (F^{-1}(y) - y)| + |x - y|)^\alpha}{|x - y|^\alpha} \\ &= [D^\beta G]_\alpha \left(1 + \sup_{x, y} \frac{|(F^{-1}(x) - x) - (F^{-1}(y) - y)|}{|x - y|} \right)^\alpha \\ &\leq [D^\beta G]_\alpha (1 + \|D(F - id)\|_0)^\alpha \\ &\leq [D^\beta G]_\alpha (1 + C\|F - id\|_1) \leq [D^\beta G]_\alpha (1 + C\delta t). \end{aligned}$$

Then by (6.21),

$$[D^\beta(G \circ F^{-1})]_\alpha \leq [D^\beta G]_\alpha (1 + C\delta t) + C\delta t \leq [D^\beta G]_\alpha + C\delta t.$$

Combining this with (6.19) and (6.22), we conclude the proof of (6.18). $\square$

Define

$$(6.23) \quad \nu_m := \det(I + f^{-1} D(f^{-1} D p_m)).$$

Lemma 6.5.

$$(6.24) \quad \|\nu_{n+1}\|_{k, \alpha} \leq \|\nu_n\|_{k, \alpha} + C_3 \delta t.$$

Here $C_3 = C_3(\|p_{n+1}\|_{k+2, \alpha}, \|p_n\|_{k+2, \alpha})$. Recall $\nu_n = \det(I + f^{-1} D(f^{-1} D p_n))$.

Proof. Write $q(x) = p_{n+1}(x)$, $p(x) = p_n(x)$. Define

$$\begin{aligned}\nu'_n &= \det(I + f^{-1}D(f^{-1}Dp)(F^{-1}(x))), \\ \nu''_n &= \det(I + f^{-1}D(f^{-1}Dp)(F^{-1}(x)) + B(x)), \\ \nu'_{n+1} &= \det(I + f^{-1}D(f^{-1}Dq) + A(x)).\end{aligned}$$

By the equation, we have $\nu'_{n+1} = \nu''_n$.

We will prove the lemma by showing

$$(6.25) \quad \|\nu'_n\|_{k,\alpha} \leq \|\nu_n\|_{k,\alpha} + C\delta t,$$

$$(6.26) \quad \|\nu'_{n+1} - \nu_{n+1}\|_{k,\alpha} \leq C\delta t,$$

$$(6.27) \quad \|\nu'_n - \nu''_n\|_{k,\alpha} \leq C\delta t.$$

First we observe that (6.25) is a direct consequence of Lemma 6.4 applied to $G = \det(I + f^{-1}D(f^{-1}Dp))$ and Lemma 6.3.

To see (6.26), we can write, upon noticing that $p_{n+1} = q$,

$$(6.28) \quad \nu'_{n+1} - \nu_{n+1} = \int_0^1 M_{ij}(I + f^{-1}D(f^{-1}Dq) + \theta A(x)) d\theta A_{ij}(x).$$

Hence

$$(6.29) \quad \|\nu'_{n+1} - \nu_{n+1}\|_{k,\alpha} \leq C_k \left\| \int_0^1 I + f^{-1}D(f^{-1}Dq) + \theta A(x) d\theta \right\|_{k,\alpha} \|A_{ij}\|_{k,\alpha}.$$

But $\|A_{ij}\|_{k,\alpha} \leq C\delta t$ by (6.9) and Lemma 6.3. So (6.26) is proved. (6.27) is proved in a similar way as (6.26), by noticing that $\|B_{ij}\|_{k,\alpha} \leq C\delta t$. This follows from (6.10), (6.11) and Lemma 6.3. $\square$

Lemma 6.6. *For $n = 0, 1, \dots$*

$$(6.30) \quad \|p_n\|_{k+2,\alpha} \leq C_4(\|\nu_n\|_{k,\alpha} + 1).$$

Here $C_4 = C_4(\|p_n\|_{2,\alpha})$.

Proof. We differentiate both sides of (6.23) with m replaced by n and obtain linear elliptic equations for the derivatives of p_n . We can then use the Schauder estimates to obtain the result. $\square$

Now we apply the estimates obtained in the Lemma to prove the following crucial estimates of approximate solutions.

Proposition 6.7. *For any $p_0 \in C^{k+2,\alpha}$, $k \geq 2$, satisfying $I + f^{-1}D(f^{-1}Dp_0) > c_0$, there exist $T_1 > 0$, $\varepsilon > 0$, depending only on $\|p_0\|_{k+2,\alpha}$, f , c_0 and k , such that for any $|\delta t| < \varepsilon$, one can construct a sequence (p_n, F_n) of solutions to (6.1), (6.2) for $n \leq \frac{T_1}{\delta t} - 1$, such that $p_n \in C_0^{k+2,\alpha}$ and $F_n : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a measure preserving diffeomorphism. Moreover, (p_n, F_n) satisfy the following estimates:*

$$(6.31) \quad \|p_n\|_{k+2,\alpha} \leq C,$$

$$(6.32) \quad \|p_{n+1} - p_n\|_{k+1,\alpha} \leq C\delta t,$$

$$(6.33) \quad \|F_{n+1}^{-1} - id\|_{k,\alpha} \leq C\delta t,$$

$$(6.34) \quad \|F_{n+1} - id\|_{k,\alpha} \leq C\delta t.$$

for $0 \leq n \leq \frac{T_1}{\delta t} - 1$. Here C depends only on $\|p_0\|_{k+2,\alpha}$, f , c_0 and k .

Proof. Let $M = 4\|\nu_0\|_{k,\alpha}$.

Let $C_5 = C_4(\|p_0\|_{2,\alpha} + 1)(M + 1)$. This is the bound for $\|p_n\|_{k+2,\alpha}$ given by Lemma 6.6 if we have the bound $\|\nu_n\|_{k,\alpha} \leq M$ and the bound $\|p_n\|_{2,\alpha} \leq \|p_0\|_{2,\alpha} + 1$.

Define C_6 to be the bound for $\|p_{n+1}\|_{k+2,\alpha}$ given by Lemma 6.1 if we have $\|p_n - p_0\|_{3,\alpha} < \varepsilon_5$, $\|p_n\|_{k+2,\alpha} \leq C_5$, and $p_{n+1} = \mathcal{H}(p_n, \delta t)$.

Now let C_7 be the constant given by Lemma 6.5 such that $\|\nu_{n+1}\|_{k,\alpha} \leq \|\nu_n\|_{k,\alpha} + C_7\delta t$ if we have the bound $\|p_n\|_{k+2,\alpha}, \|p_{n+1}\|_{k+2,\alpha} \leq C_6$.

Similarly let C_8 be the constant given by Lemma 6.2 such that $\|p_{n+1} - p_n\|_{k+1,\alpha} \leq C_8\delta t$ if we have the bound $\|p_n\|_{k+2,\alpha}, \|p_{n+1}\|_{k+2,\alpha} \leq C_6$.

Finally let $C_9 = \min(C_5, C_6)$.

We will show by induction that

$$(6.35) \quad \|\nu_n\|_{k,\alpha} \leq \|\nu_0\|_{k,\alpha} + nC_7\delta t, \quad \|p_n\|_{k+2,\alpha} \leq C_9, \quad \|p_n - p_0\|_{k+1,\alpha} \leq nC_7\delta t$$

as long as $\delta t \in (0, \varepsilon_5)$ and

$$(6.36) \quad n\delta t \leq \min\left(\frac{\varepsilon_5}{2C_8}, \frac{M}{2C_7}\right) := T_1,$$

where ε_5 is chosen according to Proposition 5.5.

We note that (6.35) obviously holds for $n = 0$. Suppose (6.35) is true for any $j \leq n$, suppose also (6.36) holds for $n+1$. By the induction hypothesis, we have $\|p_n\|_{k+2,\alpha} \leq C_9$, and also $\|p_n - p_0\|_{3,\alpha} \leq nC_7\delta t \leq \varepsilon_5/2$, where we used (6.36) in the last inequality. Hence, by Proposition 5.5, we can define $p_{n+1} = \mathcal{H}(p_n, \delta t)$. It remains to show (6.35) with $n+1$ instead of n .

First by the induction hypothesis and Lemma 6.1, since $\|p_n\|_{k+2,\alpha} \leq C_9 \leq C_5$, we can conclude that

$$(6.37) \quad \|p_{n+1}\|_{k+2,\alpha} \leq C_6.$$

By the induction hypothesis, we also have $\|p_n\|_{k+2,\alpha} \leq C_9 \leq C_6$. Hence we can apply Lemma 6.5 and the induction hypotheses (6.35), to conclude that

$$(6.38) \quad \|\nu_{n+1}\|_{k,\alpha} \leq \|\nu_n\|_{k,\alpha} + C_7\delta t \leq \|\nu_0\|_{k,\alpha} + (n+1)C_7\delta t.$$

Also, from (6.38), using that (6.36) holds for $n+1$, and that $M = 4\|\nu_0\|_{k,\alpha}$, we have $\|\nu_{n+1}\|_{k,\alpha} \leq \frac{M}{4} + \frac{M}{2} \leq M$. Now we can use Lemma 6.6 to conclude that

$$\|p_{n+1}\|_{k+2,\alpha} \leq C_4(\|p_0\|_{2,\alpha} + 1) \cdot (M + 1) = C_5$$

Combining this with (6.37), we get

$$(6.39) \quad \|p_{n+1}\|_{k+2,\alpha} \leq \min(C_5, C_6) = C_9.$$

Furthermore, from (6.35) and (6.37) we have $\|p_n\|_{k+2,\alpha}, \|p_{n+1}\|_{k+2,\alpha} \leq C_6$, thus we can apply Lemma 6.2 to get

$$(6.40) \quad \|p_{n+1} - p_n\|_{k+1,\alpha} \leq C_8\delta t.$$

Hence

$$(6.41) \quad \|p_{n+1} - p_0\|_{2,\alpha} \leq \|p_{n+1} - p_n\|_{2,\alpha} + nC_7\delta t \leq (n+1)C_8\delta t$$

since $k \geq 1$. From (6.38), (6.39) and (6.41) the induction is complete.

Thus for any $\delta t \in (0, \varepsilon_5)$ we have shown existence of (F_n, p_n) for integers $n \geq 1$ satisfying (6.1), (6.2). Moreover, the estimates (6.35) hold. We note that (6.35) directly implies (6.31), (6.32). Also, (6.33) is proved in Lemma 6.3.

To see (6.34), first observe there is no loss of generality in assuming $C\delta t < \frac{1}{200}$, so that we get $\|F_{n+1}^{-1} - id\|_{k,\alpha} \leq \frac{1}{200}$ from (6.33). Here C is the maximum of C_5 up to C_9 appearing in this proof. Then (6.34) follows from Lemma 6.8 below by taking $G = F_{n+1}^{-1}$.

The Proposition is now proved with $T = T_1$. $\square$

Lemma 6.8. *Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a diffeomorphism, such that $G(x+h) = G(x) + h$, for any $x \in \mathbb{R}^2$ and $h \in \mathbb{Z}^2$. Suppose that $G \in C_{loc}^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$ for some $k \geq 1$ and $0 < \alpha < 1$, with $\|G - id\|_{k,\alpha} \leq \frac{1}{200}$. Then $G^{-1} \in C_{loc}^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$ and $\|G^{-1} - id\|_{k,\alpha} \leq C\|G - id\|_{k,\alpha}$. Here C depends only on k, α .*

Proof. First we observe that $G^{-1} \in C_{loc}^{k,\alpha}$, with $\|D(G^{-1})\|_{k-1,\alpha} \leq C$, with C depending only on k, α . Indeed, from $\|G - id\|_{k,\alpha} \leq \frac{1}{200}$, we know that $\frac{1}{2} \leq \det DG \leq \frac{3}{2}$. From the Implicit Function Theorem, we know that G^{-1} is C^1 and $D(G^{-1}) = (DG)^{-1}(G^{-1})$. From this we get a universal $C^{0,\alpha}$ bound for $D(G^{-1})$. One keeps differentiating and get $\|D(G^{-1})\|_{k-1,\alpha} \leq C$, with C depending only on k, α .

To see that $\|G^{-1} - id\|_{k,\alpha} \leq C\|G - id\|_{k,\alpha}$, we just need to show that $\|D(G^{-1}) - I\|_{k-1,\alpha} \leq C\|G - id\|_{k,\alpha}$. Observe that $D(G^{-1}) - I = D(G^{-1})(I - (DG(G^{-1}))) = D(G^{-1})(I - DG(G^{-1}))$. Therefore,

$$\begin{aligned} \|D(G^{-1}) - I\|_{k-1,\alpha} &\leq C\|D(G^{-1})\|_{k-1,\alpha}\|I - DG \circ G^{-1}\|_{k-1,\alpha} \\ &\leq C\|I - DG \circ G^{-1}\|_{k-1,\alpha}. \end{aligned}$$

Here C depends only on k, α , and we used the calculation $\|D(G^{-1})\|_{k-1,\alpha} \leq C$. So we just need to show that $\|I - DG \circ G^{-1}\|_{k-1,\alpha} \leq C\|G - id\|_{k,\alpha}$. But if we calculate the $(k-1)$ -th derivative of $I - DG \circ G^{-1}$, it will be the sum of terms of the form $D^{\beta'}(I - DG) \circ G^{-1}$ multiplied by derivatives of G^{-1} of order $\leq k-1$, with $|\beta'| \leq k-1$. Hence we have the claimed estimate. $\square$

Remark 6.9. *We remark that the quantity T_1 in (6.36) can be chosen to be uniform for all initial data p_0 in some bounded set of $C^{k+2,\alpha}$. Indeed, the quantity M and hence C_5 depends only on $\|p_0\|_{k+2,\alpha}$, c_0 , and f , so will be C_6 . Hence the same will be true for C_7, C_8, C_9 . As for ε_5 , we have seen in Remark 5.6 that as long as p_0 is taken within some compact set in $\tilde{U}_2 \subset C_0^{3,\alpha}$, then we can choose ε_5 uniformly. But we assumed $k \geq 2$, hence any bounded set in $C_0^{k+2,\alpha}$ will be precompact in $C^{3,\alpha}$.*

Finally, we estimate the continuity in time of the flow map. Define the flow map at the n -th step to be:

$$(6.42) \quad \phi(n\delta t, \cdot) = F_n \circ F_{n-1} \cdots F_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

where $F_0 = id$. Notice that $\phi(n\delta t, \cdot)(x+h) = \phi(n\delta t, x) + h$ for any $h \in \mathbb{Z}^2$ and $\phi(n\delta t, \cdot)$ is measure preserving.

We wish to get a flow map which is Lipschitz in $C^{k,\alpha}$ norm. For this we need to estimate the continuity of the discrete flow map in time. We start with the following elementary lemma:

Lemma 6.10. *Let $G_1, G_2 \in C_{loc}^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$, $F \in C_{loc}^{k,\alpha}(\mathbb{R}^2; \mathbb{R}^2)$, with $G_i(x+h) = G_i(x) + h$, $i = 1, 2$ and $F(x+h) = F(x) + h$ for any $h \in \mathbb{Z}^2$. Then $\|G_1 \circ F - G_2 \circ F\|_{k,\alpha} \leq C_{11}(\|F\|_{k,\alpha})\|G_1 - G_2\|_{k,\alpha}$.*

Proof. First notice the obvious estimate:

$$\max_{x \in \mathbb{T}^2} |G_1 \circ F(x) - G_2 \circ F(x)| \leq \max_{x \in \mathbb{T}^2} |G_1(x) - G_2(x)|.$$

For the derivatives, notice that if β is a multi-index with $|\beta| = k$, then $D^\beta(G_1 \circ F) - D^\beta(G_2 \circ F)$ is a sum of the terms of the form $D^{\beta'}(G_1 - G_2) \circ F$ multiplied by derivatives of F of order $\leq k$, with $1 \leq |\beta'| \leq k$. $\square$

With this lemma, we are ready to prove:

Proposition 6.11. *There exist constants $T \in (0, T_1]$, $C_{12} > 0$ depending only on $\|p_0\|_{k+2, \alpha}$, f , c_0 and k , such that, for any $|\delta t| < \varepsilon_5$ and any corresponding sequence of approximate solutions constructed in Proposition 6.7, the following estimate holds for each $1 \leq n \leq \frac{T}{\delta t}$:*

$$(6.43) \quad \|\phi(n\delta t, \cdot) - \phi((n-1)\delta t)\|_{k, \alpha} \leq C_{12}\delta t.$$

Proof. Applying previous lemma to $G_1 = F_n$, $G_2 = id$, and $F = \phi((n-1)\delta t)$, we can conclude

$$(6.44) \quad \|\phi(n\delta t, \cdot) - \phi((n-1)\delta t)\|_{k, \alpha} \leq C_{11}(\|\phi((n-1)\delta t)\|_{k, \alpha})C_{10}\delta t.$$

Here C_{10} is the constant given by Proposition 6.7. Notice that $\phi(0) = id$, and put $\phi(-\delta t) = id$. Let $T' > 0$, and n_0 be the largest integer with $n_0\delta t \leq T'$, we sum (6.44) from 0 to n_0 and obtain

$$(6.45) \quad \sup_{0 \leq n \leq \frac{T'}{\delta t}} \|\phi(n\delta t, \cdot) - id\|_{k, \alpha} \leq C_{11} \left(\sup_{1 \leq n \leq \frac{T'}{\delta t}} \|\phi(n\delta t, \cdot)\|_{k, \alpha} \right) C_{10}\delta t \cdot \frac{T'}{\delta t}.$$

for any $T' > 0$. Let $\hat{C}$ be the constant C_{11} given by Lemma 6.10 when F has the bound $\|F\|_{k, \alpha} \leq 2 + \|id\|_{k, \alpha}$. Then, choosing T' is small so that $\hat{C}C_{10}T' \leq 2$, we see by (6.45) that

$$(6.46) \quad \|\phi(n\delta t, \cdot)\|_{k, \alpha} \leq 2 + \|id\|_{k, \alpha},$$

for $0 \leq n \leq \frac{T'}{\delta t}$. Let $T = \min(T_1, T')$. Then T depends only on $\|p_0\|_{k+2, \alpha}$, f , c_0 and k , and (6.46) holds for $1 \leq n \leq \frac{T}{\delta t}$. For such n , denoting $C_{12} := \hat{C}C_{10}$, we get from (6.44)

$$\|\phi(n\delta t, \cdot) - \phi((n-1)\delta t)\|_{k, \alpha} \leq C_{12}\delta t.$$

Hence the proof is complete. $\square$

7. PASSAGE TO THE LIMIT

7.1. Determining the flow map and pressure by taking the limit of time-stepping approximations. Here we make use of the estimates obtained in (6.31)–(6.34), (6.46), (6.43) and pass to the limit in the following equation

$$(7.1) \quad x + f^{-1}(x)f^{-1}(F_{n+1}^{-1}(x))\nabla p_{n+1}(x) = F_{n+1}^{-1}(x) + f^{-2}R_{f\delta t}(\nabla p_n)(F_{n+1}^{-1}(x)).$$

For $t \in [0, T]$, let $\phi_t^{\delta t} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a map defined by:

$$(7.2) \quad \phi_t^{\delta t}(x) = \phi(n\delta t, x), \quad \text{if } n\delta t \leq t < (n+1)\delta t,$$

where $\phi(n\delta t, \cdot)$ is defined by (6.42). We also define:

$$(7.3) \quad \bar{\phi}_t^{\delta t}(x) = \frac{t - n\delta t}{\delta t} \phi((n+1)\delta t, x) + \frac{(n+1)\delta t - t}{\delta t} \phi(n\delta t, x), \quad \text{if } n\delta t \leq t < (n+1)\delta t.$$

The estimates (6.43) show that the map $\bar{\phi}_t^{\delta t}$ is Lipschitz in time under the norm $C^{k,\alpha}$. Indeed, it follows from (6.43) that $\|\bar{\phi}_t^{\delta t} - \bar{\phi}_s^{\delta t}\|_{k,\alpha} \leq C_{12}|t - s|$, for any $0 \leq s \leq t \leq T$, and $|\delta t| < \varepsilon_5$. Now we can use the Arzela-Ascoli Theorem applied to $\bar{\phi}_t^{\delta t}$ restricted to $[0, 1]^2$ to get a subsequence of $\delta t_j \rightarrow 0$, and a limit map ϕ_t Lipschitz in time under the norm $C^{k,\alpha}$, such that

$$(7.4) \quad \bar{\phi}_t^{\delta t_j} \rightarrow \phi_t \text{ under the norm } C^{k,\beta} \text{ for each } 0 < \beta < \alpha, \text{ uniformly for } t \in [0, T].$$

Also one observes that for $n\delta t \leq t < (n+1)\delta t$, one has

$$\|\bar{\phi}_t^{\delta t} - \phi_t^{\delta t}\|_{k,\alpha} \leq \|\phi(n\delta t, \cdot) - \phi((n-1)\delta t, \cdot)\|_{k,\alpha} \leq C_{12}\delta t.$$

Therefore the convergence obtained in (7.4) will hold also for $\phi^{\delta t}(t)$:

$$(7.5) \quad \phi_t^{\delta t_j} \rightarrow \phi_t \text{ under the norm } C^{k,\beta} \text{ for each } 0 < \beta < \alpha, \text{ uniformly for } t \in [0, T].$$

The same argument works also when we use $\phi(n\delta t, \cdot)^{-1}$ in (7.2), (7.3), to define families of maps $\psi_t^{\delta t}$ and $\bar{\psi}_t^{\delta t}$. Then we can pass to the limit as in (7.5), and conclude the limiting map ψ_t will also be Lipschitz in time under the norm $C^{k,\alpha}$, and that $\psi_t^{\delta t_j} \rightarrow \psi_t$. Since the argument for convergence of $\psi^{\delta t_j}$ can be performed by taking a subsequence of the sequence $\{\delta t_j\}$ used in (7.5), we can assume without loss of generality that convergence $\phi_t^{\delta t_j} \rightarrow \phi_t$ and $\psi_t^{\delta t_j} \rightarrow \psi_t$ holds along the same sequence $\delta t_j \rightarrow 0$. Then it is not difficult to observe that $\psi_t = \phi_t^{-1}$ for each $t \in [0, T]$. Indeed, for each n and j , with $n\delta t_j \leq T$, one has $\phi^{\delta t_j}(n\delta t_j, \psi^{\delta t_j}(n\delta t_j, x)) = x$. Fix any $t \in [0, T]$ and $j \in \mathbb{N}$, find n such that $n\delta t_j \leq t < (n+1)\delta t_j$. Now one can conclude that $\phi(t, \psi(t, x)) = x$, which follows from the uniform convergence obtained above and the Lipschitz continuity in time of the limit. The same argument also shows $\psi(t, \phi(t, x)) = x$. Hence the maps $\psi(t)$ and $\phi(t)$ obtained above are inverse to each other. Thus we obtained that $\phi_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a diffeomorphism for each $t \in [0, T]$, and

$$(7.6) \quad \|\phi_t, \phi_t^{-1}\|_{k,\alpha} \leq C, \text{ for each } t \in [0, T].$$

Also notice that, from (6.43),

$$(7.7) \quad \|\phi_t - \phi_s\|_{k,\alpha} \leq C|t - s|, \text{ for any } s, t \in [0, T].$$

Also since the map $\phi(n\delta t, \cdot)$ in (6.42) is measure-preserving, then by (7.2) and (7.5),

$$(7.8) \quad \phi_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2 \text{ is } \mathcal{L}^2\text{-measure preserving for each } t \in [0, T].$$

Similarly we may consider $p^{\delta t}(t)$. By the estimate (6.31)-(6.32), and a similar argument to the above, we can use the Arzela-Ascoli theorem to get a limit function p_t , which is Lipschitz in time under the norm $C^{k+1,\alpha}$, such that

$$(7.9) \quad p_t^{\delta t_j} \rightarrow p_t \text{ under the norm } C^{k+1,\alpha}(\mathbb{T}^2), \text{ uniformly in } t \in [0, T].$$

Then, by (6.32),

$$(7.10) \quad \|p_t - p_s\|_{k+1,\alpha} \leq C|t - s|, \text{ for any } s, t \in [0, T].$$

Denote $\mathbf{u}_g(n\delta t, x) = f^{-1}(x)J\nabla p(n\delta t, x)$. In Lagrangian coordinates, define

$$(7.11) \quad \mathbf{v}_g^{\delta t}(t, x) = f^{-1}(\phi^{\delta t}(t, x))J\nabla p^{\delta t}(t, \phi^{\delta t}(t, x)), \quad \mathbf{v}_g(t, x) = f^{-1}(\phi(t, x))J\nabla p(t, \phi(t, x)).$$

Then $\mathbf{v}_g^{\delta t}$ is piecewise constant in time. From (7.5), (7.9) we conclude that:

$$(7.12) \quad \mathbf{v}_g^{\delta t_j}(t, \cdot) \rightarrow \mathbf{v}_g(t, \cdot) \quad \text{under the norm } C^{k,\beta}(\mathbb{T}^2; \mathbb{R}^2) \text{ for each } 0 < \beta < \alpha, \\ \text{uniformly in } t \in [0, T].$$

Also, by (7.7) and (7.10)

$$(7.13) \quad \mathbf{v}_g(t, \cdot) \text{ is Lipschitz in time under the norm } C^{k,\alpha}(\mathbb{T}^2).$$

With the above preparation, we are ready to derive the equations in the limit.

7.2. Deriving the equation in the limit. In (7.1), replace x by $\phi^{\delta t}((n+1)\delta t, x)$, and recall that $\phi^{\delta t}((n+1)\delta t, \cdot) = F_{n+1} \circ \phi^{\delta t}(n\delta t, \cdot)$. Then, recalling the notation $\phi_t(x) = \phi(t, x)$, we get for each $x \in \mathbb{R}^2$:

$$\begin{aligned} & \phi_{(n+1)\delta t}^{\delta t} + f^{-1}(\phi_{(n+1)\delta t}^{\delta t}) f^{-1}(\phi_{n\delta t}^{\delta t}) \nabla p_{n+1}^{\delta t}(\phi_{(n+1)\delta t}^{\delta t}) \\ &= \phi_{n\delta t}^{\delta t} + [f^{-2} R_{f\delta t}(\nabla p_n^{\delta t})](\phi_{n\delta t}^{\delta t}). \end{aligned}$$

In the last term on the right hand side, all the functions are evaluated at $\phi_{n\delta t}^{\delta t}(x)$. Using (7.11), we rewrite the last equality as

$$(7.14) \quad \begin{aligned} & \phi_{(n+1)\delta t}^{\delta t} - \phi_{n\delta t}^{\delta t} + f^{-1}(\phi_{n\delta t}^{\delta t})(-J)(\mathbf{v}_{g,(n+1)\delta t}^{\delta t} - \mathbf{v}_{g,n\delta t}^{\delta t}) \\ &= [f^{-1}(R_{f\delta t} - I)](\phi_{n\delta t}^{\delta t})(-J\mathbf{v}_{g,n\delta t}^{\delta t}). \end{aligned}$$

Fix $0 \leq s < t < T'$, and suppose $k_1\delta t \leq s < (k_1+1)\delta t$, and $k_2\delta t \leq t < (k_2+1)\delta t$. With δt sufficiently small, we have $k_1 < k_2$. Summing (7.14) from k_1 to k_2-1 , we get for each $x \in \mathbb{R}^2$:

$$(7.15) \quad \begin{aligned} & \phi_{k_2\delta t}^{\delta t} - \phi_{k_1\delta t}^{\delta t} + \sum_{n=k_1}^{k_2-1} f^{-1}(\phi_{n\delta t}^{\delta t})(-J)(\mathbf{v}_{g,(n+1)\delta t}^{\delta t} - \mathbf{v}_{g,n\delta t}^{\delta t}) \\ &= \sum_{n=k_1}^{k_2-1} [f^{-1}(R_{f\delta t} - I)](\phi_{n\delta t}^{\delta t})(-J\mathbf{v}_{g,n\delta t}^{\delta t}). \end{aligned}$$

Now we take the limit in (7.15) along the sequence $\delta t_j \rightarrow 0$ from (7.5) and (7.9). We will suppress the index j , and write $\delta t \rightarrow 0$ meaning the above sequence.

As $\delta t \rightarrow 0$, we get $\phi_{k_2\delta t}^{\delta t} - \phi_{k_1\delta t}^{\delta t} \rightarrow \phi_t - \phi_s$ in $C^{k,\beta}(\mathbb{T}^2)$ for any $\beta < \alpha$ by (7.5).

For the right hand side of (7.15), one can get for each $x \in \mathbb{R}^2$:

$$(7.16) \quad \begin{aligned} & \sum_{n=k_1}^{k_2-1} [f^{-1}(R_{f\delta t} - I)](\phi_{n\delta t}^{\delta t})(-J\mathbf{v}_{g,n\delta t}^{\delta t}) \\ &= \int_{k_1\delta t}^{k_2\delta t} [f^{-1} \frac{R_{f\delta t} - I}{\delta t}](\phi_\tau^{\delta t})(-J\mathbf{v}_{g,\tau}^{\delta t}) d\tau \rightarrow \int_s^t f^{-1}(\phi_\tau) Jf(\phi_\tau)(-J\mathbf{v}_{g,\tau}) d\tau \\ &= \int_s^t \mathbf{v}_{g,\tau} d\tau \text{ as } \delta t \rightarrow 0, \end{aligned}$$

where the convergence is obtained as follows. By (7.5), (7.12) there exists C such that $\|\phi_t^{\delta t_j}\|_{k,\alpha/2,\mathbb{T}^2} \leq C$ and $\|\mathbf{v}_{g,t}^{\delta t_j}\|_{k,\alpha/2,\mathbb{T}^2} \leq C$ for each $t \in [0, T]$, $j = 1, 2, \dots$. Then, using (7.5), (7.12) again, we see that for each $x \in \mathbb{T}^2$ the integrands (as functions of

$t \in [0, T]$) are uniformly bounded with respect to j , and converge pointwise on $[0, T]$ to $f^{-1}(\phi_\tau)Jf(\phi_\tau)(-J\mathbf{v}_{g,\tau})$. Then we apply the Dominated Convergence Theorem.

It only remains to deal with the second term on the left hand side of (7.15). For this we define $(\mathcal{D}_t\mathbf{v}_g)_{\delta t}(t, \cdot) = \frac{\mathbf{v}_g^{\delta t}((n+1)\delta t, \cdot) - \mathbf{v}_g^{\delta t}(n\delta t, \cdot)}{\delta t}$ for $t \in [n\delta t, (n+1)\delta t)$. Then $(\mathcal{D}_t\mathbf{v}_g)_{\delta t}(t)$ is uniformly bounded in $C^{k,\alpha}$ for $0 \leq \delta t < \varepsilon_5$ and $t \in [0, T]$. This follows from our definition (7.11) and estimates (6.32) and (6.43). Then we can write

$$\begin{aligned}
(7.17) \quad & \sum_{n=k_1}^{k_2-1} f^{-1}(\phi_{n\delta t}^{\delta t}) \left(\mathbf{v}_{g,(n+1)\delta t}^{\delta t} - \mathbf{v}_{g,n\delta t}^{\delta t} \right) \\
&= \int_{k_1\delta t}^{k_2\delta t} f^{-1}(\phi_\tau^{\delta t}) (\mathcal{D}_t\mathbf{v}_g)_{\delta t}(\tau, \cdot) d\tau \\
&= \int_{k_1\delta t}^{k_2\delta t} \left[f^{-1}(\phi_\tau^{\delta t}) - f^{-1}(\phi_\tau) \right] (\mathcal{D}_t\mathbf{v}_g)_{\delta t}(\tau, \cdot) d\tau + \int_{k_1\delta t}^{k_2\delta t} f^{-1}(\phi_\tau) (\mathcal{D}_t\mathbf{v}_g)_{\delta t}(\tau, \cdot) d\tau.
\end{aligned}$$

In the first term above, we have $f^{-1}(\phi_\tau^{\delta t}) - f^{-1}(\phi_\tau) \rightarrow 0$ in $C^{k,\beta}(\mathbb{T}^2)$ for any $\beta < \alpha$ uniformly in $\tau \in [0, T]$ by (7.5), and $(\mathcal{D}_t\mathbf{v}_g)_{\delta t}(\tau, \cdot)$ is bounded in $C^{k,\alpha}(\mathbb{T}^2)$ uniformly in $\tau \in [0, T]$ as we showed above. Thus the first term converges to zero for each $x \in \mathbb{T}^2$.

So we can focus on the second term. We do a “discrete integration by parts”:

$$\begin{aligned}
(7.18) \quad & \int_{k_1\delta t}^{k_2\delta t} f^{-1}(\phi_\tau) (\mathcal{D}_t\mathbf{v}_g)_{\delta t}(\tau, \cdot) d\tau = \sum_{n=k_1}^{k_2-1} \int_0^{\delta t} f^{-1}(\phi_{\tau+n\delta t}) \frac{\mathbf{v}_{g,(n+1)\delta t}^{\delta t} - \mathbf{v}_{g,n\delta t}^{\delta t}}{\delta t} d\tau \\
&= \frac{1}{\delta t} \left[\sum_{n=k_1+1}^{k_2} \int_0^{\delta t} f^{-1}(\phi_{\tau+(n-1)\delta t}) \mathbf{v}_{g,n\delta t}^{\delta t} - \sum_{n=k_1}^{k_2-1} \int_0^{\delta t} f^{-1}(\phi_{\tau+n\delta t}) \mathbf{v}_{g,n\delta t}^{\delta t} d\tau \right] \\
&= \delta t^{-1} \left[\int_0^{\delta t} f^{-1}(\phi_{\tau+(k_2-1)\delta t}) d\tau \cdot \mathbf{v}_{g,k_2\delta t}^{\delta t} - \int_0^{\delta t} f^{-1}(\phi_{\tau+k_1\delta t}) d\tau \cdot \mathbf{v}_{g,k_1\delta t}^{\delta t} \right] \\
&+ \sum_{n=k_1+1}^{k_2-1} \int_0^{\delta t} \frac{1}{\delta t} \left[f^{-1}(\phi_{\tau+(n-1)\delta t}) - f^{-1}(\phi_{\tau+n\delta t}) \right] d\tau \mathbf{v}_{g,n\delta t}^{\delta t}.
\end{aligned}$$

The first two terms will converge to $f^{-1}(\phi_t)\mathbf{v}_{g,t}$ and $f^{-1}(\phi_s)\mathbf{v}_{g,s}$ in $C^{k,\beta}(\mathbb{T}^2)$ for any $\beta < \alpha$ respectively. For the last term, we fix some $x \in \mathbb{T}^2$ and have

$$\begin{aligned}
(7.19) \quad & \sum_{n=k_1+1}^{k_2-1} \int_0^{\delta t} \frac{1}{\delta t} [f^{-1}(\phi_{\tau+(n-1)\delta t}) - f^{-1}(\phi_{\tau+n\delta t})] d\tau \mathbf{v}_{g,n\delta t}^{\delta t} \\
&= \sum_{n=k_1+1}^{k_2-1} \int_{(n-1)\delta t}^{n\delta t} \frac{f^{-1}(\phi_\tau) - f^{-1}(\phi_{\tau+\delta t})}{\delta t} d\tau \mathbf{v}_{g,n\delta t}^{\delta t} \\
&= \int_{k_1\delta t}^{(k_2-1)\delta t} \frac{f^{-1}(\phi_\tau) - f^{-1}(\phi_{\tau+\delta t})}{\delta t} \mathbf{v}_{g,\tau+\delta t} d\tau.
\end{aligned}$$

Now we wish to take the limit of this term as $\delta t \rightarrow 0$. We have seen in (7.7) that ϕ_t is Lipschitz in t under $C^{k,\alpha}(\mathbb{T}^2)$ norm, hence $f^{-1}(\phi_t)$ will be Lipschitz, and for a.e $\tau \in [0, T]$, we have $\frac{f^{-1}(\phi_\tau) - f^{-1}(\phi_{\tau+\delta t})}{\delta t} \rightarrow \partial_t(f^{-1}(\phi_\tau))$. Then, using also (7.13), we find that the integrand will converge to $-\partial_t(f^{-1}(\phi_\tau))\mathbf{v}_{g,\tau}$ for a.e $\tau \in [s, t]$. Also we know the integrand is

bounded on $[0, T]$ by some constant independent of δt , by the said Lipschitz continuity of $f^{-1}(\phi_t)$ and also the boundedness of $\mathbf{v}_g^{\delta t}$. Hence we can apply the Dominated Convergence Theorem to conclude that the integral converges to $-\int_s^t \partial_t(f^{-1}(\phi(t, x))\mathbf{v}_g(t, x))dt$.

Combining all calculations above, we conclude that in the limit we have for any fixed $x \in \mathbb{T}^2$:

$$(7.20) \quad \begin{aligned} & \phi_t - \phi_s + (-J)[f^{-1}(\phi_t)\mathbf{v}_{g,t} - f^{-1}(\phi_s)\mathbf{v}_g(s) \\ & - \int_s^t \partial_\tau(f^{-1}(\phi_\tau))\mathbf{v}_{g,\tau}d\tau] = \int_s^t \mathbf{v}_{g,\tau}d\tau. \end{aligned}$$

Or equivalently

$$(7.21) \quad \phi_t - \phi_s + (-J) \int_s^t f^{-1}(\phi_\tau) \partial_\tau \mathbf{v}_g(\tau, \cdot) d\tau = \int_s^t \mathbf{v}_{g,\tau} d\tau.$$

Next we take the limit as $t \rightarrow s$. In the limit, we obtain the following:

Proposition 7.1. *For $\mathcal{L}^1$ -a.e $s \in [0, T]$, and for any $x \in \mathbb{T}^2$, it holds:*

$$(7.22) \quad \partial_t \phi(s, \cdot) + (-J)f^{-1}(\phi_s) \partial_t \mathbf{v}_g(s, \cdot) = \mathbf{v}_g(s, \cdot),$$

$$(7.23) \quad \begin{aligned} \partial_t \phi(s, x) = & [I + f^{-2}D^2p + f^{-1}\nabla p \otimes \nabla(f^{-1})]^{-1}(s, \phi(s, x)) \cdot (f^{-1}J\nabla p)(s, \phi(s, x)) \\ & - f^{-2}\partial_t \nabla p(s, \phi(s, x)). \end{aligned}$$

Proof. From (7.13) we conclude that $\lim_{t \rightarrow s+} \frac{\mathbf{v}_g(t, x) - \mathbf{v}_g(s, x)}{t - s}$ exists for $\mathcal{L}^3$ -a.e $(s, x) \in [0, T) \times \mathbb{T}^2$. Now we fix some $s \in [0, T)$, such that the above limit exists for $\mathcal{L}^2$ -a.e $x \in \mathbb{T}^2$. Note that this property holds for a.e. $s \in [0, T)$, by Fubini's theorem.

For s chosen above and $t \neq s$, we divide (7.21) by $t - s$, and take a limit as $t \rightarrow s$. Then the first term in the left-hand side of (7.21) will tend to $\partial_t \phi(s, x)$, for $\mathcal{L}^2$ -a.e $x \in \mathbb{T}^2$ because of our choice of s , while the term on the right hand side of (7.21) will tend to $\mathbf{v}_g(s, x)$, because of the Lipschitz continuity of $\mathbf{v}_g$ with respect to the time-variable. It remains to consider the integral in the left-hand side. For this purpose, we calculate for each $x \in \mathbb{T}^2$, and any $0 < s < t < T$:

$$\begin{aligned} & \int_s^t f^{-1}(\phi_\tau) \partial_\tau \mathbf{v}_g(\tau, \cdot) d\tau = \int_s^t (f^{-1}(\phi_\tau) - f^{-1}(\phi_s)) \partial_\tau \mathbf{v}_g(\tau, \cdot) d\tau \\ & \quad + f^{-1}(\phi_s) \int_s^t \partial_\tau \mathbf{v}_g(\tau, \cdot) d\tau \\ & = \int_s^t (f^{-1}(\phi_\tau) - f^{-1}(\phi_s)) \partial_\tau \mathbf{v}_g(\tau, \cdot) d\tau + f^{-1}(\phi_s)(\mathbf{v}_{g,t} - \mathbf{v}_{g,s}). \end{aligned}$$

In the last line above, we used the fact that for any $x \in \mathbb{T}^2$, $t \mapsto \mathbf{v}_g(t, x)$ is Lipschitz in t , and so the usual Newton-Leibnitz formula holds. Divide last line above by $t - s$, and let $t \rightarrow s$. We observe that the first term will tend to zero, because $f^{-1}(\phi(t, x))$ is Lipschitz in t , with a Lipschitz constant independent of x , while $\partial_\tau \mathbf{v}_g(\tau, \cdot)$ is bounded. The second term will tend to $\partial_t \mathbf{v}_g(s, x)$, by the choice of (s, x) . Therefore, the last line above, divided by $t - s$, will tend to $f^{-1}(\phi_s) \partial_t \mathbf{v}_g(s, \cdot)$ as $t \rightarrow s$.

Now we can conclude from (7.21) that for $\mathcal{L}^1$ -a.e $s \in [0, T]$:

$$(7.24) \quad \partial_t \phi(s, \cdot) + (-J)f^{-1}(\phi_s) \partial_t \mathbf{v}_g(s, \cdot) = \mathbf{v}_g(s, \cdot), \text{ for } \mathcal{L}^2\text{-a.e } x \in \mathbb{T}^2.$$

Next we follow a rather standard and classical procedure to show that the higher derivatives exist. Observe that $\frac{\phi_t - \phi_s}{t-s}$ is uniformly bounded in $C^{k,\alpha}$ for any $t > s$ by (7.7). Hence for any sequence $t_j \rightarrow s$ there exists a subsequence t_{j_m} such that $\frac{\phi_{t_{j_m}} - \phi_s}{t_{j_m} - s}$ converges in $C^{k,\beta}$ for any $\beta < \alpha$, and the limit function is equal to $\partial_t \phi(s, x)$ for a.e. $x \in \mathbb{T}^2$ by the choice of s . Thus $C^{k,\beta}$ -limit on $\mathbb{T}^2$ does not depend on the choice of the sequence and subsequence. This shows that $\partial_t \phi(s, x)$ exists for all $x \in \mathbb{T}^2$. Thus we have proved that

$$(7.25) \quad \partial_t \phi(s, \cdot) \in C^{k,\alpha}(\mathbb{T}^2) \quad \text{with} \quad \|\partial_t \phi(s, \cdot)\|_{C^{k,\alpha}} \leq C \quad \text{for a.e. } s \in [0, T],$$

where C does not depend on s . We can apply similar arguments to p_t using (7.10), and thus obtain:

$$(7.26) \quad \begin{aligned} & \text{for a.e. } s \in [0, T], \partial_t p(s, x) \text{ exists for all } x \in \mathbb{T}^2 \text{ and} \\ & \partial_t p(s, \cdot) \in C^{k+1,\alpha}(\mathbb{T}^2) \quad \text{with} \quad \|\partial_t p(s, \cdot)\|_{C^{k+1,\alpha}} \leq C, \end{aligned}$$

where C does not depend on s . From (7.25), (7.26), we conclude that for a.e. $s \in [0, T]$, $\partial_t \nabla p(s, \cdot), \mathbf{v}_g(s, \cdot) \in C^{k,\alpha}$, and the usual chain rule holds. Then we obtain from (7.11):

$$(7.27) \quad \begin{aligned} \partial_t \mathbf{v}_g(s, x) = & J[f^{-1}(\phi(s, x)) \partial_t \nabla p(s, \phi(s, x)) + f^{-1}(\phi(s, x)) D^2 p(s, \phi(s, x)) \partial_t \phi(s, x) \\ & + \nabla p(s, \phi(s, x)) \otimes \nabla(f^{-1})(\phi(s, x)) \partial_t \phi(s, x)], \\ & \text{for } \mathcal{L}^2\text{-a.e } x \in \mathbb{T}^2. \end{aligned}$$

Observe that each term in both sides of (7.27) are at least $C^{k,\alpha}$, we see that (7.27) holds for every $x \in \mathbb{T}^2$. Similar arguments also works for (7.24), and we can conclude that for $\mathcal{L}^1 - a.e$ $s \in [0, T]$, (7.24) holds for every $x \in \mathbb{T}^2$.

By plugging (7.27) into (7.24), we obtain for $\mathcal{L}^1 - a.e$ $s \in [0, T]$:

$$(7.28) \quad \begin{aligned} & [I + f^{-2} D^2 p + f^{-1} \nabla p \otimes \nabla(f^{-1})](s, \phi(s, x)) \partial_t \phi(s, x) \\ & = (f^{-1} J \nabla p)(s, \phi(s, x)) - f^{-2} \partial_t \nabla p(s, \phi(s, x)), \\ & \text{for every } x \in \mathbb{T}^2. \end{aligned}$$

By our construction, we have $I + f^{-2} D^2 p(s, \cdot) + f^{-1} \nabla p(s, \cdot) \otimes \nabla(f^{-1}) > \frac{c_0}{2}$, so we can invert this matrix, and obtain

$$(7.29) \quad \begin{aligned} \partial_t \phi(s, x) = & [I + f^{-2} D^2 p + f^{-1} \nabla p \otimes \nabla(f^{-1})]^{-1}(s, \phi(s, x)) \cdot (f^{-1} J \nabla p)(s, \phi(s, x)) \\ & - f^{-2} \partial_t \nabla p(s, \phi(s, x)). \end{aligned}$$

□

7.3. Regularity in time. In this section we show that the limiting equations (7.22), (7.23) hold for any $(s, x) \in [0, T] \times \mathbb{T}^2$. For this, we need to show regularity of p and ϕ in time.

Fix any $\zeta \in C^1(\mathbb{T}^2)$. Using the measure-preserving property of the map ϕ_t in (7.8), we obtain

$$\begin{aligned}
 (7.30) \quad 0 &= \left(\frac{d}{dt} \int_{\mathbb{T}^2} \zeta(\phi(t, x)) dx \right)_{|t=s} = \int_{\mathbb{T}^2} \nabla \zeta(\phi(s, x)) \partial_t \phi(s, x) dx \\
 &= \int_{\mathbb{T}^2} \nabla \zeta(\phi(s, x)) [I + f^{-2} D^2 p + f^{-1} \nabla p \otimes \nabla(f^{-1})]^{-1} \cdot (f^{-1} J \nabla p - f^{-2} \partial_t \nabla p)(s, \phi(s, x)) dx \\
 &= \int_{\mathbb{T}^2} \nabla \zeta(x) [I + f^{-2} D^2 p + f^{-1} \nabla p \otimes \nabla(f^{-1})]^{-1} \cdot (f^{-1} J \nabla p - f^{-2} \partial_t \nabla p)(s, x) dx.
 \end{aligned}$$

That is, $\partial_t p(t, \cdot)$ solves the following uniformly elliptic equation in divergence form for a.e. $t \in [0, T]$:

$$(7.31) \quad \nabla \cdot [(I + f^{-1} D(f^{-1} D p))^{-1} f^{-2} \nabla(\partial_t p)] = \nabla \cdot [(I + f^{-1} D(f^{-1} D p))^{-1} (f^{-1} J \nabla p)].$$

Write this equation in the form

$$(7.32) \quad \nabla \cdot [A(t, x) \nabla(\partial_t p)] = \nabla \cdot F(t, x).$$

Here $A(t, x) = (I + f^{-1} D(f^{-1} D p))^{-1}(t, x)$. We have that $\|A(t, \cdot), F(t, \cdot)\|_{C^{k, \alpha}(\mathbb{T}^2)} \leq C$ for a.e. $t \in [0, T]$. Then the standard estimates for linear elliptic equations imply $\|\partial_t p(t, \cdot)\|_{C^{k+1, \alpha}(\mathbb{T}^2)} \leq C$ for a.e. $t \in [0, T]$. Next we want to obtain higher regularity in time.

Proposition 7.2. *For any $(s, x) \in [0, T] \times \mathbb{T}^2$, and $1 \leq m \leq k + 1$, $\partial_t^m p(s, x)$ exists, $\partial_t^m p \in L^\infty([0, T]; C^{k+2-m, \alpha}(\mathbb{T}^2))$, and (7.22), (7.23) hold for any $(s, x) \in [0, T] \times \mathbb{T}^2$.*

Proof. We follow the standard procedure to obtain existence of higher-order derivatives. Formally we can differentiate (7.31) in t , and obtain a similar elliptic equation for $\partial_t^2 p_t$, and apply the $C^{k, \alpha}$ estimates. To proceed rigorously, we take any $t_1 < t_2$ for which (7.31) holds true, and take their difference:

$$\begin{aligned}
 (7.33) \quad \nabla \cdot \left(A(t_1) f^{-2} \nabla \left(\frac{\partial_t p(t_1) - \partial_t p(t_2)}{t_1 - t_2} \right) \right) &= -\nabla \cdot \left(\frac{A(t_1) - A(t_2)}{t_1 - t_2} f^{-2} \partial_t \nabla p(t_2) \right) \\
 &+ \nabla \cdot \left(A(t_1) f^{-1} J \frac{\nabla p(t_1) - \nabla p(t_2)}{t_1 - t_2} + \frac{A(t_1) - A(t_2)}{t_1 - t_2} f^{-1} J \nabla p(t_2) \right),
 \end{aligned}$$

where $A(t)(x) = A(t, x)$ for $A(t, x)$ defined above. Recall that we have the estimate $A(t_1)$ in $C^{k, \alpha}(\mathbb{T}^2)$. Also, by (7.10) we have an estimate for $\frac{A(t_1) - A(t_2)}{t_1 - t_2}$ in $C^{k-1, \alpha}$ independent of $t_1, t_2 \in [0, T]$. Thus the right hand side of (7.33) has the form $\nabla \cdot G(x)$, where $\|G(\cdot)\|_{C^{k-1, \alpha}} \leq C$ for C independent of $t_1, t_2 \in [0, T]$. By $C^{1, \alpha}$ estimates for elliptic equations, we obtain

$$(7.34) \quad \left\| \frac{\partial_t p(t_1) - \partial_t p(t_2)}{t_1 - t_2} \right\|_{C^{k, \alpha}(\mathbb{T}^2)} \leq C \text{ for a.e. } t_1, t_2 \in [0, T],$$

where C is independent of $t_1, t_2 \in [0, T]$.

Hence $\partial_t p$ exists for all $(t, x) \in [0, T] \times \mathbb{T}^2$: indeed fix $x \in \mathbb{T}^2$. From (7.34), functions $g_h(t) = \frac{p(t+h, x) - p(t, x)}{h}$ defined on $[0, T - |h|]$ for $|h| < T/2$, are equiLipschitz on their common domains, thus from any sequence $h_i \rightarrow 0$ one can extract a subsequence h_{i_k} so that $g_{h_{i_k}}(\cdot)$ converge to a uniform limit $g^\infty \in C([0, T])$ on compact subsets of $[0, T]$. Also, (7.26) implies that $g^\infty(t) = \partial_t p(t, x)$ for a.e. t , thus $g^\infty(t)$ is independent of the

subsequence h_{i_j} for such t . It follows that the uniform limit $g^\infty(\cdot)$ is independent of the subsequence h_{i_k} , i.e. $\lim_{h \rightarrow 0} \frac{p(t+h, x) - p(t, x)}{h}$ exists for each $t \in [0, T]$. Thus we proved that for each $x \in \mathbb{T}^2$, $\partial_t p(\cdot, x)$ exists for all $t \in [0, T]$. Also, now (7.34) holds for all $t_1, t_2 \in [0, T]$.

Here we can see (7.23) holds for any $(s, x) \in [0, T] \times \mathbb{T}^2$, since the right hand side of (7.23) is continuous in s . The continuity of the term $\partial_t \nabla p$ comes from the estimate (7.34). From (7.27), we know $\partial_t \mathbf{v}_g$ is also continuous in s for each fixed x . Hence (7.22) is true for all s .

Then we let $t_2 \rightarrow t_1$ in (7.34) and, repeating the proof of (7.25)–(7.26), we obtain that, for a.e. $t \in [0, T]$, $\partial_t^2 p(t, x)$ exists for all $x \in \mathbb{T}^2$, and $\|\partial_t^2 p(t, \cdot)\|_{C^{k, \alpha}(\mathbb{T}^2)} \leq C$ for those t , with C independent of t . Then we can repeat the previous argument and obtain the claimed regularity for $\partial_t^{k+2-m} p(t, \cdot)$ for $0 \leq m \leq k+1$. From (7.23) we see that $\partial_t \phi(t) \in L^\infty(0, T; C^{k, \alpha}(\mathbb{T}^2))$. Also, recall that $\phi(t) : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a diffeomorphism for each t , and (7.6) holds. Hence we define

$$\mathbf{u}(t, x) = \partial_t \phi(t, \phi^{-1}(t, x)),$$

and obtain $\mathbf{u} \in L^\infty(0, T; C^{k, \alpha})$. $\square$

In (7.24), we replace x by $\phi^{-1}(t, x)$, thus get

$$(7.35) \quad \mathbf{u} + (-J)f^{-1}(x)D_t \mathbf{u}_g = \mathbf{u}_g.$$

This gives equations (2.2) and (2.3). Also, we have remarked in (7.8) that ϕ_t is measure preserving. Then it follows that $\mathbf{u}$ is divergence free, so that (2.4) holds.

8. EXTENSION TO SG SHALLOW WATER WITH VARIABLE f

In this section, we will extend our analysis to the SG shallow water (SGSW) equations with periodic boundary conditions. The SGSW equations are the following:

$$(8.1) \quad (u_g, v_g) = f^{-1}(-\frac{\partial h}{\partial y}, \frac{\partial h}{\partial x}),$$

$$(8.2) \quad D_t u_g + \frac{\partial h}{\partial x} - f v = 0,$$

$$(8.3) \quad D_t v_g + \frac{\partial h}{\partial y} + f u = 0,$$

$$(8.4) \quad \frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0,$$

with initial data $h|_{t=0} = h_0$. Recalling the assumptions made in Theorem 2.2 we require

$$(8.5) \quad h_0 \in C^{k+2, \alpha}(\mathbb{T}^2) \text{ for some } k \geq 2; \quad I + f^{-1}D(f^{-1}Dh_0) > c_0; \quad \int_{[0,1]^2} h_0 = 1; \quad h_0 \geq c_1.$$

Then as in Section 3, we can discretize in time and thus obtain a time-difference equation, and we need to solve:

$$(8.6) \quad x + f^{-1}(x)f^{-1}(F^{-1}(x))\nabla q(x) = F^{-1}(x) + f^{-2}R_{f\delta t}(\nabla h)(F^{-1}(x)).$$

In the above, $q(x) = h_{n+1}(x)$, $h(x) = h_n(x)$, and $F = F_{n+1}$. This is the same equation as (3.7). Now we require $\int_{[0,1]^2} q = \int_{[0,1]^2} h = 1$. In the shallow water setting, F does not

preserve Lebesgue measure, but should satisfy the following push-forward condition:

$$(8.7) \quad F_{\#}(hdx) = qdy,$$

which, in the case of continuous h, q and a C^1 diffeomorphism F , is equivalent to

$$(8.8) \quad \det DF^{-1} = \frac{q(x)}{h(F^{-1}(x))}.$$

Therefore, recalling (3.8), (3.11) (or its version (3.13)) we see the equation we need to solve is now

$$(8.9) \quad \frac{\det(I + f^{-1}(x)\nabla q \otimes \nabla(f^{-1}) + f^{-2}D^2q + A(x))}{\det[(I + f^{-1}\nabla h \otimes \nabla(f^{-1}) + f^{-2}D^2h)(F^{-1}(x)) + B(x)]} = \frac{q(x)}{h(F^{-1}(x))}.$$

where $A(x)$ and $B(x)$ are the same as in (3.9), (3.10), with p_{n+1} replaced by q and p_n replaced by h . Thus we need to solve (8.6), (8.9) for q and F and show that F is a diffeomorphism.

Since equation (8.6) differs from (3.12) only in notation (p changed to h), then all results of Section 4, with p replaced by h , hold for equation (8.6). In particular, the map $\mathcal{G}$ is defined as in Lemma 4.4, and satisfies the properties found in Lemmas 4.4 and 4.6, and the map $F = (id + \mathcal{G}(h, q, \delta t))^{-1}$ solves equation (8.6) for given $(h, q, \delta t) \in V_{\varepsilon'_1}(p_0) \times (-\varepsilon'_1, \varepsilon'_1)$, where ε'_1 is defined in Lemma 4.6. Lemma 5.1 also holds.

Next we use the Implicit Function Theorem to solve (8.9) for q , with F defined by $F^{-1} = id + \mathcal{G}(p, q, \delta t)$. Denote by $C_1^{2,\alpha}$ the following affine subspace of $C^{2,\alpha}(\mathbb{T}^2)$:

$$C_1^{2,\alpha} = \{w \in C^{2,\alpha}(\mathbb{T}^2) : \int_{[0,1]^2} w = 1\}.$$

Consider the following subset U_2 of $C_1^{2,\alpha/2}$:

$$(8.10) \quad U_2 = \{w \in C_1^{2,\alpha/2} : I + f^{-1}D(f^{-1}Dw) > \frac{c_0}{2}, \quad w > \frac{c_1}{2}, \quad \|w - h_0\|_{2,\alpha/2} < \varepsilon_3\}.$$

Here ε_3 is from Lemma 5.1. Then $\mathcal{G}(h, q, \delta t)$ is defined, and $id + \mathcal{G}(h, q, \delta t)$ is a $C_{loc}^{1,\alpha/2}$ diffeomorphism of $\mathbb{R}^2$, as long as $\|h - h_0\|_{3,\alpha} < \varepsilon_3$, $\|q - h_0\|_{2,\alpha/2} < \varepsilon_3$, and $|\delta t| < \varepsilon_3$.

Let $\tilde{U}_2 \subset C_1^{3,\alpha}$ be defined by

$$\tilde{U}_2 = \{w \in C_1^{3,\alpha} : I + f^{-1}D(f^{-1}Dw) > \frac{c_0}{2}, \quad w > \frac{c_1}{2}, \quad \|w - h_0\|_{3,\alpha} < \varepsilon_3\}.$$

By Lemmas 4.4 and 4.6, this guarantees that the map $F^{-1} = id + \mathcal{G}(h, q, \delta t)$ can be defined and invertible if $q \in U_2$ and $h \in \tilde{U}_2$, and that F is a Frechet differentiable map. Define a map

$$(8.11) \quad \begin{aligned} P : U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3) &\rightarrow C_0^{0,\alpha/2} \\ (q, h, \delta t) &\mapsto \frac{h \circ (id + \mathcal{G}(h, q, \delta t)) \det(I + f^{-1}\nabla q \otimes \nabla(f^{-1}) + f^{-2}D^2q + A)}{\det[(I + f^{-1}\nabla h \otimes \nabla(f^{-1}) + f^{-2}D^2h) \circ (id + \mathcal{G}(h, q, \delta t)) + B]} - q, \end{aligned}$$

where the matrices A and B are given by (3.9) and (3.10) with (p_n, p_{n+1}) replaced by (h, q) , and F_{n+1}^{-1} replaced by $id + \mathcal{G}(h, q, \delta t)$. It is clear from the formula above that P maps into $C^{0,\alpha/2}$.

Now we need to show that the right hand side of (8.11) has integral zero. This is similar to assertion (i) of Lemma 5.2. We first show the related property in a more

general setting. Fix $h_0 \in U_1 \cap C^{k+2,\alpha}$, and define more general versions of the sets U_2 and $\tilde{U}_2$ by replacing spaces $C_1^{m,\alpha}$ by spaces $C^{m,\alpha}$ (i.e. removing the condition $\int q_{[0,1]^2} = 1$ in the definitions):

$$U_2' = \{q \in C^{2,\alpha/2} : I + f^{-1}D(f^{-1}Dq) > \frac{c_0}{2}, \|q - p_0\|_{2,\alpha/2} < \varepsilon_3\};$$

$$\tilde{U}_2' = \{q \in C^{3,\alpha} : I + f^{-1}D(f^{-1}Dq) > \frac{c_0}{2}, \|q - p_0\|_{3,\alpha} < \varepsilon_3\}.$$

Then, for $(q, h, \delta t) \in U_2' \times \tilde{U}_2' \times (-\varepsilon_3, \varepsilon_3)$, the expression defining P in (8.11) is well-defined by Lemmas 4.4 and 5.1. Then we prove the following:

Lemma 8.1. *Denote $Q(q, h, \delta t) := P(q, h, \delta t) + q$, i.e.*

$$Q(q, h, \delta t) = \frac{h \circ (id + \mathcal{G}(h, q, \delta t)) \det(I + f^{-1}\nabla q \otimes \nabla(f^{-1}) + f^{-2}D^2q + A)}{\det[(I + f^{-1}\nabla h \otimes \nabla(f^{-1}) + f^{-2}D^2h) \circ (id + \mathcal{G}(h, q, \delta t)) + B]},$$

where A and B are as in (8.11). Then for any $(q, h, \delta t) \in U_2' \times \tilde{U}_2' \times (-\varepsilon_3, \varepsilon_3)$,

$$(8.12) \quad \int_{[0,1]^2} Q(q, h, \delta t)(x) dx = \int_{[0,1]^2} h(x) dx.$$

Proof. Fix $(q, h, \delta t) \in U_2' \times \tilde{U}_2' \times (-\varepsilon_3, \varepsilon_3)$.

From (3.8), we see that the map Q can be written as

$$Q(q, h, \delta t) = h \circ (id + \mathcal{G}(h, q, \delta t)) \det D_x(id + \mathcal{G}(h, q, \delta t)).$$

Also, $id + \mathcal{G}(h, q, \delta t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism by Lemma 4.6.

Denote $G := id + \mathcal{G}(h, q, \delta t)$, then, as we showed, $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism and $Q(q, h, \delta t) = (h \circ G) \det DG$. Now we calculate by changing variables:

$$\int_{[0,1]^2} h(G(x)) \det DG(x) dx = \int_{G([0,1]^2)} h(y) dy = \int_{[0,1]^2} h(x) dx,$$

where the last equality follows from $\mathbb{Z}^2$ -periodicity of $G - id = \mathcal{G}(h, q, \delta t)$ and h , and from the fact that $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism, see Lemma 10.5 for details. $\square$

Now, for $(q, h, \delta t) \in U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3)$, the right hand side of (8.11) has integral zero by Lemma 8.1 and since $\int_{[0,1]^2} h = \int_{[0,1]^2} q = 1$. Thus the map P indeed maps $U_2 \times \tilde{U}_2 \times (-\varepsilon_3, \varepsilon_3)$ into $C_0^{0,\alpha/2}$. Furthermore, similar to Lemma 5.2(iii), one can show that the map P in (8.11) is continuously Frechet differentiable.

To apply the Implicit Function Theorem, we need to show the linearized map $D_q P(h_0, h_0, 0) : C_0^{2,\alpha/2} \rightarrow C_0^{0,\alpha/2}$ is invertible. By the argument similar to the calculations before Lemma 5.4, we obtain

$$(8.13) \quad D_q P(h_0, h_0, 0)w = a_{ij}\partial_{ij}w + b_i\partial_iw - w := L(w).$$

where the coefficients $a_{ij}, b_i \in C^{0,\alpha/2}(\mathbb{T}^2)$. The injectivity follows from the strong maximum principle since the coefficient of the zero-order term is negative.

To see that $L : C_0^{2,\alpha/2} \rightarrow C_0^{0,\alpha/2}$ is surjective, we first notice $L : C^{2,\alpha/2}(\mathbb{T}^2) \rightarrow C^{0,\alpha/2}(\mathbb{T}^2)$ is injective by the strong maximum principle. Then we show that L is surjective as a map from $C^{2,\alpha/2}(\mathbb{T}^2)$ to $C^{0,\alpha/2}(\mathbb{T}^2)$. Indeed, this follows from the method of continuity similar to the proof of Lemma 5.4, except that now we use L_0 defined by $L_0w = \Delta w - w$, and show the existence of a solution to $L_0w = k$ by minimizing the functional

$I[v] = \int_{\mathbb{T}^2} \frac{1}{2}(|\nabla v|^2 + v^2) + kv$ over the space $H^1(\mathbb{T}^2) := \{v \in H_{loc}^1(\mathbb{R}^2) : v \text{ is } \mathbb{Z}^2\text{-periodic}\}$. Now, given $k_1 \in C_0^{0,\alpha/2}$, we can find $w \in C^{2,\alpha/2}$ such that $L(w) = k_1$. It only remains to show $w \in C_0^{2,\alpha/2}$, i.e. that $\int_{\mathbb{T}^2} w = 0$. We first note that $\int_{\mathbb{T}^2} L(w) = \int_{\mathbb{T}^2} k_1 = 0$. Thus, using the form of L in (8.13), it remains to show that $\int_{\mathbb{T}^2} a_{ij} \partial_{ij} w + b_i \partial_i w = 0$. This can be seen as following. Let $Q(q, h, \delta t) = P(q, h, \delta t) + q$ as in Lemma 8.1. Then, from Lemma 8.1 and since $\int_{[0,1]^2} h = 1$, we see that Q maps $U'_2 \times \tilde{U}'_2 \times (-\varepsilon_3, \varepsilon_3)$ into $C_1^{0,\alpha/2}$. From (8.5) and (8.10), $h_0 + \varepsilon w \in U'_2$ if $|\varepsilon|$ sufficiently small. Also, $a_{ij} \partial_{ij} w + b_i \partial_i w = D_q Q(h_0, h_0, 0)w$. Hence

$$\int_{\mathbb{T}^2} a_{ij} \partial_{ij} w + b_i \partial_i w = \left(\frac{d}{d\varepsilon} \int_{\mathbb{T}^2} Q(h_0 + \varepsilon w, h_0, 0) \right)_{|\varepsilon=0} = 0.$$

This proves $w \in C_0^{2,\alpha/2}$.

This discussion proves the following proposition, which is similar to Proposition 5.5 in the SG case:

Proposition 8.2. *There exist $\varepsilon_4, \varepsilon_5 \in (0, \varepsilon_3]$ with $\varepsilon_5 \leq \varepsilon_4$, such that for any $h \in C_0^{3,\alpha}(\mathbb{T}^2)$ with $\|h - h_0\|_{3,\alpha} < \varepsilon_5$ and $|\delta t| < \varepsilon_5$, there exists a unique $q \in C^{2,\alpha/2}(\mathbb{T}^2)$ which solves (3.13) with $F^{-1} = id + \mathcal{G}(p, q, \delta t)$ and satisfies $\|q - h_0\|_{2,\alpha/2} < \varepsilon_4$.*

Thus, denoting $q := \mathcal{H}(p, \delta t)$, we define a map

$$\mathcal{H} : \{\|p - p_0\|_{3,\alpha} < \varepsilon_5\} \times (-\varepsilon_5, \varepsilon_5) \rightarrow U_2.$$

With this, we can set up the iteration as in the beginning of Section 6.

Next we establish estimates for the approximate solutions, similar to Section 6. The proofs essentially repeat the corresponding proofs in Section 6, thus below we will only include the parts of the proofs which differ from the SG case.

Lemma 8.3. *Suppose $h_{n+1} = \mathcal{H}(h_n, \delta t)$ with $\|h_n - h_0\|_{3,\alpha} < \varepsilon_5$, then*

$$(8.14) \quad \|h_{n+1}\|_{k+2,\alpha} \leq C_0(\|h_n\|_{k+2,\alpha}).$$

Proof. Similar to the proof of Lemma 6.1. □

Lemma 8.4.

$$(8.15) \quad \|h_{n+1} - h_n\|_{k+1,\alpha} \leq C_1 \delta t.$$

Here the constant $C_1 = C_1(\|h_{n+1}\|_{k+2,\alpha}, \|h_n\|_{k+2,\alpha})$.

Proof. As in the proof of Lemma 6.2, the result follows from linearization.

We will write $q = h_{n+1}$, $h = h_n$. We start by rewriting the equation (8.9) as

$$(8.16) \quad \frac{\det(I + f^{-1}(x) \nabla q \otimes \nabla(f^{-1}) + f^{-2} D^2 q + A(x))}{q(x)} = \frac{\det[(I + f^{-1} \nabla h \otimes \nabla(f^{-1}) + f^{-2} D^2 h)(F^{-1}(x)) + B(x)]}{h(F^{-1}(x))}.$$

We subtract from both sides the quantity $\frac{\det(I + f^{-1} \nabla h \otimes \nabla(f^{-1}) + f^{-2} D^2 h)}{h(x)}$. The left hand side becomes

$$\frac{\det(I + f^{-1} D(f^{-1} Dq) + A(x)) - \det(I + f^{-1} D(f^{-1} Dh))}{q(x)} - \frac{q - h}{qh} \cdot \det(I + f^{-1} D(f^{-1} Dh)).$$

We can deal with the first term in the same way as we did in the proof of Lemma 6.2. That is, we can write the left hand side as $a_{ij}\partial_{ij}(q-h) + b_i\partial_i(q-h) - c(q-h)$, where $a_{ij}, b_i, c \in C^{k,\alpha}$ and $c > 0$. (Indeed $c = \frac{1}{qh} \det(I + f^{-1}D(f^{-1}Dh))$)

The right hand side becomes

$$\frac{\det[(I + f^{-1}D(f^{-1}Dh))(F^{-1}(x)) + B(x)] - \det(I + f^{-1}D(f^{-1}Dh))}{h(F^{-1}(x))} + \det(I + f^{-1}D(f^{-1}Dh)) \cdot \left(\frac{1}{h(F^{-1}(x))} - \frac{1}{h(x)} \right).$$

In the same way as in the proof of Lemma 6.2, we can write the first term above as $\tilde{b}_i\partial_i(q-h) + \delta t f$, with $\tilde{b}_i \in C^{k-1,\alpha}$, $f \in C^{k,\alpha}$. The second term can be represented in terms of $F^{-1}(x) - x$, hence in terms of $\nabla q - \nabla h$, as was done in the proof of Lemma 5.13. To summarize, we will get an elliptic equation of the form

$$a_{ij}\partial_{ij}(q-h) + c_i\partial_i(q-h) - c(q-h) = \delta t f.$$

with $a_{ij}, c \in C^{k,\alpha}$, $c_i \in C^{k-1,\alpha}$, $f \in C^{k,\alpha}$, and $c > 0$. Noting the sign of the zero-order term, we obtain the desired estimate. $\square$

Lemma 8.5.

$$(8.17) \quad \|F^{-1} - id\|_{k,\alpha} \leq C_2 \delta t.$$

Here $C_2 = C_2(\|h_{n+1}\|_{k+2,\alpha}, \|h_n\|_{k+2,\alpha})$.

Proof. Exactly the same proof as in Lemma 6.3. $\square$

Denote

$$\nu_n = \frac{\det(I + f^{-1}D(f^{-1}h_n))}{h_n}.$$

Lemma 8.6.

$$(8.18) \quad \|\nu_{n+1}\|_{k,\alpha} \leq \|\nu_n\|_{k,\alpha} + C_3 \delta t.$$

Here $C_3 = C_3(\|h_{n+1}\|_{k+2,\alpha}, \|h_n\|_{k+2,\alpha})$. Recall $\nu_n = \frac{\det(I + f^{-1}D(f^{-1}Dh_n))}{h_n}$.

Proof. We follow the proof of Lemma 6.5.

We use the notation $q(x) = h_{n+1}(x)$, $h(x) = h_n(x)$. Define

$$\begin{aligned} \nu'_{n+1} &= \frac{\det(I + f^{-1}D(f^{-1}Dq) + A(x))}{q(x)}, \\ \nu'_n &= \frac{\det[(I + f^{-1}D(f^{-1}Dh))(F^{-1}(x))]}{h(F^{-1}(x))}, \\ \nu''_n &= \frac{\det[(I + f^{-1}D(f^{-1}Dh))(F^{-1}(x)) + B(x)]}{h(F^{-1}(x))}. \end{aligned}$$

By equation (8.9), $\nu'_{n+1} = \nu''_n$. We prove the result by showing

$$(8.19) \quad \|\nu'_{n+1} - \nu_{n+1}\|_{k,\alpha} \leq C \delta t,$$

$$(8.20) \quad \|\nu'_n\|_{k,\alpha} \leq \|\nu_n\|_{k,\alpha} + C \delta t,$$

$$(8.21) \quad \|\nu'_n - \nu''_n\|_{k,\alpha} \leq C \delta t.$$

(8.19), (8.21) are proved in exactly the same way as in Lemma 6.5. (8.20) is the consequence of Lemma 6.4 applied to $G = \frac{\det(I + f^{-1}D(f^{-1}Dp))}{p}$. $\square$

Lemma 8.7.

$$(8.22) \quad \|h_{n+1}\|_{k+2,\alpha} \leq C_4(\|\nu_{n+1}\|_{k,\alpha} + 1).$$

Here $C_4 = C_4(\|p_{n+1}\|_{2,\alpha})$.

So we can obtain exactly the same estimates as in the SG case. Then we repeat the proof of Proposition 6.7, to get the following

Proposition 8.8. *For any $h_0 \in C^{k+2,\alpha}$, $k \geq 2$, satisfying $I + f^{-1}D(f^{-1}Dh_0) > c_0$, there exists $T_1 > 0$, $\varepsilon > 0$, depending only on $\|h_0\|_{k+2,\alpha}$, f , c_0 and k , such that for any $|\delta t| < \varepsilon$, one can construct a sequence (h_n, F_n) of solutions to (8.6), (8.8) (with $h = h_n$, $q = h_{n+1}$ and $F = F_{n+1}$) for $n \leq \frac{T_1}{\delta t} - 1$, such that $h_n \in C_1^{k+2,\alpha}$ and $F_n : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a diffeomorphism satisfying $\det DF_{n+1} = \frac{h_n}{h_{n+1}(F_{n+1})}$. Moreover, (h_n, F_n) satisfy the following estimates:*

$$(8.23) \quad \|h_n\|_{k+2,\alpha} \leq C,$$

$$(8.24) \quad \|h_{n+1} - p_n\|_{k+1,\alpha} \leq C\delta t,$$

$$(8.25) \quad \|F_{n+1}^{-1} - id\|_{k,\alpha} \leq C\delta t,$$

$$(8.26) \quad \|F_{n+1} - id\|_{k,\alpha} \leq C\delta t.$$

for $0 \leq n \leq \frac{T_1}{\delta t} - 1$. Here C depends only on $\|h_0\|_{k+2,\alpha}$, f , c_0 and k .

Define the time-discretized map $\phi(n\delta t, \cdot) = F_n \circ \dots \circ F_1$ as in (6.42). Then similarly to the SG case, we can show that $\phi(n\delta t, x + h) = \phi(n\delta t, x) + h$ for $h \in \mathbb{Z}^2$ and the following estimate holds:

$$(8.27) \quad \|\phi(n\delta t, \cdot) - \phi((n-1)\delta t, \cdot)\|_{k,\alpha} \leq C\delta t.$$

for $1 \leq n \leq \frac{T}{\delta t}$. Here T depends only on $\|h_0\|_{k+2,\alpha}$, f , c_0, c_1 and k .

Also $\phi(n\delta t, \cdot)_{\#}(h_0 dx) = h_n dx$. All the convergence facts obtained in Section 7.1, i.e. (7.5) and (7.9) for h instead of p , hold true here. Similarly to (7.11), we define

$$\mathbf{v}_{g,\delta t}(t, x) = f^{-1}(\phi_{\delta t}(t, x))J\nabla h_{\delta t}(t, \phi_{\delta t}(t, x)), \quad \mathbf{v}_g(t, x) = f^{-1}(\phi(t, x))J\nabla h(t, \phi(t, x)).$$

Then we obtain (7.12) and (7.13). With this, following the argument in Sections 7.2, 7.3, we obtain as in Proposition 7.1:

$$(8.28) \quad \partial_t \phi(t, x) + (-J)f^{-1}(\phi_t(x))\partial_t \mathbf{v}_g(t, x) = \mathbf{v}_g(t, x), \text{ for any } (t, x) \in [0, T] \times \mathbb{T}^2.$$

In Eulerian coordinates this equation is equivalent to

$$(8.29) \quad \mathbf{u}_t(x) + (-J)f^{-1}(x)D_t \mathbf{u}_g(t, x) = \mathbf{u}_g(t, x).$$

This gives (8.2), (8.3). It only remains to show (8.4). To show this, we only need to show

$$(8.30) \quad \phi_t \# h_0 = h_t.$$

Since ϕ_t is a diffeomorphism, this is equivalent to

$$(8.31) \quad \det D\phi_t = \frac{h_0}{h_t(\phi_t)}.$$

To see (8.31), notice ϕ_t is the limit of $\phi_t^{\delta t}$ under the norm $C^{k,\beta}$ for every $\beta < \alpha$, and $h(t)$ is the limit of $h^{\delta t}(t)$ under the norm $C^{k+1,\beta}$, with any $\beta < \alpha$. In the discrete setting: $\det D\phi(n\delta t, \cdot) = \frac{h_0}{h_n(\phi(n\delta t, \cdot))}$. Now one can use the Lipschitz continuity in time of ϕ_n and h_n and the convergence results above to pass to the limit and obtain (8.31).

9. UNIQUENESS

In this section, we prove uniqueness of smooth solutions to the SG and SG shallow water equations. In the following, C will denote a generic constant which may change from line to line, but its dependence will be made clear.

First we briefly explain the idea of the proof of uniqueness. Take the SG case as an example. We employ the Lagrangian formulation (2.28). Let $(p_1, \phi_1), (p_2, \phi_2)$ be two smooth solutions with the same data. We derive a Gronwall-type inequality for the function $a(t) = \sup_{0 \leq s \leq t} \|\phi_{1,s} - \phi_{2,s}\|_{L^2}$. First we take (2.21) for p_1, p_2 and subtract. Then we obtain a linear elliptic equation for the function $\partial_t p_{1,t} - \partial_t p_{2,t}$. This allows us to estimate $\|\partial_t \nabla p_{1,t} - \partial_t \nabla p_{2,t}\|_{L^2}$ in terms of $\|p_{1,t} - p_{2,t}\|_{H^2}$, using linear elliptic estimates. This is done in Lemma 9.4. Next, using equation (2.26) and the measure preserving property of ϕ_t , after some calculations we obtain equation (9.17). Then we take (9.17) for p_1, p_2 and subtract. This gives us a linear elliptic equation for $p_{1,t} - p_{2,t}$. This allows us to control $\|p_{1,t} - p_{2,t}\|_{H^2}$ in terms of $\sup_{0 \leq s \leq t} \|\phi_{1,s} - \phi_{2,s}\|_{L^2}$. This is done in Proposition 9.5. Next we use (2.20) and the fact that $\partial_t \phi = \mathbf{u}(\phi)$ for both solutions to get an estimate for $\|\partial_t \phi_{1,t} - \partial_t \phi_{2,t}\|_{L^2}$ in terms of $\|p_{1,t} - p_{2,t}\|_{H^2}$ and $\|\partial_t \nabla p_{1,t} - \partial_t \nabla p_{2,t}\|_{L^2}$. Integrating in time, we get in Lemma 9.6 an estimate for $\|\phi_{1,t} - \phi_{2,t}\|_{L^2}$ in terms of the time-integral of $\|p_{1,t} - p_{2,t}\|_{H^2}$ and $\|\partial_t \nabla p_{1,t} - \partial_t \nabla p_{2,t}\|_{L^2}$, and noting that these last quantities are estimated above in terms of $\sup_{0 \leq s \leq t} \|\phi_{1,s} - \phi_{2,s}\|_{L^2}$, we get a Gronwall-type inequality. This allows us to conclude uniqueness.

9.1. Uniqueness for SG. In this subsection, we prove the uniqueness part of Theorem 2.1. Suppose we have two solutions of SG, $(p_1, \mathbf{u}_{1,g}, \mathbf{u}_1)$ and $(p_2, \mathbf{u}_{2,g}, \mathbf{u}_2)$ defined on $[0, T_0] \times \mathbb{T}^2$, with regularity $p_i \in L^\infty(0, T_0; C^3(\mathbb{T}^2))$ and $\partial_t p_i \in L^\infty(0, T_0; C^2(\mathbb{T}^2))$. Then by (7.35), we see $\mathbf{u}_i \in L^\infty(0, T_0; C^1(\mathbb{T}^2))$. This regularity allows us to define the flow map $\phi_i(t, x) (i = 1, 2)$ which satisfies $\phi_i(t, x + h) = \phi_i(t, x) + h$ for any $h \in \mathbb{Z}^2$. Also, from (2.6) for p_1 and p_2 , there exists a constant $\lambda > 0$ such that

$$(9.1) \quad I + f^{-1}D(f^{-1}Dp_i) \geq \lambda I \text{ for } i = 1, 2 \text{ on } [0, T_0] \times \mathbb{T}^2.$$

In the following, the argument is made for either one of the solutions if there is no subscript. Also we will denote $\phi_t(x) = \phi(t, x)$, $\phi_{1,t}(x) = \phi_1(t, x)$ with similar notation for other functions.

Recall that (2.28) holds for both solutions. Write (2.26) as

$$(9.2) \quad \partial_t \phi_t(x) + f^{-1}(\phi_t(x)) \partial_t (f^{-1}(\phi_t(x)) \nabla p_t(\phi_t(x))) = f^{-1} J \nabla p_t(\phi_t(x)).$$

We can solve (9.2) for $\partial_t \phi_t$ to obtain

$$(9.3) \quad \partial_t \phi_t(x) = (I + f^{-1}D(f^{-1}Dp_t))^{-1}(\phi_t(x))(f^{-1}J \nabla p_t - f^{-2} \partial_t \nabla p_t)(\phi_t(x)).$$

On the other hand, we can rewrite (9.2) as

$$(9.4) \quad \partial_t \phi_t(x) + \partial_t (f^{-2} \nabla p_t(\phi_t(x))) = f^{-1} \nabla p_t \otimes \nabla (f^{-1})(\phi_t(x)) \partial_t \phi_t(x) + f^{-1} J \nabla p_t(\phi_t(x)).$$

Integrate in t to get

$$\begin{aligned} \phi_t(x) + f^{-2} \nabla p_t(\phi_t(x)) &= x + f^{-2} \nabla p_0(x) \\ &+ \int_0^t f^{-1} \nabla p_s \otimes \nabla (f^{-1})(\phi_s(x)) \partial_s \phi_s(x) ds + \int_0^t f^{-1} J \nabla p_s(\phi_s(x)) ds. \end{aligned}$$

Differentiating on both sides, we obtain

$$\begin{aligned}
 (9.5) \quad & (I + D(f^{-2}Dp_t))(\phi_t)D\phi_t(x) = (I + D(f^{-2}Dp_0))(x) \\
 & + \int_0^t f^{-1}\nabla p_s \otimes \nabla(f^{-1})(\phi_s(x))\partial_s D\phi_s(x)ds + \int_0^t D(f^{-1}J\nabla p_s)(\phi_s(x))D\phi_s(x)ds \\
 & + \int_0^t \sum_{k,m=1}^2 \partial_m(f^{-1}\partial_i p_s \partial_k(f^{-1}))(\phi_s(x))\partial_{x_j} \phi_s^m \partial_s \phi_s^k ds.
 \end{aligned}$$

In the above, ∂_m , ∂_k means ∂_{x_m} and ∂_{x_k} . We do an integration by parts in the first integral of the right hand side:

$$\begin{aligned}
 & \int_0^t f^{-1}\nabla p_s \otimes \nabla(f^{-1})(\phi_s(x))\partial_s D\phi_s(x)ds \\
 & = f^{-1}\nabla p_t \otimes \nabla(f^{-1})(\phi_t(x))D\phi_t(x) - f^{-1}\nabla p_0 \otimes \nabla(f^{-1})(x) \\
 & \quad - \int_0^t \partial_s [f^{-1}\nabla p_s \otimes \nabla(f^{-1})(\phi_s(x))]D\phi_s(x)ds.
 \end{aligned}$$

Plugging this into (9.5), we obtain

$$\begin{aligned}
 (9.6) \quad & (I + f^{-1}D(f^{-1}Dp_t))(\phi_t(x))D\phi_t(x) = (I + f^{-1}D(f^{-1}Dp_0))(x) \\
 & - \int_0^t f^{-1}\partial_s \nabla p_s \otimes \nabla(f^{-1})(\phi_s(x))D\phi_s(x)ds \\
 & + \int_0^t D(f^{-1}J\nabla p_s)(\phi_s(x))D\phi_s(x)ds \\
 & + \int_0^t \sum_{k,m=1}^2 \partial_m(f^{-1}\partial_i p_s \partial_k(f^{-1}))(\phi_s(x))(\partial_{x_j} \phi_s^m \partial_s \phi_s^k - \partial_{x_j} \phi_s^k \partial_s \phi_s^m)ds.
 \end{aligned}$$

Define

$$(9.7) \quad E^1(s, x) = -f^{-1}\partial_s \nabla p_s \otimes \nabla(f^{-1})(x),$$

$$(9.8) \quad E^2(s, x) = D(f^{-1}J\nabla p_s)(x),$$

$$\begin{aligned}
 (9.9) \quad & E_{ik}^3(s, x) = \sum_{m=1}^2 \partial_s \phi_s^m(\phi_s^{-1}(x))[-\partial_m(f^{-1}\partial_i p_s \partial_k(f^{-1}))(x) \\
 & + \partial_k(f^{-1}\partial_i p_s \partial_m(f^{-1}))(x)] \\
 & = \sum_{m=1}^2 \mathbf{u}^m(x)[- \partial_m(f^{-1}\partial_i p_s \partial_k(f^{-1}))(x) + \partial_k(f^{-1}\partial_i p_s \partial_m(f^{-1}))(x)].
 \end{aligned}$$

and let $E(s, x) = E^1 + E^2 + E^3$. Notice that we can use (9.4) to plug in (9.9) for $\partial_s \phi_s^m$. So $E(s, x)$ involves space derivatives of p up to second order and also $\partial_t \nabla p_t$. We can write (9.6) as

$$(9.10) \quad (I + f^{-1}D(f^{-1}Dp_t))(\phi_t(x))D\phi_t(x) = (I + f^{-1}D(f^{-1}Dp_0))(x) + \int_0^t E(s, \phi_s(x))D\phi_s(x)ds.$$

Multiplying (9.10) by $E(t, \phi_t(x))(I + f^{-1}D(f^{-1}Dp_t))^{-1}(\phi_t(x))$ from the left on both sides, we obtain

$$(9.11) \quad \begin{aligned} E(t, \phi_t(x))D\phi_t(x) &= E(t, \phi_t(x))(I + f^{-1}D(f^{-1}Dp_t))^{-1}(\phi_t(x)) \cdot \\ &\quad [(I + f^{-1}D(f^{-1}Dp_0))(x) + \int_0^t E(s, \phi_s(x))D\phi_s(x)ds]. \end{aligned}$$

Denote

$$(9.12) \quad H(t, x) = \int_0^t E(s, \phi_s(x))D\phi_s(x)ds.$$

We see that it solves the following initial value problem.

$$(9.13) \quad \frac{dH(t, x)}{dt} = G(t, \phi_t(x))(D(x) + H(t, x)),$$

$$(9.14) \quad H(0) = 0,$$

where

$$(9.15) \quad G(t, x) = E(t, x)(I + f^{-1}D(f^{-1}Dp))^{-1}(t, x),$$

and

$$(9.16) \quad D(x) = I + f^{-1}D(f^{-1}Dp_0)(x).$$

Finally we conclude from the measure preserving property of ϕ_t and (9.10) that

$$(9.17) \quad \det(I + f^{-1}D(f^{-1}Dp_t)) = \det[I + f^{-1}D(f^{-1}Dp_0)(\phi_t^{-1}(x)) + H(t, \phi_t^{-1}(x))].$$

Let ϕ_1, ϕ_2 be the flow maps corresponding to the two solutions introduced at the beginning of this section, with initial data p_0 . We start with some preliminary estimates:

Lemma 9.1. *Let $H(t, x) : [0, T] \times \mathbb{R}^2 \rightarrow M_{2 \times 2}(\mathbb{R})$ be the solution to the problem:*

$$\begin{aligned} \frac{dH(t, x)}{dt} &= A(t, x) \cdot H(t, x) + B(t, x), \\ H(0) &= 0. \end{aligned}$$

Here $A(t, x), B(t, x), H(t, x)$ are 2×2 matrix valued functions, periodic with respect to $\mathbb{Z}^2$, and we assume $\sup_{0 \leq t \leq T} \|A(t, \cdot)\|_{C^1} + \|B(t, \cdot)\|_{C^1} \leq M$. Then we have

$$(9.18) \quad \|H(t, \cdot)\|_{L^\infty} + \|D_x H(t, \cdot)\|_{L^\infty} \leq Ct.$$

Here C depends only on M and T .

Proof. The estimate of $\|H(t, \cdot)\|_{L^\infty}$ follows from the Gronwall inequality. To estimate $D_x H$, we first differentiate and then use the Gronwall inequality and the estimate already obtained for H . $\square$

We use Lemma 9.1 to estimate C^1 -norm of the function $H(t, x)$ defined in (9.12):

Lemma 9.2. *Let H be as defined in (9.12), then:*

$$(9.19) \quad \|\phi(t, \cdot)\|_{C^1(\mathbb{T}^2)} \leq C,$$

$$(9.20) \quad \|H(t, \cdot)\|_{C^1(\mathbb{T}^2)} \leq Ct.$$

Here C depends on $\|p\|_{L^\infty(0, T_0; C^3)}, \|\partial_t p\|_{L^\infty(0, T_0; C^2)}, \lambda, T_0$.

Proof. To show (9.19), we observe that due to (7.35), $\mathbf{u}_t(\cdot)$ is bounded in $C^1(\mathbb{T}^2)$ for each t , with a bound which has the same dependence as in the Lemma. Thus the flow map ϕ_t of $\mathbf{u}_t(\cdot)$ has an estimate in $C^1(\mathbb{T}^2)$ for each t by a constant with the same dependence.

To prove (9.20), according to (9.13), (9.14) and Lemma 9.1, we only need to show

$$(9.21) \quad \sup_{0 \leq t \leq T_0} \|G(t, \phi_t(\cdot))\|_{C^1(\mathbb{T}^2)} \leq C,$$

since the estimate of $\|D\|_{C^1(\mathbb{T}^2)}$ follows from (9.16).

To prove (9.21), we only need to show $\sup_{0 \leq t \leq T_0} \|G(t, \cdot)\|_{C^1} + \|\phi_t\|_{C^1} \leq C$. The term $\|\phi_t\|_{C^1}$ is estimated in (9.19). For the estimate of $G(t, \cdot)$, we need to go back to (9.15), and (9.7)-(9.9). From the explicit expressions given there and (9.1), we see that the C^1 norm of $G(t, \cdot)$ is bounded by a constant which has the dependence as in the Lemma. $\square$

Below, we write L^2 and H^2 for $L^2(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$.

Lemma 9.3. *For any $t \in [0, T_0]$, we have*

$$(9.22) \quad \|\phi_{1,t}^{-1} - \phi_{2,t}^{-1}\|_{L^2} \leq C \|\phi_{1,t} - \phi_{2,t}\|_{L^2}.$$

and

$$(9.23) \quad \|\phi_{1,t} - \phi_{2,t}\|_{L^2} \leq C \|\phi_{1,t}^{-1} - \phi_{2,t}^{-1}\|_{L^2}.$$

Here C depends on $\|p_i\|_{L^\infty(0, T_0; C^3(\mathbb{T}^2))}$, $\|\partial_t p_i\|_{L^\infty(0, T_0; C^2(\mathbb{T}^2))}$, λ, T_0 , $i = 1, 2$.

Proof. From estimate (9.19) for ϕ_1 and ϕ_2 , we can bound the eigenvalues of $D\phi_{i,t}$, $i = 1, 2$ from above. Since $\det D\phi_{i,t} = 1$, we also obtain the bound for the eigenvalues from below. This gives an estimate for $\|D\phi_t^{-1}\|_{L^\infty}$. Now, since $\phi_{i,t}$ is measure preserving, we have

$$(9.24) \quad \|\phi_{1,t}^{-1} - \phi_{2,t}^{-1}\|_{L^2} = \|\phi_{1,t}^{-1} \circ \phi_{2,t} - id\|_{L^2} \leq \|D\phi_{1,t}^{-1}\|_{L^\infty} \|\phi_{1,t} - \phi_{2,t}\|_{L^2}.$$

This gives (9.22). The proof for (9.23) is similar. $\square$

Lemma 9.4. *For each $t \in [0, T_0]$, we have*

$$(9.25) \quad \|\partial_t \nabla p_{1,t} - \partial_t \nabla p_{2,t}\|_{L^2} \leq C \|p_{1,t} - p_{2,t}\|_{H^2}.$$

Here C depends on $\lambda, \|p_{i,t}\|_{C^2}, \|\partial_t p_{i,t}\|_{C^1}$, $i = 1, 2$.

Proof. We take (2.21) for p_1, p_2 and subtract to get

$$(9.26) \quad \begin{aligned} & \nabla \cdot [(I + f^{-1}D(f^{-1}Dp_{1,t}))^{-1} f^{-2}(\partial_t \nabla p_{1,t} - \partial_t \nabla p_{2,t})] \\ &= \nabla \cdot [(I + f^{-1}D(f^{-1}p_{1,t}))^{-1} (f^{-1}J\nabla p_{1,t} - f^{-1}J\nabla p_{2,t})] \\ &+ \nabla \cdot [(I + f^{-1}D(f^{-1}Dp_{1,t}))^{-1} \\ &- (I + f^{-1}D(f^{-1}Dp_{2,t}))^{-1} (f^{-1}J\nabla p_{2,t} - f^{-2}\partial_t \nabla p_{2,t})]. \end{aligned}$$

Multiplying both sides of (9.26) by $\partial_t p_{1,t} - \partial_t p_{2,t}$ and integrating by parts, and using (9.1), we obtain (9.25) $\square$

Proposition 9.5. *For any $t \in [0, T_0]$*

$$\sup_{0 \leq s \leq t} \|p_{1,s} - p_{2,s}\|_{H^2} \leq C \sup_{0 \leq s \leq t} \|\phi_{1,s} - \phi_{2,s}\|_{L^2}.$$

Here C depends on $\|p_i\|_{L^\infty(0, T_0; C^3(\mathbb{T}^2))}$, $\|\partial_t p_i\|_{L^\infty(0, T_0; C^2(\mathbb{T}^2))}$, λ, T_0 , $i = 1, 2$.

Proof. From (9.17) for p_1 and p_2 , by subtraction we get

$$(9.27) \quad a_{ij} f^{-1} \partial_j (f^{-1} \partial_i (p_{1,t} - p_{2,t})) = F(x),$$

where

$$\begin{aligned} a_{ij} &= \int_0^1 M_{ji} (I + \theta f^{-1} D(f^{-1} D p_{1,t}) + (1 - \theta) f^{-1} D(f^{-1} D p_{2,t})) d\theta, \\ F(x) &= b_{ij} [f^{-1} \partial_j (f^{-1} \partial_i p_0)(\phi_{1,t}^{-1}) - f^{-1} \partial_i (f^{-1} \partial_j p_0)(\phi_{2,t}^{-1}(x)) \\ &\quad + H_{1,ij}(t, \phi_{1,t}^{-1}(x)) - H_{2,ij}(t, \phi_{2,t}^{-1}(x))], \end{aligned}$$

with

$$\begin{aligned} b_{ij} &= \int_0^1 M_{ji} [I + \theta f^{-1} D(f^{-1} D p_0)(\phi_{1,t}^{-1}(x)) + (1 - \theta) f^{-1} D(f^{-1} D p_0)(\phi_{2,t}^{-1}(x)) \\ &\quad + \theta H_1(t, \phi_{1,t}^{-1}(x)) + (1 - \theta) H_2(t, \phi_{2,t}^{-1}(x))] d\theta. \end{aligned}$$

Here $[M_{ij}(A)]$ denotes the cofactor matrix of the matrix A .

We assumed p_1, p_2 to be C^3 smooth, and $f \in C^2$, from which we conclude that the coefficients of equation (9.27), written in the non-divergence form, are in $C^1(\mathbb{T}^2)$, and the C^1 norms depend on $\|p_1, p_2\|_{L^\infty(0, T_0; C^3(\mathbb{T}^2))}$. Also, equation (9.27) is uniformly elliptic (by (9.1)), and has no zero-th order term. Then we apply the H^2 estimate to get

$$(9.28) \quad \|p_{1,t} - p_{2,t}\|_{H^2} \leq C \|F\|_{L^2}.$$

Next notice that b_{ij} is bounded. Indeed, we just need to observe H_1, H_2 are bounded, this follows from Lemma 9.2. Then we estimate

$$\begin{aligned} \|F\|_{L^2} &\leq C \|f^{-1} D(f^{-1} D p_0)(\phi_{1,t}^{-1}) - f^{-1} D(f^{-1} D p_0)(\phi_{2,t}^{-1})\|_{L^2} \\ &\quad + C \|H_{1,ij}(t, \phi_{1,t}^{-1}(\cdot)) - H_{2,ij}(t, \phi_{2,t}^{-1}(\cdot))\|_{L^2} \\ &\leq C \|\phi_{1,t} - \phi_{2,t}\|_{L^2} + C \|D_x H_1(t, \cdot)\|_{L^\infty} \|\phi_{1,t} - \phi_{2,t}\|_{L^2} + C \|H_1 - H_2\|_{L^2}. \end{aligned}$$

Here we used that $p_0 \in C^3$ and Lemma 9.3. Now we use that $\|DH_1\|_{L^\infty}$ is bounded, which follows from Lemma 9.2, to obtain

$$(9.29) \quad \|F\|_{L^2} \leq C \|\phi_{1,t} - \phi_{2,t}\|_{L^2} + C \|H_1 - H_2\|_{L^2}.$$

We can deduce from (9.13) that

$$\frac{d}{dt}(H_1 - H_2) = G_1(t, \phi_{1,t}(x))(H_1 - H_2) + (G_1(t, \phi_{1,t}) - G_2(t, \phi_{2,t}))(H_2 + D).$$

Thus we have

$$\begin{aligned} (9.30) \quad \|H_{1,t} - H_{2,t}\|_{L^2} &\leq \sup_{0 \leq t \leq T_0} \|G_1(t, \cdot)\|_{L^\infty} \int_0^t \|H_{1,s} - H_{2,s}\|_{L^2} \\ &\quad + (\sup_{0 \leq t \leq T_0} \|H\|_{L^\infty} + \|D\|_{L^\infty}) \int_0^t \|G_1(s, \phi_{1,s}(\cdot)) - G_2(s, \phi_{2,s}(\cdot))\|_{L^2}. \end{aligned}$$

To deal with the second term we have

$$\begin{aligned} (9.31) \quad &\|G_1(s, \phi_{1,s}(\cdot)) - G_2(s, \phi_{2,s}(\cdot))\|_{L^2} \\ &\leq \|G_1(s, \phi_{1,s}(\cdot)) - G_1(s, \phi_{2,s}(\cdot))\|_{L^2} + \|G_1(s, \phi_{2,s}(\cdot)) - G_2(s, \phi_{2,s}(\cdot))\|_{L^2} \\ &\leq \|DG_1\|_{L^\infty} \|\phi_{1,s} - \phi_{2,s}\|_{L^2} + \|G_1 - G_2\|_{L^2}. \end{aligned}$$

Recall (9.15) for the definition of G_1 , and the explicit expressions given in (9.7)-(9.9), we see $\|D_x G_1\|_{L^\infty} \leq C$. We still need to estimate $\|G_1 - G_2\|_{L^2}$. Now

$$G_1 - G_2 = (E_1 - E_2)(I + f^{-1}D(f^{-1}Dp_1))^{-1} + E_2[(I + f^{-1}D(f^{-1}Dp_1))^{-1} - (I + f^{-1}D(f^{-1}Dp_2))^{-1}].$$

Then we estimate

(9.32)

$$\begin{aligned} \|G_1 - G_2\|_{L^2} &\leq \|(I + f^{-1}D(f^{-1}Dp))^{-1}\|_{L^\infty} \|E_1 - E_2\|_{L^2} \\ &\quad + \|E_2\|_{L^\infty} \cdot \|(I + f^{-1}D(f^{-1}Dp_1))^{-1} - (I + f^{-1}D(f^{-1}Dp_2))^{-1}\|_{L^2} \\ &\leq C\|E_1 - E_2\|_{L^2} + C\|p_1 - p_2\|_{H^2}. \end{aligned}$$

It only remains to estimate $\|E_1 - E_2\|_{L^2}$. For this we go back to (9.7)-(9.9). One sees easily from (9.7) and the previous lemma

$$(9.33) \quad \|E_{1,s}^1 - E_{2,s}^1\|_{L^2} \leq C\|\partial_s \nabla p_1 - \partial_s \nabla p_2\|_{L^2} \leq C\|p_1 - p_2\|_{H^2}.$$

Also from (9.8):

$$(9.34) \quad \|E_{1,s}^2 - E_{2,s}^2\|_{L^2} \leq C\|p_{1,s} - p_{2,s}\|_{H^2}$$

One can see from (9.9) that

$$(9.35) \quad \|E_{1,s}^3 - E_{2,s}^3\|_{L^2} \leq C\|\mathbf{u}_{1,s}^m - \mathbf{u}_{2,s}^m\|_{L^2}\|p_1\|_{C^2} + C\|\mathbf{u}_{2,s}^m\|_{L^\infty}\|p_1 - p_2\|_{H^2}$$

From (7.35) we observe that $\|\mathbf{u}_t\|_{L^\infty}$ can be estimated by $\|p\|_{C^2}$ and $\|\partial_t p\|_{C^1}$. On the other hand, from Lemma 9.4:

$$(9.36) \quad \|\mathbf{u}_{1,t} - \mathbf{u}_{2,t}\|_{L^2} \leq C(\|p_1 - p_2\|_{H^2} + \|\partial_t \nabla p_1 - \partial_t \nabla p_2\|_{L^2}) \leq C\|p_{1,t} - p_{2,t}\|_{H^2}.$$

Collect (9.32)-(9.36) to get

$$\|G_1 - G_2\|_{L^2} \leq C\|p_1 - p_2\|_{H^2}.$$

Recalling (9.30) and (9.31), we get

$$\|H_{1,t} - H_{2,t}\|_{L^2} \leq C \int_0^t \|H_{1,s} - H_{2,s}\|_{L^2} ds + C \int_0^t \|\phi_{1,s} - \phi_{2,s}\|_{L^2} ds + C \int_0^t \|p_{1,s} - p_{2,s}\|_{H^2} ds.$$

Use the Gronwall inequality to get

$$\|H_{1,t} - H_{2,t}\|_{L^2} \leq C \int_0^t \|\phi_{1,s} - \phi_{2,s}\|_{L^2} ds + C \int_0^t \|p_{1,s} - p_{2,s}\|_{H^2} ds.$$

Recalling (9.28) and (9.29), we have

$$\|p_{1,t} - p_{2,t}\|_{H^2} \leq C\|\phi_{1,t} - \phi_{2,t}\|_{L^2} + \int_0^t \|\phi_{1,s} - \phi_{2,s}\|_{L^2} ds + \int_0^t \|p_{1,s} - p_{2,s}\|_{H^2} ds.$$

Using the Gronwall inequality now completes the proof. $\square$

We present a final lemma which follows directly from (9.3).

Lemma 9.6.

$$\|\phi_{1,t} - \phi_{2,t}\|_{L^2} \leq C \int_0^t (\|p_{1,s} - p_{2,s}\|_{H^2} + \|\partial_t \nabla p_{1,s} - \partial_t \nabla p_{2,s}\|_{L^2}) ds.$$

Here C depends on $\|p_i\|_{L^\infty(0,T_0;C^3(\mathbb{T}^2))}$, $\|\partial_t p_i\|_{L^\infty(0,T_0;C^2(\mathbb{T}^2))}$, λ , T_0 , $i = 1, 2$.

Proof. From (9.3), using (9.1) and the measure-preserving property of $\phi_{2,t}$, we have

$$\begin{aligned}
& \|\partial_t \phi_{1,t} - \partial_t \phi_{2,t}\|_{L^2} \\
& \leq \|(I + f^{-1}D(f^{-1}Dp_{1,t}))^{-1}(f^{-1}J\nabla p_{1,t} - f^{-2}\partial_t \nabla p_{1,t})(\phi_{1,t}) \\
& \quad - (I + f^{-1}D(f^{-1}Dp_{1,t}))^{-1}(f^{-1}J\nabla p_{1,t} - f^{-2}\partial_t \nabla p_{1,t})(\phi_{2,t})\|_{L^2} \\
& \quad + \|(I + f^{-1}D(f^{-1}Dp_{1,t}))^{-1}(f^{-1}J\nabla p_{1,t} - f^{-2}\partial_t \nabla p_{1,t}) \\
& \quad - (I + f^{-1}D(f^{-1}Dp_{2,t}))^{-1}(f^{-1}J\nabla p_{2,t} - f^{-2}\partial_t \nabla p_{2,t})\|_{L^2} \\
& \leq \|D[(I + f^{-1}D(f^{-1}Dp_{1,t}))^{-1}(f^{-1}J\nabla p_{1,t} - f^{-2}\partial_t \nabla p_{1,t})]\|_{L^\infty} \|\phi_{1,t} - \phi_{2,t}\|_{L^2} \\
& \quad + C\|p_{1,t} - p_{2,t}\|_{H^2} + C\|\partial_t \nabla p_{1,t} - \partial_t \nabla p_{2,t}\|_{L^2}.
\end{aligned}$$

Applying the Gronwall inequality now completes the proof. $\square$

Combining all the above estimates, we can now conclude uniqueness. Indeed, from Lemma 9.6 and Lemma 9.4, we get

$$\|\phi_{1,t} - \phi_{2,t}\|_{L^2} \leq C \int_0^t \|p_{1,s} - p_{2,s}\|_{H^2} ds.$$

Let $a(t) = \sup_{0 \leq s \leq t} \|\phi_{1,s} - \phi_{2,s}\|_{L^2}$, and $b(t) = \sup_{0 \leq s \leq t} \|p_{1,s} - p_{2,s}\|_{H^2}$. Therefore

$$a(t) \leq C \int_0^t b(s) ds.$$

Using Proposition 9.5 we obtain

$$a(t) \leq C \int_0^t b(s) ds \leq C \int_0^t a(s) ds.$$

Also $a(0) = 0$ since $\phi_{i,0} = id$. Then from the Gronwall inequality, $a(t) = 0$ for all $t > 0$, that is $\phi_{1,t} = \phi_{2,t}$ for all $t > 0$, which implies $\mathbf{u}_1 = \mathbf{u}_2$. Also, from $a(t) \equiv 0$ and Proposition 9.5, $b(t) = 0$ for all $t > 0$, that is $p_{1,t} = p_{2,t}$ for all $t > 0$. Now uniqueness is proved.

9.2. Uniqueness for the SG shallow water equations with variable f . The uniqueness for the SG shallow water case is very similar to the SG case. Let $\phi(t)$ be the flow map, and $h(t, x)$ be the fluid depth field. Assume that (2.10) is satisfied. Then similar to (9.2), we have

$$(9.37) \quad \partial_t \phi_t(x) + f^{-1}(\phi_t(x)) \partial_t (f^{-1}(\phi_t(x))) \nabla h_t(\phi_t(x)) = f^{-1} J \nabla h_t(\phi_t(x)).$$

The same calculation as in the SG case shows (9.3)-(9.6) still holds with p_t replaced by h_t . In particular, we have

$$(9.38) \quad (I + f^{-1}D(f^{-1}Dh_t))(\phi_t(x)) D\phi_t(x) = (I + f^{-1}D(f^{-1}Dh_0))(x) + H(t, x).$$

Here H is given by the same expression as in (9.12) and solves an initial value problem similar to (9.13)-(9.16) with E given by (9.7)-(9.9), the only difference being p replaced by h . A similar comment applies to Lemma 9.2. We can conclude $\|H(t, \cdot)\|_{C^1} \leq Ct$. Here C depends on $\|h\|_{L^\infty(0, T_0; C^3(\mathbb{T}^2))}$, $\|\partial_t h\|_{L^\infty(0, T_0; C^2(\mathbb{T}^2))}$, λ , T_0 , where $\lambda > 0$ is such number that $I + f^{-1}D(f^{-1}Dh) \geq \lambda I$ on $[0, T_0] \times \mathbb{T}^2$, which exists by (2.10).

The difference arises in (9.17). Since ϕ_t does not preserve Lebesgue measure, but satisfies $(\phi_t)_\# h_0 = h_t$, we have

$$(9.39) \quad \det D\phi_t(x) = \frac{h_0(x)}{h_t(\phi_t(x))}.$$

Taking the determinant of both sides of (9.38), we obtain

$$\frac{\det(I + f^{-1}D(f^{-1}Dh_t)(\phi_t(x)))}{h_t(\phi_t(x))} = \frac{\det(I + f^{-1}D(f^{-1}Dh_0)(x) + H(t, x))}{h_0(x)}.$$

Applying this at the point $\phi_t^{-1}(x)$, we get

$$(9.40) \quad \frac{\det(I + f^{-1}D(f^{-1}Dh_t)(x))}{h_t(x)} = \frac{\det(I + f^{-1}D(f^{-1}Dh_0)(\phi_t^{-1}(x)) + H(t, \phi_t^{-1}(x)))}{h_0(\phi_t^{-1}(x))}.$$

As in the SG case, we solve (9.37) for $\partial_t \phi_t$:

$$\partial_t \phi_t(x) = (I + f^{-1}D(f^{-1}Dh_t))^{-1}(\phi_t(x))[f^{-1}J\nabla h_t - f^{-2}\partial_t \nabla h_t](\phi_t(x)).$$

Equivalently, writing this at the point $\phi_t^{-1}(x)$, we get an expression for the physical velocity:

$$(9.41) \quad \mathbf{u}_t = (I + f^{-1}D(f^{-1}Dh_t))^{-1}(f^{-1}J\nabla h_t - f^{-2}\partial_t \nabla h_t).$$

Applying the equation $\partial_t h + \nabla \cdot (\mathbf{u}_t h_t) = 0$, we obtain an elliptic equation for $\partial_t h$, namely

$$(9.42) \quad \nabla \cdot [(I + f^{-1}D(f^{-1}Dh_t))^{-1}f^{-2}h_t\partial_t \nabla h_t] - \partial_t h = \nabla \cdot [(I + f^{-1}D(f^{-1}Dh_t))^{-1}f^{-1}h_tJ\nabla h_t].$$

Suppose we have two solutions of the SG shallow water equations, $(h_1, \mathbf{u}_{1,g}, \mathbf{u}_1)$ and $(h_2, \mathbf{u}_{2,g}, \mathbf{u}_2)$, which satisfy the assumptions of Theorem 2.2, and are defined on $[0, T_0] \times \mathbb{R}^2$. Then there exist constants $\lambda, \mu > 0$ such that

$$(9.43) \quad I + f^{-1}D(f^{-1}Dh_i) \geq \lambda I \text{ and } h_i \geq \mu \text{ for } i = 1, 2 \text{ on } [0, T_0] \times \mathbb{T}^2.$$

Due to (9.41), we have $\mathbf{u}_i \in L^\infty(0, T_0; C^1)$, hence we can define the flow map $\phi_i(t)$.

Next we present the estimates in the shallow water case which lead to uniqueness, parallel to the previous subsection.

As a preparation, we first observe that the analogue of Lemma 9.3 also holds for the shallow water case.

Lemma 9.7. *For any $t \in [0, T_0]$, we have*

$$(9.44) \quad \|\phi_{1,t}^{-1} - \phi_{2,t}^{-1}\|_{L^2} \leq C\|\phi_{1,t} - \phi_{2,t}\|_{L^2}$$

and

$$(9.45) \quad \|\phi_{1,t} - \phi_{2,t}\|_{L^2} \leq C\|\phi_{1,t}^{-1} - \phi_{2,t}^{-1}\|_{L^2}.$$

Here C depends only on $\lambda, \mu, \|h_{i,t}\|_{L^\infty(0, T_0; C^2(\mathbb{T}^2))}, \|\partial_t h_i\|_{L^\infty(0, T_0; C^1(\mathbb{T}^2))}, i = 1, 2$.

Proof. From (9.41), $\mathbf{u}_t \in L^\infty(0, T_0; C^1(\mathbb{T}^2))$. Since the flow map ϕ_i is generated by $\mathbf{u}_i$, it follows that $\|D\phi_{i,t}\|_{L^\infty}$ are uniformly bounded in t , with a bound having the dependence as in the Lemma. Also, from (9.39) combined with (2.9) and (9.43), it follows that $\frac{c_1}{\|h_{i,t}\|_0} \leq \det D\phi_{i,t} \leq \frac{\|h_0\|_0}{\mu}$, for $i = 1, 2$. From this we conclude $\|D\phi_{i,t}^{-1}\|_{L^\infty}$ are also uniformly bounded in $t \in [0, T_0]$, with a bound having the dependence as in the lemma.

To prove (9.44), we can calculate

$$\begin{aligned} \int_{[0,1]^2} |\phi_{1,t}^{-1}(x) - \phi_{2,t}^{-1}(x)|^2 dx &= \int_{[0,1]^2} |x - \phi_{2,t}^{-1}(\phi_{1,t}(x))|^2 \det D\phi_{1,t}(x) dx \\ &= \int_{[0,1]^2} |\phi_{2,t}^{-1}(\phi_{2,t}(x)) - \phi_{2,t}^{-1}(\phi_{1,t}(x))|^2 \det D\phi_{1,t}(x) dx \\ &\leq \|D\phi_{2,t}^{-1}\|_{L^\infty}^2 \|\det D\phi_{1,t}\|_{L^\infty} \|\phi_{1,t} - \phi_{2,t}\|_{L^2}^2. \end{aligned}$$

The proof for (9.45) is similar. $\square$

Similar to Lemma 9.4, we can show

Lemma 9.8. *For each $t \in [0, T_0]$, we have*

$$(9.46) \quad \|\partial_t \nabla h_{1,t} - \partial_t \nabla h_{2,t}\|_{L^2} \leq C \|h_{1,t} - h_{2,t}\|_{H^2}.$$

Here C depends on $\lambda, \mu, \|h_{i,t}\|_{L^\infty(0,T_0;C^2)}, \|\partial_t h_i\|_{L^\infty(0,T_0;C^1)}, i = 1, 2$.

Proof. Similar to the proof of Lemma 9.4, we take equation (9.42) for h_1 and h_2 , and subtract. Multiplying both sides of the resulting equation by $\partial_t h_{1,t} - \partial_t h_{2,t}$ and integrating by parts, we obtain (9.46), noticing that the zero-th order term has the right sign so we can argue in the same way as in Lemma 9.4. $\square$

The key proposition holds also in SG shallow water case. It is an analogue to Proposition 9.5.

Proposition 9.9. *For any $t \in [0, T_0]$*

$$\sup_{0 \leq s \leq t} \|h_{1,s} - h_{2,s}\|_{H^2} \leq C \sup_{0 \leq s \leq t} \|\phi_{1,s} - \phi_{2,s}\|_{L^2}.$$

Here C depends on $\lambda, \mu, \|h_i\|_{L^\infty(0,T_0;C^3)}, \|\partial_t h_i\|_{L^\infty(0,T_0;C^2)}, T_0, i = 1, 2$.

Proof. Similar to the proof of Proposition 9.5, we take equation (9.40) for h_1, h_2 and subtract. Then we obtain equation for $h_{1,t} - h_{2,t}$, with the left-hand side

$$\begin{aligned} &\frac{\det(I + f^{-1}D(f^{-1}Dh_{1,t})(x))}{h_{1,t}(x)} - \frac{\det(I + f^{-1}D(f^{-1}Dh_{2,t})(x))}{h_{2,t}(x)} \\ &= \frac{a_{ij}f^{-1}\partial_j(f^{-1}\partial_i(h_{1,t} - h_{2,t}))}{h_{1,t}} - \frac{\det(I + f^{-1}D(f^{-1}Dh_{2,t}))}{h_{1,t}h_{2,t}}(h_{1,t} - h_{2,t}). \end{aligned}$$

Here

$$a_{ij} = \int_0^1 M_{ji}(I + \theta f^{-1}D(f^{-1}Dh_{1,t}) + (1 - \theta)f^{-1}D(f^{-1}Dh_{2,t}))d\theta.$$

Property (9.43) and regularity assumptions for h_1, h_2 imply that the equation is uniformly elliptic, and coefficients are in $C^1(\mathbb{T}^2)$, with ellipticity constant and C^1 norms depending on the parameters listed in the lemma. The sign of the coefficient of zero-th order term is negative, by property (9.43). Hence we can apply the H^2 estimates to conclude

$$\|h_{1,t} - h_{2,t}\|_{H^2} \leq C \|F\|_{L^2},$$

where F is the right-hand side of the equation. Now we can do a calculation as in the proof of Proposition 9.5 to estimate $\|F\|_{L^2}$. $\square$

Finally, we have

Lemma 9.10.

$$\|\phi_{1,t} - \phi_{2,t}\|_{L^2} \leq C \int_0^t (\|h_{1,s} - h_{2,s}\|_{H^2} + \|\partial_t \nabla h_{1,s} - \partial_t \nabla h_{2,s}\|_{L^2}) ds.$$

Here C depends on $\|h_i\|_{L^\infty(0,T_0;C^3)}$, $\|\partial_t h_i\|_{L^\infty(0,T_0;C^2)}$, $i = 1, 2$.

Proof. We argue exactly as in the proof of Lemma 9.6, using (9.39) for $\phi_{2,t}$ instead of the measure-preserving property. $\square$

With the above estimates, we can conclude uniqueness in the same way as for the SG case.

10. APPENDIX

10.1. Frechet derivatives for maps between Hölder spaces.

Lemma 10.1. *Let $\alpha, \beta \in (0, 1)$ and $\alpha > \beta$. Then,*

(i) *The map $G : C^{2,\alpha}(\mathbb{T}^2) \times C^{1,\beta}(\mathbb{T}^2; \mathbb{R}^2) \rightarrow C^{1,\beta}(\mathbb{T}^2)$ defined by*

$$G(p, w) = p \circ (id + w)$$

is continuously Frechet-differentiable.

(ii) *The map G above considered as a map $C^{1,\alpha}(\mathbb{T}^2) \times C^{1,\beta}(\mathbb{T}^2; \mathbb{R}^2) \rightarrow C^{0,\beta}(\mathbb{T}^2)$ is also continuously Frechet differentiable.*

Proof. In the argument below, the constant C may change from line to line, and depends only on $\alpha, \beta, \|p\|_{C^{2,\alpha}(\mathbb{T}^2)}$ and $\|w\|_{C^{1,\beta}(\mathbb{T}^2; \mathbb{R}^2)}$.

We only prove the first statement. The second statement is proved in a similar manner and is simpler.

That G maps into $C^{1,\beta}(\mathbb{T}^2)$ is clear from the expression. We show Frechet differentiability, with differential $DG(p, w)$ for any $(p, w) \in C^{2,\alpha}(\mathbb{T}^2) \times C^{1,\beta}(\mathbb{T}^2; \mathbb{R}^2)$ given by

$$(10.1) \quad \begin{aligned} DG(p, w) : C^{2,\alpha}(\mathbb{T}^2) \times C^{1,\beta}(\mathbb{T}^2; \mathbb{R}^2) &\rightarrow C^{1,\beta}(\mathbb{T}^2), \\ (h_1, h_2) &\mapsto h_1 \circ (id + w) + (\nabla p \circ (id + w)) \cdot h_2. \end{aligned}$$

Here the second term above denotes the dot product of 2-vectors in $\mathbb{R}^2$.

By definition, we need to show that

$$\frac{\|G(p + \varepsilon h_1, w + \delta h_2) - G(p, w) - DG(p, w)(\varepsilon h_1, \delta h_2)\|_{1,\beta}}{\varepsilon + \delta} \rightarrow 0,$$

as $\varepsilon, \delta \rightarrow 0$, uniformly for any $\|h_1\|_{2,\alpha}, \|h_2\|_{1,\beta} \leq 1$. we can write

$$\begin{aligned} &[G(p + \varepsilon h_1, w + \delta h_2) - G(p, w) - DG(p, w)(\varepsilon h_1, \delta h_2)](x) \\ &= \int_0^1 (\nabla p(x + w(x) + \theta \delta h_2(x)) - \nabla p(x + w(x))) d\theta \cdot \delta h_2(x) \\ &\quad + \varepsilon (h_1(x + w(x) + \delta h_2(x)) - h_1(x + w(x))). \end{aligned}$$

Therefore,

$$\begin{aligned}
(10.2) \quad & \|G(p + \varepsilon h_1, w + \delta h_2) - G(p, w) - DG(p, w)(\varepsilon h_1, \delta h_2)\|_{1,\beta} \\
& \leq \delta \sup_{0 \leq \theta \leq 1} \|\nabla p(id + w + \theta \delta h_2) - \nabla p(id + w)\|_{1,\beta} \|h_2\|_{1,\beta} \\
& \quad + \varepsilon \|h_1(id + w + \delta h_2) - h_1(id + w)\|_{1,\beta} \\
& \leq \delta \sup_{0 \leq \theta \leq 1} \|\nabla p(id + w + \theta \delta h_2) - \nabla p(id + w)\|_{1,\beta} + \varepsilon \|h_1(id + w + \delta h_2) - h_1(id + w)\|_{1,\beta}.
\end{aligned}$$

Now

$$\begin{aligned}
& D[\nabla p(id + w + \theta \delta h_2) - \nabla p(id + w)] \\
& = D^2 p(id + w + \theta \delta h_2)(I + Dw + \theta \delta Dh_2) - D^2 p(id + w)(I + Dw) \\
& = [D^2 p(id + w + \theta \delta h_2) - D^2 p(id + w)](I + Dw + \theta \delta Dh_2) + \theta \delta D^2 p(id + w) Dh_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|D[\nabla p(id + w + \theta \delta h_2) - \nabla p(id + w)]\|_{0,\beta} \\
& \leq \|D^2 p(id + w + \theta \delta h_2) - D^2 p(id + w)\|_{0,\beta} \|I + Dw + \theta \delta Dh_2\|_{0,\beta} \\
& \quad + \delta \|D^2 p(id + w)\|_{0,\beta} \|Dh_2\|_{0,\beta} \\
& \leq \|D^2 p(id + w + \theta \delta h_2) - D^2 p(id + w)\|_{0,\beta} (1 + \|w\|_{1,\beta} + \delta) + \delta C \\
& \leq C \|D^2 p(id + w + \theta \delta h_2) - D^2 p(id + w)\|_{0,\beta} + \delta C
\end{aligned}$$

Finally, we use the interpolation inequality to estimate the first term above:

$$\begin{aligned}
& \|D^2 p(id + w + \theta \delta h_2) - D^2 p(id + w)\|_{0,\beta} \\
& \leq \|D^2 p(id + w + \theta \delta h_2) - D^2 p(id + w)\|_{0,\alpha}^{\beta/\alpha} \cdot \|D^2 p(id + w + \theta \delta h_2) - D^2 p(id + w)\|_0^{1-\beta/\alpha} \\
& \leq C(\|p\|_{2,\alpha} \|h_2\|_0^\alpha \delta^\alpha)^{1-\beta/\alpha} \\
& \leq C \delta^{\alpha-\beta} \rightarrow 0.
\end{aligned}$$

The second term in (10.2) can be handled in a similar way.

Also using the explicit expression of $DG(p, w)$ in (10.1), and arguing in a similar way as above, we show that DG is continuous in p, w under the operator norm. $\square$

Lemma 10.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^3 function, then the map*

$$\begin{aligned}
& F : C^{0,\alpha}(\mathbb{T}^2) \times C^{0,\alpha} \dots C^{0,\alpha}(\mathbb{T}^2) \rightarrow C^{0,\alpha}(\mathbb{T}^2) \\
& (p_1, p_2, \dots, p_n) \mapsto f(p_1, p_2, \dots, p_n).
\end{aligned}$$

is continuously Frechet differentiable.

Proof. For simplicity, we just prove this Lemma for $n = 2$. The general n can be handled in a similar way. From Taylor's formula, one can write

$$f(p_1 + \varepsilon h_1, p_2 + \delta h_2) = f(p_1, p_2) + \varepsilon h_1 \partial_1 f(p_1) + \delta h_2 \partial_2 f(p_2) + \int_0^1 A(x, t)(1-t) dt,$$

where

$$A(x, t) = \partial_{11} f(\varepsilon h_1)^2 + 2\partial_{12} f(\varepsilon h_1)(\delta h_2) + \partial_{22} f(\delta h_2)^2.$$

Here $\partial_{ij} f$ is evaluated at $(p_1 + t\varepsilon h_1, p_2 + t\delta h_2)$. Now observe that if $\|h_1\|_{0,\alpha}, \|h_2\|_{0,\alpha} \leq 1$, one has a uniform $C^{0,\alpha}$ bound for $\frac{A(x,t)}{\varepsilon^2 + \delta^2}$. This proves that the map F is differentiable. The

continuity of the differential follows readily from the expression $D_{p_i}F(p_1, p_2, \dots, p_n)h_i = \partial_i f(p)h_i$.

□

Corollary 10.3. *The map*

$$\det : C^{0,\alpha}(M_{2 \times 2}) \rightarrow C^{0,\alpha}$$

$$A \mapsto \det A.$$

is continuously Frechet differentiable. Here $C^{0,\alpha}(M_{2 \times 2})$ denotes the space of 2×2 matrices with entries in $C^{0,\alpha}(\mathbb{T}^2)$.

Similar to Lemma 10.1, one can also show

Lemma 10.4. *Let $f \in C^2(\mathbb{T}^2)$, then the map*

$$F : C^{1,\alpha}(\mathbb{T}^2; \mathbb{R}^2) \rightarrow C^{0,\alpha}(\mathbb{T}^2)$$

$$w \mapsto f(id + w).$$

is continuously Frechet differentiable.

Proof. Let $\|h\|_{1,\alpha} \leq 1$, we can write

$$\begin{aligned} & f(id + w + \varepsilon h) - f(id + w) - \nabla f(id + w)\varepsilon h \\ &= \varepsilon h \int_0^1 [\nabla f(id + w + \varepsilon \theta h) - \nabla f(id + w)] d\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|\nabla f(id + w + \varepsilon h) - \nabla f(id + w) - \nabla f(id + w)\varepsilon h\|_{0,\alpha} \\ & \leq \varepsilon \|h\|_{0,\alpha} \sup_{0 \leq \theta \leq 1} \|\nabla f(id + w + \varepsilon \theta h) - \nabla f(id + w)\|_{0,\alpha}. \end{aligned}$$

Now we use the interpolation inequality:

$$\begin{aligned} & \|\nabla f(id + w + \varepsilon \theta h) - \nabla f(id + w)\|_{0,\alpha} \\ & \leq \|\nabla f(id + w + \varepsilon \theta h) - \nabla f(id + w)\|_{C^1}^\alpha \cdot \|\nabla f(id + w + \varepsilon \theta h) - \nabla f(id + w)\|_0^{1-\alpha}. \\ & \leq C(\varepsilon \|D^2 f\|_0 \|h\|_0)^{1-\alpha}. \end{aligned}$$

□

10.2. Integration on $\mathbb{T}^2$. We note the following elementary property of $\mathbb{Z}^2$ -periodic functions:

Lemma 10.5. *Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 diffeomorphism, $h \in C(\mathbb{R}^2)$, and let $G - id$ and h be $\mathbb{Z}^2$ -periodic. Then*

$$\int_{G([0,1]^2)} h(y) dy = \int_{[0,1]^2} h(x) dx.$$

Proof. We can write

$$G([0,1]^2) = \cup_{(i,j) \in \mathbb{Z}^2} A_{ij}, \quad \text{where } A_{ij} \text{ are disjoint, and } A_{ij} - (i,j) \subset [0,1]^2.$$

Denote $B_{ij} := A_{ij} - (i,j)$. We show that

$$(10.3) \quad B_{ij} \text{ are disjoint, and } [0,1]^2 = \cup_{(i,j) \in \mathbb{Z}^2} B_{ij}.$$

Indeed, suppose $x \in B_{ij} \cap B_{kl}$. Then $x + (i,j) = G(y)$ and $x + (k,l) = G(z)$ for $y, z \in [0,1]^2$. Then using $\mathbb{Z}^2$ -periodicity of $G - id$, we have $G(y) - y = G(y - (i,j)) - (y - (i,j))$

and $G(z) - z = G(z - (k, l)) - (z - (k, l))$, from which $x = G(y) - (i, j) = G(y - (i, j))$ and similarly $x = G(z - (k, l))$. Since $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism, we get $y - (i, j) = z - (k, l)$, which implies $y = z$ and $(i, j) = (k, l)$ since $y, z \in [0, 1]^2$. Thus B_{ij} are disjoint. Next, let $x \in [0, 1]^2$. Since $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism, then $x = G(y)$ for some $y \in \mathbb{R}^2$. Then $y = z - (i, j)$ for some $z \in [0, 1]^2$ and $(i, j) \in \mathbb{Z}^2$. Using again $\mathbb{Z}^2$ -periodicity of $G - id$, we get $x = G(z - (i, j)) = G(z) - (i, j)$, from which, using that $x, z \in [0, 1]^2$, we get $x \in B_{ij}$. Thus $[0, 1]^2 = \cup_{(i,j) \in \mathbb{Z}^2} B_{ij}$. Now (10.3) is proved.

With this, using $\mathbb{Z}^2$ -periodicity of h , we get

$$\int_{G([0,1]^2)} h(y) dy = \sum_{(i,j) \in \mathbb{Z}^2} \int_{A_{ij}} h(y) dy = \sum_{(i,j) \in \mathbb{Z}^2} \int_{B_{ij}} h(x) dx = \int_{[0,1]^2} h(x) dx.$$

□

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