

EXISTENCE AND STABILITY OF MULTIDIMENSIONAL TRANSONIC FLOWS THROUGH AN INFINITE NOZZLE OF ARBITRARY CROSS-SECTIONS

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ABSTRACT. We establish the existence and stability of multidimensional steady transonic flows with transonic shocks through an infinite nozzle of arbitrary cross-sections, including a slowly varying de Laval nozzle. The transonic flow is governed by the inviscid potential flow equation with supersonic upstream flow at the entrance, uniform subsonic downstream flow at the exit at infinity, and the slip boundary condition on the nozzle boundary. Our results indicate that, if the supersonic upstream flow at the entrance is sufficiently close to a uniform flow, there exists a solution that consists of a $C^{1,\alpha}$ subsonic flow in the unbounded downstream region, converging to a uniform velocity state at infinity, and a $C^{1,\alpha}$ multidimensional transonic shock dividing the subsonic flow from the supersonic upstream flow; the uniform velocity state at the exit at infinity in the downstream direction is uniquely determined by the supersonic upstream flow; and the shock is orthogonal to the nozzle boundary at every point of their intersection. In order to construct such a transonic flow, we reformulate the multidimensional transonic nozzle problem into a free boundary problem for the subsonic phase, in which the equation is elliptic and the free boundary is a transonic shock. The free boundary conditions are determined by the Rankine-Hugoniot conditions along the shock. We further develop a nonlinear iteration approach and employ its advantages to deal with such a free boundary problem in the unbounded domain. We also prove that the transonic flow with a transonic shock is unique and stable with respect to the nozzle boundary and the smooth supersonic upstream flow at the entrance.

1. INTRODUCTION

We are concerned with the existence and stability of multidimensional steady transonic flows with transonic shocks through multidimensional infinite nozzles of arbitrary cross-sections. Such problems naturally arise in many physical situations, especially in the de Laval nozzles which have widely been used in the design of steam turbines and modern rocket engines (see Courant-Friedrichs [11], Whitham [47], and the references cited therein). Since the nozzles in applications are usually much longer with respect to their cross-sections, the problem is often formulated mathematically as an infinite nozzle problem. Correspondingly, such a multidimensional infinite nozzle problem has extensively been studied experimentally, computationally, and asymptotically (see [11, 15, 18, 21, 22, 23, 47] and the references cited therein). The stability issue for transonic gas flow through a nozzle for the one-dimensional model was analyzed in [40, 41]. The existence and stability of multidimensional steady transonic flows through multidimensional infinite nozzles has remained open; see [5, 11, 12, 43, 47].

In this paper, we focus on the infinity nozzle to establish the existence and stability of multidimensional transonic flows with supersonic upstream flows at the entrance, uniform subsonic downstream flows at the exit at infinity, and the slip boundary condition on the nozzle boundary. The potential flow equation for the velocity potential $\varphi : \Omega \subset \mathbf{R}^n \rightarrow \mathbf{R}$ is a second-order

Date: November 1, 2005.

1991 Mathematics Subject Classification. 35M10, 35J65, 35R35, 76H05, 76L05, 35B45.

Key words and phrases. Elliptic-hyperbolic, nonlinear equations, second order, mixed type, multidimensional transonic shocks, free-boundary problems, unbounded domains, infinite nozzles, cross-sections, existence, uniqueness, stability, Euler equations, inviscid potential flow.

nonlinear mixed equation of elliptic-hyperbolic type:

$$\operatorname{div}(\rho(|D\varphi|^2)D\varphi) = 0, \quad x \in \Omega \subset \mathbf{R}^n, \quad (1.1)$$

where the density function $\rho(q^2)$ is

$$\rho(q^2) = (1 - \theta q^2)^{\frac{1}{2\theta}} \quad (1.2)$$

and $\theta = \frac{\gamma-1}{2} > 0$ with adiabatic exponent $\gamma > 1$.

The nonlinear equation (1.1) is elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0 \quad (1.3)$$

and hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \quad (1.4)$$

For this infinite nozzle problem, we first seek a multidimensional transonic flow containing a multidimensional transonic shock which divides a subsonic downstream flow from the supersonic upstream flow. In order to construct such a transonic flow, we first formulate this infinite nozzle problem into a free boundary problem: The multidimensional transonic shock is a free boundary which divides two phases of $C^{1,\alpha}$ flow in the infinite nozzle, and the equation is hyperbolic in the supersonic upstream phase and elliptic in the subsonic downstream phase. Since the existence of the supersonic upstream phase is a direct corollary of the standard local existence theory of the initial-boundary value problem with the slip boundary condition for nonlinear wave equations (cf. [27, 28, 30]), we can further formulate the free boundary problem into a one-phase free boundary problem for a nonlinear elliptic equation by shiffmanization, a cut-off technique (see [4, 45]).

We further develop a nonlinear iteration approach and employ its advantages to deal with such a free boundary problem in the unbounded domain and to solve the multidimensional transonic nozzle problem in a direct fashion. Our results indicate that, for the transonic nozzle problem, there exists a transonic flow such that the flow is divided into a $C^{1,\alpha}$ subsonic flow up to the nozzle boundary in the unbounded downstream region from the supersonic upstream flow by a $C^{1,\alpha}$ multidimensional transonic shock that is orthogonal to the nozzle boundary at every intersection point, and the uniform velocity state at the exit at infinity in the downstream direction is uniquely determined by the supersonic upstream flow at the entrance which is sufficiently close to a uniform flow. The uniform velocity state at the exit can not be apriori prescribed from the corresponding pressure for such a flow to exist. We further prove that the transonic flow with a transonic shock is unique and stable with respect to the nozzle boundary and the smooth supersonic upstream flow at the entrance.

One of the advantages to employ our nonlinear iteration approach is that, as long as we know how the corresponding fixed conormal problem for (1.1) can be solved and estimated, the solution of the free boundary problem and its estimates directly follow, even for more complicated geometry of the domain under consideration. In this sense, the iteration approach is more efficient than the partial hodograph approach developed in [6], since we do not need to change equation (1.1) with fine features. This indeed plays an important role to achieve the $C^{1,\alpha}$ -regularity of both the solution and the free boundary up to the nozzle boundary and to allow the nozzle boundary of arbitrary cross-sections, by employing the features of equation (1.1). Also, this approach enables us to deal with the multidimensional nozzle problem in a direct fashion, which especially applies to the bounded nozzle problems with various different compatible boundary conditions at the exit by solving the elliptic problems with nonsmooth fixed boundaries, as discussed in Section 3 and in [9, 10, 13, 24].

Another new ingredient of this paper is to develop a $C^{1,\alpha}$ -estimate framework for solutions of the conormal problem for elliptic equations of divergence form with C^α -coefficients in the domains with corners and wedges. In our nozzle problem, the corners and wedges are formed in

the intersection of the free boundary with the fixed boundary, so it is necessary to develop such an estimate framework to deal with the new difficulty of the intersection of a free boundary with a fixed boundary, especially when $n \geq 3$. In [5], we considered a transonic flow in an infinite cylinder with a cross-section of the form $\Lambda = (0, 1)^{n-1}$, i.e., a flow in the domain $(0, 1)^{n-1} \times \mathbf{R}$, in which, by reflections, we can reduce the problem to the domain $\mathbf{T}^{n-1} \times \mathbf{R}$, where $\mathbf{T}^{n-1}$ is the $(n-1)$ -dimensional torus. Thus, in studying the free boundary problems in [4, 5], we were able to avoid studying the intersection of the free boundary with the fixed boundary. In the case of arbitrary smooth cross-section in dimension $n \geq 3$, this can not be done even for the flow in the unperturbed cylinder: Indeed, if we locally flatten the boundary, the equation in the new coordinates changes its structure and the reflection leads to an equation with discontinuous coefficients. When we consider a flow in the nozzle (i.e. in a perturbed cylinder), we face a similar problem. Thus, in the present paper, we have to consider the intersection of a free boundary with a fixed boundary, that is, a nonlinear conormal problem in the domain with nonsmooth boundary for any dimension ($n \geq 2$). The problem in a fixed domain with wedges was considered in [36], where the $C^{1,\beta}$ -regularity of solutions near the wedges for a general oblique derivative problem for linear elliptic equations was obtained. On the other hand, the results in [36] require certain conditions on the relative geometry of the wedge and the vector fields in the oblique derivative conditions; and, moreover, the Hölder exponent β of the solution depends on the geometry of wedges, in addition to the Hölder exponent of the coefficients and the right-hand sides of the equation and boundary conditions. These features present the difficulties in using the results of [36] in constructing solutions for our nonlinear problem. Thus, in this paper, we first study the existence and regularity of solutions, by restricting the class of equations and considering only the conormal boundary conditions and cylindrical domains, and then we obtain sharp results that are used for constructing the solution of the free boundary problem for the nonlinear elliptic equation in the subsonic domain.

A further new ingredient is to develop a useful approach for obtaining uniform estimates of solutions in the (perturbed) cylinders of length $R \rightarrow \infty$ and showing that the velocity (the gradient of the solution) converges to an asymptotic constant state at infinity. In [5], we reduced the problem to the domain $\mathbf{T}^{n-1} \times \mathbf{R}$ so that the symmetry allows us to use the pointwise estimates of solutions to study the asymptotics at infinity. In the present case, the domains are of more general geometry, so our approach is now to use natural integral estimates of solutions to achieve the asymptotics of solutions at infinity. This approach should be useful to other problems in unbounded domains with similar features.

The uniqueness and stability of the nozzle problem is a question of great importance to know under what circumstances a steady transonic flow involving transonic shocks is uniquely determined by the boundary condition and the conditions at the entrance and when further conditions at the exit are appropriate (see Courant-Friedrichs [11]). In this paper, we identify that the steady transonic flow involving a transonic shock is uniquely determined by the Cauchy data at the entrance for the nonlinear wave equation and the uniform subsonic flow condition at the exit at infinity, which are natural physical conditions. Note that the pressure can not be apriori prescribed at the exit at infinity even for the existence; otherwise, it is clear from our results that there is no weak solution for this problem (also see [5]).

Some efforts have been made for solving the nonlinear equation (1.1) of mixed type. In particular, Shiffman [45], Bers [2], and Finn-Gilbarg [17] proved the existence and uniqueness of solutions for the problem of subsonic flows of (1.1) past an obstacle (also see [14]). Morawetz in [42] showed that the flows of (1.1) past the obstacle may contain transonic shocks in general. In [4, 5, 6], we developed two approaches to deal with multidimensional transonic shocks. Also see Canic-Keyfitz-Lieberman [3] for the transonic small disturbance equation and Zheng [51] for the pressure-gradient equations. For the bounded transonic nozzle problem, see [4] for the multidimensional channel case and [48] for the 2-dimensional case. Non-transonic shocks

(hyperbolic-hyperbolic shocks) were analyzed in [7, 8, 25, 32, 39, 44, 49] and the references cited therein.

In Section 2, we set up the infinite nozzle problem involving multidimensional transonic shocks and present the main theorems of this paper. In Section 3, we develop a $C^{1,\alpha}$ estimate framework up to the wedge boundary of the cylinder for the conormal problem for second-order linear elliptic equations of divergence form with C^α coefficients for subsequent development. In Section 4, we reformulate the transonic nozzle problem into a free boundary problem by introducing a multidimensional transonic shock as a free boundary which divides the upstream and downstream phases of $C^{1,\alpha}$ flow in the infinite nozzle. We introduce an iteration procedure to construct approximate solutions of the free boundary problem in Section 4 and make uniform estimates of the solutions on the bounded domains with wedge boundary in Section 6. In Section 7, we establish the existence of multidimensional transonic flows with transonic shocks via solving the corresponding free boundary problem in the infinite nozzle. In Section 8, we establish the uniqueness and stability of multidimensional transonic flows with transonic shocks in the infinite nozzle.

2. TRANSONIC SHOCKS, NOZZLE PROBLEMS, AND MAIN THEOREMS

In this section, we first set up the infinite nozzle problem involving multidimensional transonic shocks and present the main theorems of this paper.

A function $\varphi \in C^{0,1}(\Omega)$ is a weak solution of (1.1) in an unbounded domain Ω if

- (i) $|D\varphi(x)| \leq 1/\sqrt{\theta}$ a.e.
- (ii) For any $\zeta \in C_0^\infty(\Omega)$,

$$\int_{\Omega} \rho(|D\varphi|^2) D\varphi \cdot D\zeta \, dx = 0. \quad (2.1)$$

We are interested in weak solutions with shocks. Let Ω^+ and Ω^- be open subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \emptyset, \quad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega},$$

and $S = \partial\Omega^+ \cap \Omega$. Let $\varphi \in C^{0,1}(\Omega)$ be a weak solution of (1.1) and be in $C^1(\Omega^\pm \cup S)$ so that $D\varphi$ experiences a jump across S that is an $(n-1)$ -dimensional smooth surface. Then φ satisfies the following Rankine-Hugoniot conditions on S :

$$[\varphi]_S = 0, \quad \left[\rho(|D\varphi|^2) D\varphi \cdot \nu \right]_S = 0, \quad (2.2)$$

where ν is the unit normal to S from Ω^- to Ω^+ , and the bracket denotes the difference between the values of the function along S on the $\Omega^\pm$ sides. Moreover, a function $\varphi \in C^1(\Omega^\pm \cup S)$, which satisfies $|D\varphi| \leq 1/\sqrt{\theta}$, (2.2), and equation (1.1) in $\Omega^\pm$ respectively, is a weak solution of (1.1), i.e., φ satisfies (2.1) in the whole domain Ω .

Set $\varphi^\pm = \varphi|_{\Omega^\pm}$. Then we can also write (2.2) as

$$\varphi^+ = \varphi^- \quad \text{on } S \quad (2.3)$$

and

$$\rho(|D\varphi^+|^2) \varphi_\nu^+ = \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu \quad \text{on } S, \quad (2.4)$$

where $\varphi_\nu^+ = D\varphi^+ \cdot \nu$ is the normal derivative on the Ω^+ side.

Note that the function

$$\Phi(p) := (1 - \theta p^2)^{\frac{1}{2\theta}} p \quad (2.5)$$

is continuous on $[0, \sqrt{1/\theta}]$ and satisfies

$$\Phi(p) > 0 \quad \text{for } p \in (0, \sqrt{1/\theta}), \quad \Phi(0) = \Phi(\sqrt{1/\theta}) = 0, \quad (2.6)$$

$$0 < \Phi'(p) < 1 \quad \text{on } (0, c_*), \quad \Phi'(p) < 0 \quad \text{on } (c_*, \sqrt{1/\theta}), \quad (2.7)$$

$$\Phi''(p) < 0 \quad \text{on } (0, c_*], \quad (2.8)$$

where

$$c_* = \sqrt{1/(\theta + 1)} = \sqrt{2/(\gamma + 1)} \quad (2.9)$$

is the sonic speed, for which a flow is called supersonic if $|D\varphi| > c_*$ and subsonic if $|D\varphi| < c_*$.

Suppose that $\varphi \in C^{0,1}(\Omega) \cap C^1(\Omega^\pm \cup S)$ is a weak solution satisfying

$$|D\varphi| < c_* \quad \text{in } \Omega^+, \quad |D\varphi| > c_* \quad \text{in } \Omega^-, \quad D\varphi^\pm \cdot \nu|_S > 0. \quad (2.10)$$

Then φ is a *transonic shock solution* with *transonic shock* S dividing Ω into the *subsonic region* Ω^+ and the *supersonic region* Ω^- and satisfying the physical entropy condition (see Courant-Friedrichs [11]):

$$\rho(|D\varphi^-|^2) < \rho(|D\varphi^+|^2) \quad \text{along } S.$$

Note that equation (1.1) is elliptic in the subsonic region and hyperbolic in the supersonic region.

Let (x', x_n) be the coordinates in $\mathbf{R}^n$, where $x_n \in \mathbf{R}$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $q_0^- \in (c_*, 1/\sqrt{\theta})$ and $\varphi_0^-(x) := q_0^- x_n$. Then φ_0^- is a supersonic solution in Ω . According to (2.6)–(2.7), there exists a unique $q_0^+ \in (0, c_*)$ such that

$$\rho((q_0^+)^2) q_0^+ = \rho((q_0^-)^2) q_0^-. \quad (2.11)$$

Thus, the function

$$\varphi_0(x) = \begin{cases} q_0^- x_n, & x \in \Omega_0^- := \Omega \cap \{x : x_n < 0\}, \\ q_0^+ x_n, & x \in \Omega_0^+ := \Omega \cap \{x : x_n > 0\} \end{cases} \quad (2.12)$$

is a plane transonic shock solution in Ω , Ω_0^+ and Ω_0^- are its subsonic and supersonic regions respectively, and $S = \{x_n = 0\}$ is a transonic shock.

Defining $\varphi_0^+(x) := q_0^+ x_n$ in Ω , we have

$$\varphi_0(x) = \min(\varphi_0^+(x), \varphi_0^-(x)) \quad \text{for } x \in \Omega. \quad (2.13)$$

In this paper, we focus on the following infinite nozzle Ω with arbitrary smooth cross-sections:

$$\Omega = \Psi(\Lambda \times (-\infty, \infty)) \cap \{x_n \geq -1\}, \quad (2.14)$$

where $\Lambda \subset \mathbf{R}^{n-1}$ is an arbitrary open bounded, connected set with a smooth boundary, and $\Psi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a smooth map, which is close to the identity map. For simplicity, we assume that

$$\partial\Lambda \text{ is in } C^{[\frac{n}{2}]+3, \alpha} \quad (2.15)$$

and

$$\|\Psi - Id\|_{[\frac{n}{2}]+3, \alpha, \mathbf{R}^n} \leq \sigma \quad (2.16)$$

for some $\alpha \in (0, 1)$ and small $\sigma > 0$, where $[s]$ is the integer part of s , $Id : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is the identity map, and $\|u\|_{m, \alpha, \mathcal{D}}$ is the norm in the Hölder space $C^{m, \alpha}(\overline{\mathcal{D}})$ in the domain $\mathcal{D}$. Such nozzles especially include the slowly varying de Laval nozzles [11, 47]. For concreteness, we also assume that there exists $L > L_0$ (say, $L_0 > 10$ without loss of generality) such that

$$\Psi(x) = x \quad \text{for any } x = (x', x_n) \text{ with } x_n > L, \quad (2.17)$$

that is, the nozzle slowly varies locally in a bounded domain as the de Laval nozzles.

In the two-dimensional case, the domain Ω defined above has the following simple form:

$$\Omega = \{(x_1, x_2) : x_1 \geq -1, f_-(x_2) < x_1 < f_+(x_2)\},$$

where $\|f_{\pm} - d_{\pm}\|_{4,\alpha,\mathbf{R}} \leq \sigma$ and $f_{\pm} \equiv d_{\pm}$ on $[L, \infty)$ for some constants $d_{\pm}$ satisfying $d_+ > d_-$. For the multidimensional case, the geometry of the nozzles is much richer.

Note that our setup implies that

$$\partial\Omega = \overline{\partial_o\Omega} \cup \partial_l\Omega$$

with

$$\begin{aligned}\partial_l\Omega &:= \Psi[\partial\Lambda \times (-\infty, \infty)] \cap \{(x', x_n) : x_n > -1\}, \\ \partial_o\Omega &:= \Psi(\Lambda \times (-\infty, \infty)) \cap \{(x', x_n) : x_n = -1\}.\end{aligned}$$

Then our transonic nozzle problem can be formulated into the following form:

Problem (TN): Transonic Nozzle Problem. Given the supersonic upstream flow at the entrance $\partial_o\Omega$:

$$\varphi = \varphi_e^-, \quad \varphi_{x_n} = \psi_e^- \quad \text{on } \partial_o\Omega, \quad (2.18)$$

the slip boundary condition on the nozzle boundary $\partial_l\Omega$:

$$\partial_\nu\varphi = 0 \quad \text{on } \partial_l\Omega, \quad (2.19)$$

and the uniform subsonic flow condition at the exit $x_n = \infty$:

$$\|\varphi(\cdot) - \omega x_n\|_{C^1(\Omega \cap \{x_n > R\})} \rightarrow 0 \quad \text{as } R \rightarrow \infty, \text{ for some } \omega \in (0, c_*), \quad (2.20)$$

find a solution φ of problem (1.1) and (2.18)–(2.20) in Ω .

As we will see below, the supersonic upstream part φ^- of the solution can be constructed in $\Omega_1 := \Omega \cap \{x_n < 1\}$ from the data on the nozzle entrance (2.18) by using the standard results on initial-boundary value problems for hyperbolic equations. Thus, assuming that the $C^{1,\alpha}$ supersonic solution φ^- is given in Ω_1 , we can formulate the transonic nozzle problem (TN) as the following one-phase free boundary problem.

Problem (FB): Free Boundary Problem. Given a supersonic upstream flow φ^- , a weak solution of (1.1) in Ω_1 , which is a $C^{1,\alpha}$ perturbation of φ_0^- :

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_1} \leq C_0\sigma \quad (2.21)$$

with $\sigma > 0$ small, for some constant C_0 , and satisfies (2.19), find a multidimensional subsonic flow φ^+ of (1.1) satisfying (2.19)–(2.20) and identify a free boundary $x_n = f(x')$ dividing the subsonic flow φ^+ from the given supersonic flow φ^- so that

$$\varphi(x) = \begin{cases} \varphi^+(x), & x_n > f(x'), \\ \varphi^-(x), & x_n < f(x') \end{cases} \quad (2.22)$$

is a transonic shock solution.

Note that, for a solution φ of Problem (FB), the subsonic region Ω^+ and supersonic region Ω^- in (2.10) are of the form

$$\Omega^+(\varphi) = \{x_n > f(x')\}, \quad \Omega^-(\varphi) = \{x_n < f(x')\}. \quad (2.23)$$

Our main theorem for the free boundary problem, Problem (FB), is the following.

Theorem 2.1 (Existence). *There exist $\sigma_0 > 0$, C , and $\hat{C}$, depending only on n , α , γ , q_0^+ , C_0 , Λ , and L , such that, for every $\sigma \in (0, \sigma_0)$, any map Ψ satisfying (2.16)–(2.17), and any supersonic upstream flow φ^- of (1.1) satisfying (2.19) and (2.21), there exists a solution $\varphi \in C^{0,1}(\Omega) \cap C^\infty(\Omega^+)$ of Problem (FB) such that*

$$\|D\varphi - q_0^+ e_n\|_{0,0,\Omega^+} \leq \hat{C}\sigma. \quad (2.24)$$

Moreover, the solution φ satisfies the following properties:

(i) The constant ω in (2.20) must be q^+ :

$$\omega = q^+, \quad (2.25)$$

where q^+ is the unique solution in the interval $(0, c_*)$ of the equation

$$\rho((q^+)^2)q^+ = Q^+ \quad (2.26)$$

with

$$Q^+ = \frac{1}{|\Lambda|} \int_{\partial_o \Omega} \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu d\mathcal{H}^{n-1}. \quad (2.27)$$

Thus, φ satisfies

$$\|\varphi - q^+ x_n\|_{C^1(\Omega \cap \{x_n > R\})} \rightarrow 0 \quad \text{as } R \rightarrow \infty, \quad (2.28)$$

and q^+ satisfies

$$|q^+ - q_0^+| \leq C\sigma; \quad (2.29)$$

(ii) The function $f(x')$ in (2.22) satisfies

$$\|f\|_{1,\alpha,\mathbf{R}^{n-1}} \leq C\sigma, \quad (2.30)$$

and the surface $S = \{(x', f(x')) : x' \in \mathbf{R}^{n-1}\} \cap \Omega$ is orthogonal to $\partial_l \Omega$ at every intersection point;

(iii) Furthermore, $\varphi \in C^{1,\alpha}(\overline{\Omega^+})$ with

$$\|\varphi - q^+ x_n\|_{1,\alpha,\Omega_+} \leq C\sigma. \quad (2.31)$$

If the supersonic upstream flow φ^- has a higher regularity, then we have the following uniqueness theorem.

Theorem 2.2 (Uniqueness). *There exists a constant $\sigma_0 > 0$ depending only on $n, \alpha, \gamma, C_0, \Lambda, L$, and q_0^+ such that, if $\sigma < \sigma_0$ and the supersonic solution φ^- in Problem (FB) additionally satisfies*

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega_1} \leq C_0\sigma, \quad (2.32)$$

then the solution φ of Problem (FB) satisfying (2.24) is unique.

The standard local existence theory of smooth solutions for the initial-boundary value problem (2.18)–(2.19) for second-order quasilinear hyperbolic equations implies (see Appendix) that, as σ is sufficiently small in (2.16) and (2.21), there exists a supersonic solution φ^- of (1.1) in Ω_1 , which is a C^{k+1} perturbation of φ_0^- : For any $\alpha \in (0, 1]$,

$$\|\varphi^- - \varphi_0^-\|_{k,\alpha,\Omega_1} \leq C_0\sigma, \quad k = 1, 2, \quad (2.33)$$

for some constant $C_0 > 0$, and satisfies

$$\partial_\nu \varphi^- = 0 \quad \text{on } \partial_l \Omega_1, \quad (2.34)$$

provided that (φ_e^-, ψ_e^-) on $\partial_o \Omega$ satisfying

$$\|\varphi_e^- - q_0^- x_n\|_{H^{s+k}} + \|\psi_e^- - q_0^-\|_{H^{s+k-1}} \leq \sigma, \quad k = 1, 2, \quad (2.35)$$

for some integer $s > n/2 + 1$ and the compatibility conditions up to the $(s+k)^{th}$ -order, where the norm $\|\cdot\|_{H^s}$ is the Sobolev norm with $H^s = W^{s,2}$. In particular, as a direct corollary of Theorems 2.1–2.2 and Proposition A.1 in Appendix, we obtain the following existence and uniqueness result for the transonic nozzle problem, Problem (TN):

Theorem 2.3. *Let $q_0^+ \in (0, c_*)$ and $q_0^- \in (c_*, 1/\sqrt{\theta})$ satisfy (2.11), and let φ_0 be the transonic shock solution (2.12). Then there exist $\sigma_0 > 0$, C , and $\hat{C}$, depending only on $n, \alpha, \gamma, q_0^+, \Lambda$, and L such that, for every $\sigma \in (0, \sigma_0)$, any map Ψ satisfying (2.16)–(2.17), and any supersonic upstream flow (φ_e^-, ψ_e^-) on $\partial_o \Omega$ satisfying*

$$\|\varphi_e^- - q_0^- x_n\|_{H^{s+2}} + \|\psi_e^- - q_0^-\|_{H^{s+1}} \leq \sigma \quad (2.36)$$

for $s > n/2 + 1$ and the compatibility conditions up to the $(s + 2)^{th}$ -order, there exists a unique solution $\varphi \in C^{0,1}(\Omega)$ of Problem (TN) satisfying (2.23)–(2.24) and

$$\|\varphi - \varphi_0^-\|_{2,\alpha,\Omega^-} \leq C\sigma.$$

Moreover, this solution satisfies $\varphi \in C^{0,1}(\Omega) \cap C^\infty(\Omega^+)$ and properties (i)–(iii) of Theorem 2.1, where (2.27) is replaced by

$$Q^+ = \frac{1}{|\Lambda|} \int_{\partial_o\Omega} \rho(|D_{x'}\varphi_e^-|^2 + (\psi_e^-)^2) \psi_e^- d\mathcal{H}^{n-1}. \quad (2.37)$$

Remark 2.1. The smoothness conditions (2.15)–(2.16) are just for simplicity of presentation and can be relaxed: For example, in Theorems 2.1–2.2, we require only $C^{2,\alpha}$ in (2.15)–(2.16). The localization condition (2.17) is not essential to achieve Theorem 2.1; In general, it can be replaced by an appropriate decay condition.

Remark 2.2. When the initial data $(\varphi_e^-, \psi_e^-) \equiv (-\psi_e^-, \psi_e^-)$ is constant and the nozzle $\Omega \cap \{-1 \leq x_n \leq -1 + \epsilon\} = \Lambda \times [-1, -1 + \epsilon]$ for some $\epsilon > 0$ as a de Laval nozzle, then the compatibility conditions are automatically satisfied. In fact, in this case, $\varphi^-(x) = \psi_e^- x_n$ is a solution near $x_n = -1$ in the nozzle.

Remark 2.3. When $n = 2$, condition (2.36) for the supersonic upstream flow (φ_e^-, ψ_e^-) on $\partial_o\Omega$ in Theorem 2.3 can be replaced by the C^3 -condition:

$$\|\varphi_e^- - q_0^- x_n\|_{C^3} + \|\psi_e^- - q_0^-\|_{C^2} \leq \sigma. \quad (2.38)$$

This can be achieved by following the arguments in Li-Yu [33].

Remark 2.4. There exist σ_0 and C depending only on the data such that equation (2.26) with Q^+ defined by (2.27) has a unique solution $q^+ \in (0, c_*)$ satisfying (2.29).

This can be seen as follows: From (2.16), it follows that $||\partial_o\Omega| - |\Lambda|| \leq C\sigma$ and $|\nu - e_n| \leq C\sigma$ on $\partial_o\Omega$, and hence that $|\nu \cdot e_n - 1| \leq C\sigma$ on $\partial_o\Omega$. Then

$$\begin{aligned} & |Q^+ - \rho((q_0^+)^2)q_0^+| \\ & \leq \frac{1}{|\Lambda|} \int_{\partial_o\Omega} |\rho(|D\varphi^-|^2)D\varphi^- \cdot \nu - \rho((q_0^+)^2)q_0^+| dx' + \left|1 - \frac{|\partial_o\Omega|}{|\Lambda|}\right| \rho((q_0^+)^2)q_0^+ \\ & = \frac{1}{|\Lambda|} \int_{\partial_o\Omega} |\rho(|D\varphi^-|^2)D\varphi^- \cdot \nu - \rho((q_0^-)^2)q_0^-| dx' + \left|1 - \frac{|\partial_o\Omega|}{|\Lambda|}\right| \rho((q_0^+)^2)q_0^+ \\ & \leq \frac{1}{|\Lambda|} \int_{\partial_o\Omega} (|\rho(|D\varphi^-|^2)D\varphi^- \cdot \nu - \rho(|D\varphi_0^-|^2)D\varphi_0^- \cdot \nu| + \rho((q_0^-)^2)q_0^- |\nu \cdot e_n - 1|) dx' + C\sigma \\ & \leq C\sigma, \end{aligned}$$

where we used (2.11) and (2.21). Thus, by (2.6)–(2.7), we obtain that, if σ is small depending only on the data, then there exists a unique solution $q^+ \in (0, c_*)$ of equation (2.26) such that (2.29) holds.

Remark 2.5. In fact, the solution with a transonic shock of Problem (TN) in Theorem 2.3 is also stable with respect to the nozzle boundary and the smooth supersonic upstream flow at the entrance; see Theorem 8.1.

Remark 2.6. For the isothermal gas $\gamma = 1$, the same results can be obtained by following the same arguments in this paper.

3. CONORMAL PROBLEMS FOR LINEAR ELLIPTIC EQUATIONS IN A CYLINDER

In this section, we develop a $C^{1,\alpha}$ estimate framework up to the boundary of the cylinder for a conormal problem for linear elliptic equations of divergence form with C^α coefficients.

Let $\Lambda \subset \mathbf{R}^{n-1}$ be an open bounded, connected set with $C^{2,\alpha}$ boundary.

We fix the set Λ and a number $\alpha \in (0, 1)$ through this section.

For $Q > 0$, denote

$$\mathcal{C}_Q := \Lambda \times (0, Q), \quad \Sigma_Q^1 := \Lambda \times \{x_n = 0\}, \quad \Sigma_Q^2 := \partial\Lambda \times (0, Q), \quad \Sigma_Q^3 := \Lambda \times \{x_n = Q\}.$$

Consider the problem

$$\sum_{i,j=1}^n (a_{ij}u_{x_j})_{x_i} = f \quad \text{in } \mathcal{C}_Q, \quad (3.1)$$

$$\sum_{i,j=1}^n a_{ij}u_{x_j}\nu_i = g_k \quad \text{on } \Sigma_Q^k, \quad k = 1, 2, \quad (3.2)$$

$$u = h \quad \text{on } \Sigma_Q^3, \quad (3.3)$$

where ν is the inward unit normal on Σ_Q^1 and Σ_Q^2 to $\mathcal{C}_Q$.

The main issue in the argument below is to show the existence and uniqueness of solutions which are $C^{1,\alpha}$ up to the edge $\partial\Lambda \times \{0\}$ of the cylinder. Lieberman [36] studied general oblique derivative problems for linear elliptic equations in the domains with wedges in which the $C^{1,\beta}$ regularity of solutions is obtained near the wedges. However, these results require certain conditions on the relative geometry of the wedge and the vector fields in the oblique derivative conditions; Moreover, in [36], the Hölder exponent β of the solution gradients depends on the geometry of wedges and the coefficients of the equation and the right-hand sides of the boundary conditions, and $\beta < \alpha$ in general where α is the Hölder exponent of the coefficients and the right-hand sides. In this section, by restricting the class of equations and considering only conormal boundary conditions and cylindrical domains, we obtain the sharp $C^{1,\alpha}$ regularity of solutions up to $\partial\Lambda \times \{0\}$, i.e., $\beta = \alpha$. This is important for our approach developed in Sections 4–8.

Theorem 3.1. *Let $\lambda_1, \dots, \lambda_n$, and λ are the constants satisfying $0 < \lambda \leq \lambda_i \leq \lambda^{-1}$ for $i = 1, \dots, n$. Then there exist $\kappa > 0$ and $C > 0$, depending only on n, Λ, Q , and λ , such that, when $a_{ij}(x)$ satisfy*

$$\|a_{ij} - \lambda_i \delta_j^i\|_{0,\alpha,\mathcal{C}_Q} \leq \kappa \quad \text{for } i, j = 1, \dots, n, \quad (3.4)$$

and

$$f \in L^\infty(\mathcal{C}_Q), \quad h \in H^1(\mathcal{C}_Q), \quad (3.5)$$

$$g_1 \in C^{0,\alpha}(\overline{\Sigma_Q^1}), \quad g_2 \in C^{0,\alpha}(\overline{\Sigma_Q^2}), \quad (3.6)$$

there exists a unique weak solution $u \in H^1(\mathcal{C}_Q)$ of (3.1)–(3.3) in the sense that (3.3) holds as the trace and

$$\int_{\mathcal{C}_Q} \left(\sum_{i,j=1}^n a_{ij}u_{x_j}w_{x_i} + fw \right) dx + \sum_{k=1}^2 \int_{\Sigma_Q^k} g_k w d\mathcal{H}^{n-1} = 0 \quad (3.7)$$

for any function $w \in H^1(\mathcal{C}_Q)$ with $w = 0$ on $\partial\mathcal{C}_Q \cap \{x_n = Q\}$. Moreover, the solution u is $C^{1,\alpha}(\overline{\mathcal{C}_{Q/2}})$ and satisfies

$$\|u\|_{1,\alpha,\mathcal{C}_{Q/2}} \leq C(\|u\|_{L^2(\mathcal{C}_Q)} + \|f\|_{0,0,\mathcal{C}_Q} + \|g_1\|_{0,\alpha,\Sigma_Q^1} + \|g_2\|_{0,\alpha,\Sigma_Q^2}). \quad (3.8)$$

Theorem 3.1 is a direct corollary of Theorem 3.2 below, stated in more technical terms. In order to state Theorem 3.2, we introduce some notations and weighted Hölder norms.

Denote

$$\mathcal{C}'_Q = \overline{\mathcal{C}_Q} \setminus \{x_n = Q\}, \quad (\Sigma_Q^2)' = \overline{\Sigma_Q^2} \setminus \{x_n = Q\}. \quad (3.9)$$

We will use the following weighted Hölder semi-norms and norms in the cylinder $\mathcal{C}_Q$, in which the weight is the distance to the portion of the boundary $\overline{\Sigma_Q^3} = \partial\mathcal{C}_Q \setminus \mathcal{C}'_Q$. Denote $\delta_x = \text{dist}(x, \overline{\Sigma_Q^3}) = Q - x_n$ for $x = (x', x_n) \in \mathcal{C}_Q$ and $\delta_{x,y} = \min(\delta_x, \delta_y)$ for $x, y \in \mathcal{C}_Q$. For $k \in \mathbf{R}$, $\alpha \in (0, 1)$, and $m \in \mathbf{N}$ (the set of nonnegative integers), define

$$|u|_{m;0;\mathcal{C}'_Q}^{(k)} = \sum_{0 \leq j \leq m} \sum_{|\beta|=m} \sup_{x \in \mathcal{C}'_Q} \left(\delta_x^{\max(m+k,0)} |D^\beta u(x)| \right), \quad (3.10)$$

$$[[u]]_{m;\alpha;\mathcal{C}'_Q}^{(k)} = \sum_{|\beta|=m} \sup_{x,y \in \mathcal{C}'_Q, x \neq y} \left(\delta_{x,y}^{\max(m+\alpha+k,0)} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \right),$$

$$|u|_{m;\alpha;\mathcal{C}'_Q}^{(k)} = |u|_{m;0;\mathcal{C}'_Q}^{(k)} + [[u]]_{m;\alpha;\mathcal{C}'_Q}^{(k)},$$

where $D^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_n}^{\beta_n}$, $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index with $\beta_j \geq 0$, $\beta_j \in \mathbf{N}$, and $|\beta| = \beta_1 + \dots + \beta_n$. The weighted Hölder norms $|u|_{m;\alpha;(\Sigma_Q^2)'}^{(k)}$ on the boundary part Σ_Q^2 are defined similarly.

Theorem 3.2. *Let $\lambda_1, \dots, \lambda_n$, and λ be the constants satisfying $0 < \lambda \leq \lambda_i \leq \lambda^{-1}$ for $i = 1, \dots, n$. Then there exist $\kappa > 0$ and $C > 0$, depending only on n, Λ, Q , and λ , such that, when $a_{ij}(x)$ satisfy (3.4) and*

$$f \in L^2(\mathcal{C}_Q) \cap L_{loc}^\infty(\mathcal{C}'_Q), \quad h \in H^1(\mathcal{C}_Q), \quad (3.11)$$

$$g_1 \in L^2(\Sigma_Q^1) \cap C^{0,\alpha}(\overline{\Sigma_Q^1}), \quad g_2 \in L^2(\Sigma_Q^2) \cap C^{0,\alpha}((\Sigma_Q^2)') \quad (3.12)$$

with

$$|f|_{0,0;\mathcal{C}'_Q}^{(2+n/2)} + |g_2|_{0,\alpha;(\Sigma_Q^2)'}^{(1+n/2)} < \infty,$$

there exists a unique weak solution $u \in H^1(\mathcal{C}_Q)$ of (3.1)–(3.3) in the sense that (3.3) holds as the trace and (3.7) holds for any $w \in H^1(\mathcal{C}_Q)$ with $w = 0$ on $\partial\mathcal{C}_Q \cap \{x_n = Q\}$. Moreover, the solution u is $C^{1,\alpha}(\mathcal{C}')$ and satisfies

$$|u|_{1,\alpha;\mathcal{C}'_Q}^{(n/2)} \leq C(\|u\|_{L^2(\mathcal{C}_Q)} + |f|_{0,0;\mathcal{C}'_Q}^{(2+n/2)} + \|g_1\|_{0,\alpha;\Sigma_Q^1} + |g_2|_{0,\alpha;(\Sigma_Q^2)'}^{(1+n/2)}). \quad (3.13)$$

Now we prove Theorem 3.2. We first consider an auxiliary problem:

$$\sum_{i=1}^n \lambda_i u_{x_i x_i} = f + \sum_{i=1}^n \psi_{x_i}^i \quad \text{in } \mathcal{C}_Q, \quad (3.14)$$

$$\sum_{i=1}^n \lambda_i u_{x_i} \nu_i \equiv \lambda_n u_{x_n} = g_1 \quad \text{on } \Sigma_Q^1, \quad (3.15)$$

$$\sum_{i=1}^n \lambda_i u_{x_i} \nu_i \equiv \sum_{i=1}^{n-1} \lambda_i u_{x_i} \nu_i = g_2 + \sum_{i=1}^{n-1} \psi^i \nu_i \quad \text{on } \Sigma_Q^2, \quad (3.16)$$

$$u = h \quad \text{on } \Sigma_Q^3, \quad (3.17)$$

where we used that $\nu = e_n$ on Σ_Q^1 in (3.15) and $\nu \cdot e_n = 0$ on Σ_Q^2 in (3.16). Note that smooth solutions of (3.14)–(3.17) satisfy

$$\begin{aligned} & \int_{\mathcal{C}_Q} \left(\sum_{i=1}^n \lambda_i u_{x_i} w_{x_i} - \sum_{i=1}^n \psi^i w_{x_i} + f w \right) dx \\ & + \int_{\Sigma_Q^1} (g_1 - \psi^n) w dx' - \int_{\Sigma_Q^2} g_2 w d\mathcal{H}^{n-1} = 0 \end{aligned} \quad (3.18)$$

for any function $w \in H^1(\mathcal{C}_Q)$ with $w = 0$ on $\mathcal{C}_Q \cap \{x_n = Q\}$.

Lemma 3.1. *Let $\lambda_1, \dots, \lambda_n$, and λ be the constants satisfying $0 < \lambda \leq \lambda_i \leq \lambda^{-1}$ for $i = 1, \dots, n$. Let*

$$\begin{aligned} f &\in L^2(\mathcal{C}_Q) \cap L_{loc}^\infty(\mathcal{C}'_Q), \quad \psi = (\psi^1, \dots, \psi^n) \in H^1(\mathcal{C}_Q) \cap C^\alpha(\mathcal{C}'_Q), \quad h \in H^1(\mathcal{C}_Q), \\ g_1 &\in L^2(\Sigma_Q^1) \cap C^{0,\alpha}(\overline{\Sigma_Q^1}), \quad g_2 \in C(\mathcal{C}'_Q) \cap L^2(\Sigma_Q^2) \cap C^{0,\alpha}(\mathcal{C}'_Q) \end{aligned} \quad (3.19)$$

with

$$|f|_{0,0,\mathcal{C}'_Q}^{(2+n/2)} + |\psi|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} + |g_2|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} < \infty. \quad (3.20)$$

Then there exists a unique weak solution $u \in H^1(\mathcal{C}_Q)$ of (3.14)–(3.17) in the sense that (3.17) holds as the trace and (3.18) holds for any $w \in H^1(\mathcal{C}_Q)$ with $w = 0$ on $\partial\mathcal{C}_Q \cap \{x_n = Q\}$. The solution u is $C^{1,\alpha}(\mathcal{C}'_Q)$ and satisfies

$$\begin{aligned} &\|u\|_{H^1(\mathcal{C}_Q)} + |u|_{1,\alpha,\mathcal{C}'_Q}^{(n/2)} \\ &\leq C(\|f\|_{L^2(\mathcal{C}_Q)} + |f|_{0,0,\mathcal{C}'_Q}^{(2+n/2)} + \|\psi\|_{L^2(\mathcal{C}_Q)} + |\psi|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} + \|g_1\|_{0,\alpha,\Sigma_Q^1} \\ &\quad + \|g_2\|_{L^2(\Sigma_Q^2)} + |g_2|_{0,\alpha,(\Sigma_Q^2)'}^{(1+n/2)} + \|h\|_{H^1(\mathcal{C}_Q)}) \end{aligned} \quad (3.21)$$

and

$$|u|_{1,\alpha,\mathcal{C}'_Q}^{(n/2)} \leq C(\|u\|_{L^2(\mathcal{C}_Q)} + |f|_{0,0,\mathcal{C}'_Q}^{(2+n/2)} + |\psi|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} + \|g_1\|_{0,\alpha,\Sigma_Q^1} + |g_2|_{0,\alpha,(\Sigma_Q^2)'}^{(1+n/2)}), \quad (3.22)$$

where C depends only on n, Λ, Q , and λ .

Proof. We prove this lemma in six steps. The constant C in the argument below is a universal constant depending only on n, Λ, Q , and λ , and may be different at each occurrence.

Step 1. First we show the uniqueness of weak solutions. Suppose that u_1 and u_2 are two weak solutions of (3.14)–(3.17). Then $u_1 - u_2 = 0$ on Σ_Q^3 . Thus, writing the weak forms (3.18) for u_1 and u_2 respectively and subtracting the expressions with $w = u_1 - u_2$ yield

$$0 = \int_{\mathcal{C}_Q} \sum_{i,j=1}^n \lambda_i |\partial_{x_i}(u_1 - u_2)|^2 dx.$$

Since $u_1 - u_2 = 0$ on Σ_Q^3 and $\lambda_i > 0$, we conclude $u_1 = u_2$ a.e. in $\mathcal{C}_Q$.

Step 2. We now focus on the existence of solutions and estimate (3.21). First we show that the problem can be reduced to the case where the right-hand side of the boundary condition (3.15) vanishes. We can extend g_1 from $\Sigma_Q^1 = \Lambda \times \{0\}$ to the whole hyperplane $\{(x', 0) : x' \in \mathbf{R}^{n-1}\}$ such that the extension $g_1^* \in C_0^\alpha(\mathbf{R}^{n-1})$ satisfies $\|g_1^*\|_{0,\alpha,\mathbf{R}^{n-1}} \leq C\|g_1\|_{0,\alpha,\Lambda}$. Now, as in [20, page 124-125], we choose a nonnegative $\eta \in C_0^2(\mathbf{R}^{n-1})$ with $\int_{\mathbf{R}^{n-1}} \eta(y') dy' = 1$ and define

$$G(x', x_n) = \lambda_n^{-1} x_n \int_{\mathbf{R}^{n-1}} g_1^*(x' - x_n y') \eta(y') dy' \quad \text{for } x_n > 0.$$

Then $G \in C^{1,\alpha}(\overline{\mathbf{R}_+^n})$ with

$$\|G\|_{1,\alpha,\mathcal{C}_Q} \leq C\|g_1^*\|_{0,\alpha,\mathbf{R}^{n-1}} \leq C\|g_1\|_{0,\alpha,\Lambda} \quad (3.23)$$

and

$$\lambda_n G_{x_n} = g_1^* = g_1 \quad \text{on } \Lambda \times \{x_n = 0\}.$$

Thus, u is a weak solution of (3.14)–(3.17) if and only if $v = u - G$ is a weak solution of the same problem with modified right-hand sides: f, ψ, g_1 , and g_2 are replaced by $\tilde{f} = f, \tilde{\psi}^i = \psi^i - \lambda_i G_{x_i}$, $\tilde{g}_1 = 0$, and $\tilde{g}_2 = g_2$. Using (3.23), it is easy to check that (3.21)–(3.22) for v in terms of $\tilde{f}, \tilde{\psi}, \tilde{g}_1$, and $\tilde{g}_2$ implies (3.21)–(3.22) for u in terms of f, ψ, g_1 , and g_2 .

Thus, from now on, we assume

$$g_1 = 0 \quad \text{on } \Sigma_Q^1. \quad (3.24)$$

Step 3. We also note that we can assume without loss of generality that

$$\psi^n = 0 \quad \text{on } \Sigma_Q^1. \quad (3.25)$$

Indeed, if u is a weak solution of (3.14)–(3.17), then u is also a weak solution of (3.14)–(3.17) with ψ^n replaced by $\hat{\psi}^n$: $\hat{\psi}^n(x', x_n) = \psi^n(x', x_n) - \psi^n(x', 0)$ for $(x', x_n) \in \mathcal{C}_Q$. Since

$$\|\hat{\psi}^n\|_{L^2(\mathcal{C}_Q)} + |\hat{\psi}|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} \leq C(\|\psi^n\|_{L^2(\mathcal{C}_Q)} + |\psi|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)}),$$

it follows that (3.21)–(3.22) in terms of the right-hand sides with ψ^n replaced by $\hat{\psi}^n$ implies (3.21)–(3.22) in terms of the original functions.

Clearly, $\hat{\psi}^n$ satisfies (3.25). Thus we can assume (3.25) from now on.

Step 4. With (3.24)–(3.25), we extend problem (3.14)–(3.17) to the cylinder $\mathcal{C}_{(-Q,Q)} = \Lambda \times (-Q, Q)$. Set $\Sigma_{(-Q,Q)}^2 := \partial\Lambda \times (-Q, Q)$.

Note that, with (3.24)–(3.25), u is a weak solution of (3.14)–(3.17) if (3.17) holds as the trace and, for any $w \in H^1(\mathcal{C}_Q)$ with $w = 0$ on $\partial\mathcal{C}_Q \cap \{x_n = Q\}$,

$$\int_{\mathcal{C}_Q} \left(\sum_{i=1}^n \lambda_i u_{x_i} w_{x_i} - \sum_{i=1}^n \psi^i w_{x_i} + fw \right) dx - \int_{\Sigma_Q^2} g_2 w d\mathcal{H}^{n-1} = 0. \quad (3.26)$$

We use the even reflection to extend $u, f, h, g_2, \psi^1, \dots, \psi^{n-1}$ to $\mathcal{C}_{(-Q,Q)}$ and the odd reflection to extend ψ^n to $\mathcal{C}_{(-Q,Q)}$. That is, for $x' \in \Lambda$ and $x_n \in (0, Q)$, we use the same notations for the original and extended functions to define

$$\begin{aligned} \varphi(x', -x_n) &= \varphi(x', x_n) \quad \text{for } \varphi \in \{u, f, h, g_2, \psi^1, \dots, \psi^{n-1}\}, \\ \psi^n(x', -x_n) &= -\psi^n(x', x_n). \end{aligned} \quad (3.27)$$

Obviously, for $u \in H^1(\mathcal{C}_Q)$ and f, ψ, h , and g_2 as in Lemma 3.1 with (3.25) for ψ^n , the extended functions satisfy

$$u \in H^1(\mathcal{C}_{(-Q,Q)}), \quad f \in L^2(\mathcal{C}_{(-Q,Q)}) \cap L_{loc}^\infty(\mathcal{C}'_{(-Q,Q)}), \quad \psi \in H^1(\mathcal{C}_{(-Q,Q)}) \cap C^\alpha(\mathcal{C}'_{(-Q,Q)}),$$

$$h \in H^1(\mathcal{C}_{(-Q,Q)}), \quad g_2 \in C^\alpha(\Sigma_{(-Q,Q)}^2) \cap L^2(\Sigma_{(-Q,Q)}^2),$$

where $\mathcal{C}'_{(-Q,Q)} = \mathcal{C}_{(-Q,Q)} \cup \Sigma_{(-Q,Q)}^2$. Moreover, defining the weighted Hölder semi-norms and norms $|\cdot|_{m;\alpha;\mathcal{C}'_{(-Q,Q)}}^{(k)}$ and $|\cdot|_{m;\alpha;\Sigma_{(-Q,Q)}^2}^{(k)}$ by (3.10) with the supremums taken over the domains $\mathcal{C}'_{(-Q,Q)}$ and $\Sigma_{(-Q,Q)}^2$ respectively and

$$\delta_x = \text{dist}(x, \partial\mathcal{C}_{(-Q,Q)} \setminus \Sigma_{(-Q,Q)}^2) = \min(Q - x_n, Q + x_n) \quad \text{for } x = (x', x_n) \in \mathcal{C}_{(-Q,Q)},$$

we obtain

$$\begin{aligned} |f|_{0,0,\mathcal{C}'_{(-Q,Q)}}^{(2+n/2)} &\leq C|f|_{0,0,\mathcal{C}'_Q}^{(2+n/2)}, \quad |\psi|_{0,\alpha,\mathcal{C}'_{(-Q,Q)}}^{(1+n/2)} \leq C|\psi|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)}, \\ |g_2|_{0,\alpha,\Sigma_{(-Q,Q)}^2}^{(1+n/2)} &\leq C|g_2|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)}, \end{aligned} \quad (3.28)$$

where the norms on the right-hand sides are finite by (3.20).

Then it follows from (3.26)–(3.27) that the extended functions satisfy

$$\int_{\mathcal{C}_{(-Q,Q)}} \left(\sum_{i=1}^n \lambda_i u_{x_i} w_{x_i} - \sum_{i=1}^n \psi^i w_{x_i} + fw \right) dx - \int_{\Sigma_{(-Q,Q)}^2} g_2 w d\mathcal{H}^{n-1} = 0 \quad (3.29)$$

for any $w \in H^1(\mathcal{C}_{(-Q,Q)})$ satisfying $w = 0$ on $\{x_n = \pm Q\}$. Indeed, (3.29) holds separately when the integration is over subdomains $\{x_n > 0\}$ and $\{x_n < 0\}$ (the last follows by the change of

variable $x_n \rightarrow -x_n$). Thus, the extended function u is a weak solution of

$$\sum_{i=1}^n \lambda_i u_{x_i x_i} = f + \sum_{i=1}^n \psi_{x_i}^i \quad \text{in } \mathcal{C}_{(-Q,Q)}, \quad (3.30)$$

$$\sum_{i=1}^n \lambda_i u_{x_i} \nu_i \equiv \sum_{i=1}^{n-1} \lambda_i u_{x_i} \nu_i = g_2 + \sum_{i=1}^{n-1} \psi^i \nu_i \quad \text{on } \Sigma_{(-Q,Q)}^2, \quad (3.31)$$

$$u = h \quad \text{on } (\Lambda \times \{-Q\}) \cup (\Lambda \times \{Q\}). \quad (3.32)$$

Conversely, if u is a weak solution of (3.30)–(3.32) (i.e. (3.29) holds for w as above), then u satisfies $u(x', -x_n) = u(x', x_n)$ for $x' \in \Lambda$ and $x_n \in (0, Q)$. To see this, we first note that a weak solution of (3.29) is unique: The proof similarly follows the proof of uniqueness for (3.1)–(3.3) by using the fact that, if u_1 and u_2 are two weak solutions of (3.30)–(3.32), then we can use $w = u_1 - u_2$ in (3.29). Now, the properties in (3.27) for f, h, ψ , and g_2 imply that, if $u(x', x_n)$ is a solution of (3.29), then $u(x', -x_n)$ is also a solution. Thus, by uniqueness, $u(x', -x_n) = u(x', x_n)$. Now, let $w \in H^1(\mathcal{C}_Q)$ with $w = 0$ on $\{x_n = Q\}$. Extend w by the even reflection to $\mathcal{C}_{(-Q,Q)}$. We can use this extended w in (3.29). Then, using the properties in (3.27) for u, f, h, ψ , and g_2 , we see that, for all the integrals in (3.29), the integrals over subdomains $\{x_n > 0\}$ and $\{x_n < 0\}$ are equal. Thus, (3.29) implies (3.26). Then, in order to solve (3.14)–(3.17), it suffices to solve (3.30)–(3.32) for the extended right-hand sides.

Step 5. We now derive some estimates for the weak solution $u \in H^1(\mathcal{C}_{(-Q,Q)})$ of (3.30)–(3.32). We first derive the energy estimate. We can use $w = u - h$ in (3.29) and then use the ellipticity and the Hölder inequality to obtain

$$\|u\|_{H^1(\mathcal{C}_{(-Q,Q)})} \leq C(\|(f, \psi)\|_{L^2(\mathcal{C}_{(-Q,Q)})} + \|h\|_{H^1(\mathcal{C}_{(-Q,Q)})} + \|g_2\|_{L^2(\Sigma_{(-Q,Q)}^2)}). \quad (3.33)$$

Note that, by the standard local regularity results for the conormal problem for elliptic equations in domains with $C^{1,\alpha}$ boundary (e.g., [35]), we have $u \in C^{1,\alpha}(\mathcal{C}'_{(-Q,Q)})$.

Now we derive the estimates for the Hölder norms of u . We first recall the standard estimates for the conormal problem for elliptic equations in a unit ball $B_1 = B_1(0)$ and a half-ball $B_1^+ = B_1(0) \cap \{x_n > 0\}$. If $u \in H^1(B_1)$ is a weak solution of the equation

$$\sum_{i,j=1}^n (a_{ij} u_{x_j})_{x_i} = f + \sum_{i=1}^n \psi_{x_i}^i \quad \text{in } B_1 \quad (3.34)$$

with $a_{ij} \in C^\alpha(B_1)$ satisfying the ellipticity condition: there exists some $\lambda > 0$ such that, for any $x \in B_1$,

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2 \quad \text{for any } \xi \in \mathbf{R}^n, \quad (3.35)$$

and, if $f \in L^\infty(B_1)$ and $\psi \in C^\alpha(\overline{B_1})$, then $u \in C^{1,\alpha}(\overline{B_{1/2}})$ with

$$\|u\|_{1,\alpha,B_{1/2}} \leq C(\|u\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|\psi\|_{0,\alpha,B_1}), \quad (3.36)$$

where C depends only on n, λ , and $\|a_{ij}\|_{0,\alpha,B_1}$. This follows from [20, Theorem 8.32], combined with [20, Theorem 8.17]. Similar estimates can be obtained for the conormal problem, in which $u \in H^1(B_1^+)$ is a weak solution of (3.34) in B_1^+ with the conormal condition

$$\sum_{j=1}^n a_{nj} u_{x_j} = g + \psi^n \quad \text{on } \partial' B_1^+ := \partial B_1^+ \cap \{x_n = 0\}, \quad (3.37)$$

where $g \in C^\alpha(\overline{B^+})$. Then $u \in C^{1,\alpha}(\overline{B_{1/2}^+})$ with

$$\|u\|_{1,\alpha,B_{1/2}^+} \leq C(\|u\|_{L^2(B_1^+)} + \|f\|_{L^\infty(B_1^+)} + \|\psi\|_{0,\alpha,B_1^+} + \|g\|_{0,\alpha,\partial' B_1^+}), \quad (3.38)$$

where C depends only on n, λ , and $\|a_{ij}\|_{0,\alpha,B_1^+}$. This can be inferred by simplifying the estimates of [35] to the case of linear elliptic equations.

Let $\rho \in (0, 1/2)$. By scaling, we obtain the corresponding local estimates for weak solutions of (3.34) in B_ρ and for weak solutions of the conormal problem (3.34) and (3.37) in B_ρ^+ . We write only the estimate for the conormal problem in B_ρ^+ , since the interior estimate in B_ρ differs only in that it does not have the terms with the norms of g on the right-hand side. Thus, if u is a weak solution of (3.34) in $B_{\rho/2}^+$ and satisfies (3.37) in $B_\rho^+ \cap \{x_n = 0\}$, then $u \in C^{1,\alpha}(\overline{B_{\rho/2}^+})$ with

$$\begin{aligned} & \|u\|_{0,0,B_{\rho/2}^+} + \rho \|Du\|_{0,0,B_{\rho/2}^+} + \rho^{1+\alpha} [Du]_{0,\alpha,B_{\rho/2}^+} \\ & \leq C(\rho^{-n/2} \|u\|_{L^2(B_\rho^+)} + \rho^2 \|f\|_{L^\infty(B_\rho^+)} + \rho \|\psi\|_{0,0,B_\rho^+} \\ & \quad + \rho^{1+\alpha} [\psi]_{0,\alpha,B_\rho^+} + \rho \|g\|_{0,0,\partial' B_\rho^+} + \rho^{1+\alpha} [g]_{0,\alpha,\partial' B_\rho^+}). \end{aligned} \quad (3.39)$$

This estimate is obtained by introducing the function $v(x) = u(\rho x)$ in B_1^+ , writing the equation and the conormal condition satisfied by u in terms of v , using estimate (3.38) for v with corresponding right-hand sides of the equation and conormal condition satisfied by v , and then rewriting that estimate in terms of u .

Now we get similar estimates for weak solutions of (3.30)–(3.32). Using that $\partial\Lambda$ is $C^{1,\alpha}$ and compact, there exists $r > 0$ and $M > 0$ such that, for any $z \in \Sigma_{(-Q,Q)}^2$ and any positive $\rho < \min(r, \delta_z)$, there is a $C^{1,\alpha}$ diffeomorphism F that flattens $\Sigma_{(-Q,Q)}^2$ in $B_\rho(z) \cap \mathcal{C}_{(-Q,Q)}$ so that $B_{\rho/M}^+ \subset F(B_\rho(z) \cap \mathcal{C}_{(-Q,Q)}) \subset B_{M\rho}^+$ and $\|(F, F^{-1})\|_{C^{1,\alpha}}$ depends only on Λ . The transformed function $v(y) := u(F^{-1}(y))$ satisfies a conormal problem of the form (3.34) and (3.37) in $B_{\rho/M}^+$, where the equation is elliptic (with ellipticity constant $\lambda/2$ and the C^α norms of the coefficients a_{ij} depending only on the $C^{1,\alpha}$ norms of $\partial\Lambda$) and the appropriate norms of right-hand sides of the original and transformed problems are estimated by one in terms of another with a constant depending only on the $C^{1,\alpha}$ norms of $\partial\Lambda$. This can be seen by choosing the function w supported in $B_\rho(z)$ in (3.29) and changing the variables $x \rightarrow y = F(x)$ in (3.29). Now we obtain estimate (3.39) for v , which implies the following estimate for u :

$$\begin{aligned} & \|u\|_{0,0,\Omega_{\rho/2}(z)} + \rho \|Du\|_{0,0,\Omega_{\rho/2}(z)} + \rho^{1+\alpha} [Du]_{0,\alpha,\Omega_{\rho/2}(z)} \\ & \leq C(\rho^{-n/2} \|u\|_{L^2(\Omega_\rho(z))} + \rho^2 \|f\|_{L^\infty(\Omega_\rho(z))} + \rho \|\psi\|_{0,0,\Omega_\rho(z)} + \rho^{1+\alpha} [\psi]_{0,\alpha,\Omega_\rho(z)} \\ & \quad + \rho \|g_2\|_{0,0,\partial'\Omega_\rho(z)} + \rho^{1+\alpha} [g_2]_{0,\alpha,\partial'\Omega_\rho(z)}) \end{aligned} \quad (3.40)$$

for any $z \in \Sigma_{(-Q,Q)}^2$, $0 < \rho < \min(r, \delta_z)$, with C depending only on n, Λ, L , and λ , where $\Omega_\rho(z) := B_\rho(z) \cap \mathcal{C}_{(-Q,Q)}$ and $\partial'\Omega_\rho(z) := B_\rho(z) \cap \Sigma_{(-Q,Q)}^2$. We also have the corresponding interior estimates: For any $z \in \mathcal{C}_{(-Q,Q)}$ and $0 < \rho < \text{dist}(z, \partial\mathcal{C}_{(-Q,Q)})$, we obtain (3.40) for $\Omega_\rho(z) = B_\rho(z)$ without the terms involving the norms of g_2 on the right-hand side. Multiplying the interior and boundary estimates (3.40) by $\rho^{n/2}$ and using a standard argument (e.g., [20, Theorem 4.8]) yields

$$|u|_{1,\alpha,\mathcal{C}'_{(-Q,Q)}}^{(n/2)} \leq C(\|u\|_{L^2(\mathcal{C}_{(-Q,Q)})} + |f|_{0,0,\mathcal{C}'_{(-Q,Q)}}^{(2+n/2)} + |\psi|_{0,\alpha,\mathcal{C}'_{(-Q,Q)}}^{(1+n/2)} + |g_2|_{0,\alpha,\Sigma_{(-Q,Q)}^2}^{(1+n/2)}). \quad (3.41)$$

Obviously, estimates (3.33) and (3.41) imply (3.21). Also, (3.41) implies (3.22).

Step 6. It remains to prove the existence of a weak solution $u \in H^1(\mathcal{C}_{(-Q,Q)})$ of (3.30)–(3.32). We first assume that $f, \psi \in C^\infty(\overline{\mathcal{C}_{(-Q,Q)}})$ and $g_2 \in C^\infty(\overline{\Sigma_{(-Q,Q)}^2})$. Then, since $\Sigma_{(-Q,Q)}^2$ is a $C^{2,\alpha}$ surface and $g_2 + \sum_{i=1}^{n-1} \psi^i \nu_i \in C^{1,\alpha}(\overline{\Sigma_{(-Q,Q)}^2})$, we can find $G \in C^{2,\alpha}(\mathcal{C}'_{(-Q,Q)})$ such that

$$\sum_{i=1}^{n-1} \lambda_i G_{x_i} \nu_i = g_2 + \sum_{i=1}^{n-1} \psi^i \nu_i \quad \text{on } \Sigma_{(-Q,Q)}^2. \quad (3.42)$$

To construct such a function G , we first extend the right-hand side of (3.42) to $\Sigma_{(-2Q, 2Q)}^2 = \partial\Lambda \times (-2Q, 2Q)$ so that $\tilde{g} \in C^{1,\alpha}(\overline{\Sigma_{(-2Q, 2Q)}^2})$ with $\tilde{g} = g_2 + \sum_{i=1}^{n-1} \psi^i \nu_i$ on $\Sigma_{(-Q, Q)}^2$. Then, for every $x_* \in \partial\Lambda \times (-3Q/2, 3Q/2)$, we can locally flatten $\Sigma_{(-2Q, 2Q)}^2$ by a $C^{2,\alpha}$ diffeomorphism $\Psi : \mathbf{R}^n \rightarrow \mathbf{R}^n$, i.e., for some $r > 0$, $\Psi(B_r(x_*) \cap (\Lambda \times \mathbf{R})) = \Psi(B_r(x_*)) \cap \{x_n > 0\}$ with $\Psi(B_r(x_*) \cap (\partial\Lambda \times \mathbf{R})) = \Psi(B_r(x_*)) \cap \{x_n = 0\}$. Then condition (3.42) is transformed into

$$\sum_{i=1}^n c_i(x') \hat{G}_{x_i} = \hat{g} \quad \text{on } \{x_n = 0\}, \quad (3.43)$$

where $c_i \in C^{1,\alpha}(\mathbf{R}^{n-1})$ with $c_n \geq \min(\lambda_1, \dots, \lambda_n)/2 > 0$ if r is chosen sufficiently small, and $\hat{g} \in C^{1,\alpha}(\mathbf{R}^{n-1})$ with compact support is obtained by transforming $\tilde{g}$ by Ψ and extending to $\mathbf{R}^{n-1}$. Now, by the argument similar to [20, page 124-125], we choose a nonnegative $\eta \in C_0^2(\mathbf{R}^{n-1})$ with $\int_{\mathbf{R}^{n-1}} \eta(y') dy' = 1$, define the function

$$\hat{G}(x', x_n) = x_n \int_{\mathbf{R}^{n-1}} \frac{\hat{g}(x' - x_n y')}{c_n(x' - x_n y')} \eta(y') dy',$$

and show that $\hat{G}$ is $C^{2,\alpha}(\overline{\mathbf{R}_+^n})$ and satisfies (3.43). Transforming $\hat{G}$ back by Ψ^{-1} , we obtain $G \equiv G_{x_*} \in C^{2,\alpha}((\overline{\Lambda \times \mathbf{R}}) \cap B_r(x_*))$ satisfying (3.42) and $G \equiv 0$ on $\Sigma_{(-2Q, 2Q)}^2 \cap B_r(x_*)$. Gluing these local functions by using a partition of unity, and taking into account that the local functions G vanish on $\Sigma_{(-2Q, 2Q)}^2$, we obtain a $C^{2,\alpha}$ function G in a neighborhood of the boundary part $\Sigma_{(-Q, Q)}^2$ in $\mathcal{C}$, which satisfies (3.42) on $\Sigma_{(-Q, Q)}^2$. Finally, we extend G to the whole cylinder $\mathcal{C}_{(-Q, Q)}$ with $G \in C^{2,\alpha}(\overline{\mathcal{C}_{(-Q, Q)}})$.

Now, denoting $\tilde{f} = f + \sum_{i=1}^n (\psi_{x_i}^i - \lambda_i G_{x_i x_i})$, we have $\tilde{f} \in C(\overline{\mathcal{C}_{(-Q, Q)}})$. Thus we can solve the variational problem of minimizing

$$I[v] = \int_{\mathcal{C}_{(-Q, Q)}} \left(\frac{1}{2} \sum_{i=1}^n \lambda_i v_{x_i}^2 + \tilde{f} v \right) dx$$

on the set $\{v \in H^1(\mathcal{C}_{(-Q, Q)}) : v = h - G \text{ on } \Lambda \times \{x_n = \pm Q\}\}$. If v is a minimizer, then $u = v + G$ is a weak solution of (3.30)–(3.32).

For general $f, \psi \in L^2(\mathcal{C}_{(-Q, Q)})$ and $g_2 \in L^2(\Sigma_{(-Q, Q)}^2)$, we approximate by $f_l, \psi_l \in C^\infty(\overline{\mathcal{C}_{(-Q, Q)}})$ and $g_2^l \in C^\infty(\overline{\Sigma_{(-Q, Q)}^2})$ such that $(f_l, \psi_l) \rightarrow (f, \psi)$ in $L^2(\mathcal{C}_{(-Q, Q)})$ and $g_2^l \rightarrow g_2$ in $L^2(\Sigma_{(-Q, Q)}^2)$. Then, for each l , we can find a weak solution $u_l \in H^1(\mathcal{C}_{(-Q, Q)})$ of (3.30)–(3.32) with functions f_l, ψ_l , and g_2^l on the right-hand side. Since $u_k - u_l$ satisfies (3.30)–(3.32) with $f_k - f_l, \psi_k - \psi_l, g_2^k - g_2^l$, and $h = 0$ on the right-hand side, we apply (3.33) to see that $\{u_l\}$ is a Cauchy sequence in $H^1(\mathcal{C}_{(-Q, Q)})$ and its limit u is a solution of (3.30)–(3.32) with the original right-hand sides. Now Lemma 3.1 is proved. $\square$

Proof of Theorem 3.2. If κ is a small constant depending only on λ , then (3.4) implies the ellipticity (3.35) with $\frac{3}{4}\lambda$ instead of λ . We choose such κ below.

First we show the uniqueness of weak solutions. Suppose that u_1 and u_2 are two weak solutions of (3.1)–(3.3). Then $u_1 - u_2 = 0$ on Σ_Q^3 . Writing the weak form (3.7) for u_1 and u_2 respectively, subtracting the expressions with $w = u_1 - u_2$, and then using the ellipticity yields

$$0 = \int_{\mathcal{C}_Q} \sum_{i,j=1}^n a_{ij} \partial_{x_i} (u_1 - u_2) \partial_{x_j} (u_1 - u_2) dx \geq \frac{3}{4} \lambda \int_{\mathcal{C}_Q} |D(u_1 - u_2)|^2 dx.$$

Since $u_1 - u_2 = 0$ on Σ_Q^3 , we conclude $u_1 = u_2$ a.e. in $\mathcal{C}_Q$.

We now apply the Banach contraction fixed point theorem to prove the existence and regularity of the weak solution $u \in H^1(\mathcal{C}_Q)$ of (3.1)–(3.3) for sufficiently small $\kappa > 0$ in the space:

$$\mathcal{K} = \{v \in H^1(\mathcal{C}_Q) \cap C^{1,\alpha}(\mathcal{C}'_Q) : \|v\|_{\mathcal{K}} := \|v\|_{H^1(\mathcal{C}_Q)} + |v|_{1,\alpha,\mathcal{C}'_Q}^{(n/2)} < \infty\}.$$

We now define a mapping $J : \mathcal{K} \rightarrow \mathcal{K}$. For $v \in \mathcal{K}$, consider problem (3.14)–(3.17) with functions $\tilde{f}$, $\tilde{h}$, $\tilde{g}_1$, $\tilde{g}_2$, and $\tilde{\psi}$ on the right-hand sides, where

$$\tilde{f} = f, \quad \tilde{h} = h, \quad \tilde{g}_2 = g_2, \quad (3.44)$$

$$\tilde{g}_1(x) = g_1(x) + \sum_{i,j=1}^n (\lambda_i \delta_i^j - a_{ij}(x)) v_{x_j}(x) \nu_i(x) \quad \text{for } x \in \Sigma_Q^1 = \Lambda \times \{0\}, \quad (3.45)$$

$$\tilde{\psi}^i(x) = \sum_{j=1}^n (\lambda_i \delta_i^j - a_{ij}(x)) v_{x_j}(x) \quad \text{for } x \in \mathcal{C}_Q, \quad i = 1, \dots, n, \quad (3.46)$$

with f, g_1, g_2 , and h from (3.1)–(3.3). Since $\nu = e_n$ on Σ_Q^1 , we have

$$\tilde{g}_1 = g_1 + \sum_{j=1}^n (\lambda_n \delta_n^j - a_{nj}) v_{x_j} \equiv g_1 + \tilde{\psi}^n \quad \text{on } x \in \Sigma_Q^1. \quad (3.47)$$

By (3.4) and $v \in \mathcal{K}$, the functions $\tilde{g}_1$ and $\tilde{\psi}$ satisfy the conditions of Lemma 3.1. Thus, by Lemma 3.1, there exists a unique weak solution u of (3.14)–(3.17) with the right-hand sides described above, which satisfies (3.21) and thus implies $u \in \mathcal{K}$. We define the mapping $J : \mathcal{K} \rightarrow \mathcal{K}$ by setting $Jv = u$.

Now we show that J is a contraction mapping in the norm $\|\cdot\|_{\mathcal{K}}$ if $\kappa > 0$ is small. Let $v^1, v^2 \in \mathcal{K}$ and $u^k = Jv^k$ for $k = 1, 2$. Then $u^1 - u^2$ is a weak solution of (3.14)–(3.17) with the following functions $\hat{f}$, $\hat{h}$, $\hat{g}_1$, $\hat{g}_2$, and $\hat{\psi}$ on the right-hand sides:

$$\begin{aligned} \hat{f} &= 0, \quad \hat{g}_2 = 0, \quad \hat{h} = 0, \\ \hat{\psi}^i(x) &= \sum_{j=1}^n (\lambda_i \delta_i^j - a_{ij}(x)) (v_{x_j}^1(x) - v_{x_j}^2(x)) \quad \text{for } x \in \mathcal{C}_Q, \quad i = 1, \dots, n, \\ \hat{g}_1(x) &= \sum_{j=1}^n (\lambda_n \delta_n^j - a_{nj}(x)) (v_{x_j}^1(x) - v_{x_j}^2(x)) \quad \text{for } x \in \Sigma_Q^1. \end{aligned}$$

By (3.4), we have

$$\begin{aligned} \|\hat{\psi}\|_{L^2(\mathcal{C}_Q)} &\leq C\kappa \|Dv^1 - Dv^2\|_{L^2(\mathcal{C}_Q)} \leq C\kappa \|v^1 - v^2\|_{\mathcal{K}}, \\ |\hat{\psi}|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} &\leq C\kappa |Dv^1 - Dv^2|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} \leq C\kappa |v^1 - v^2|_{1,\alpha,\mathcal{C}'_Q}^{(n/2)} \leq C\kappa \|v^1 - v^2\|_{\mathcal{K}}, \\ \|\hat{g}_1\|_{0,\alpha,\Sigma_Q^1} &\leq C\kappa \|Dv^1 - Dv^2\|_{0,\alpha,\Sigma_Q^1} \leq C\kappa |Dv^1 - Dv^2|_{0,\alpha,\mathcal{C}'_Q}^{(1+n/2)} \\ &\leq C\kappa |v^1 - v^2|_{1,\alpha,\mathcal{C}'_Q}^{(n/2)} \leq C\kappa \|v^1 - v^2\|_{\mathcal{K}}. \end{aligned}$$

Thus, from (3.21),

$$\|u^1 - u^2\|_{\mathcal{K}} \leq C\kappa \|v^1 - v^2\|_{\mathcal{K}}.$$

Therefore, the mapping $J : \mathcal{K} \rightarrow \mathcal{K}$ is a contraction mapping in the norm $\|\cdot\|_{\mathcal{K}}$ if $\kappa < 1/C$, i.e., κ is small depending only on n, Λ, Q , and λ .

For such κ , there exists a fixed point $u \in \mathcal{K}$ satisfying $Ju = u$. Then u is a weak solution of (3.1)–(3.3). Indeed, since $Ju = u$, then u satisfies (3.18) with right-hand sides given by (3.44)–(3.46) computed with $v = u$. Rearranging and taking into account the last expression in (3.47) yields (3.7). Also, since u satisfies (3.18) with right-hand sides given by (3.44)–(3.46), u satisfies (3.22) with these right-hand sides, which implies

$$|u|_{1,\alpha,\mathcal{C}'_Q}^{(n/2)} \leq C(\|u\|_{L^2(\mathcal{C}_Q)} + |f|_{0,0,\mathcal{C}'_Q}^{(2+n/2)} + \|g_1\|_{0,\alpha,\Sigma_Q^1} + |g_2|_{0,\alpha,(\Sigma_Q^2)'}^{(1+n/2)}) + C\kappa |u|_{1,\alpha,\mathcal{C}'_Q}^{(n/2)},$$

by estimating $\tilde{\psi}$ and $\tilde{g}_1$ similar to the estimates of $\hat{\psi}$ and $\hat{g}_1$ above and with $v = u$. If κ is small, this implies (3.13). Then Theorem 3.2 is proved.

Proposition 3.2. *Let $\lambda_1, \dots, \lambda_n$, and λ be as in Theorem 3.2. Let $L > 0$. Then there exist $\kappa > 0$, depending only on n, Λ, λ , and L , such that for any $Q \geq L$, if $a_{ij}(x)$ satisfy (3.4) and f, g_1, g_2, h satisfy (3.5), (3.6), there exists a unique weak solution $u \in H^1(\mathcal{C}_Q)$ of (3.1)–(3.3).*

Proof. By Theorem 3.2, if $\kappa > 0$ is small depending only on n, Λ, λ , and L , there exists $u_1 \in H^1(\mathcal{C}_L) \cap C^{1,\alpha}(\mathcal{C}'_L)$ which is a weak solution of (3.1) in $\mathcal{C}_L$, (3.2) on Σ_L^k , and $u_1 = 0$ on Σ_L^3 , in the sense of (3.7) with Q replaced by L . Then we can extend u_1 from $\mathcal{C}_{L/2}$ to $\mathcal{C}_Q$ so that the extended function $\tilde{u}_1$ satisfies $\tilde{u}_1 \in C^{1,\alpha}(\overline{\mathcal{C}_Q})$ and $\tilde{u}_1 = u_1$ on $\mathcal{C}_{L/2}$.

Denote

$$\tilde{g}_2 := g_2 - \sum_{i,j=1}^n a_{ij}(\tilde{u}_1)_{x_j} \nu_i \quad \text{on } \Sigma_Q^2.$$

Then $\tilde{g}_2 \in C^{0,\alpha}(\overline{\Sigma_Q^2})$, and $\tilde{g}_2 \equiv 0$ on $\Sigma_{L/2}^2$. Now, repeating the construction of Step 6 of proof of Lemma 3.1, we show existence of $G \in C^{1,\alpha}(\overline{\mathcal{C}_Q})$ satisfying

$$\sum_{i,j=1}^n a_{ij} G_{x_j} \nu_i = \tilde{g}_2 \quad \text{and} \quad G = 0 \quad \text{on } \Sigma_Q^2. \quad (3.48)$$

Also, choosing by $\eta \in C^\infty(\mathbf{R}^1)$ satisfying $\eta \equiv 0$ on $(-\infty, L/4)$ and $\eta \equiv 1$ on $(L/2, \infty)$, and using that $\tilde{g}_2 \equiv 0$ on $\Sigma_{L/2}^2$ and $G \equiv 0$ on Σ_Q^2 , we see that the function $\tilde{G}(x) := \eta(x_n)G(x)$ also satisfies (3.48). Thus, replacing G by $\tilde{G}$, we get a function $G \in C^{1,\alpha}(\overline{\mathcal{C}_Q})$ satisfying (3.48) and

$$\sum_{i,j=1}^n a_{ij} G_{x_j} \nu_i = 0 \quad \text{on } \Sigma_Q^1.$$

Now, for $f, \psi^i \in C^{0,\alpha}(\overline{\mathcal{C}_Q})$ for $i = 1, \dots, n$, and $\tilde{h} \in H^1(\mathcal{C}_Q)$, we construct a weak solution $v \in H^1(\mathcal{C}_Q)$ of the problem

$$\begin{aligned} \sum_{i,j=1}^n (a_{ij} v_{x_j})_{x_i} &= f + \sum_{i=1}^n \psi^i_{x_i} \quad \text{in } \mathcal{C}_Q, \\ \sum_{i,j=1}^n a_{ij} v_{x_j} \nu_i &= 0 \quad \text{on } \Sigma_Q^1, \Sigma_Q^2, \quad v = \tilde{h} \quad \text{on } \Sigma_Q^3 \end{aligned}$$

in the sense that $v = \tilde{h}$ on Σ_Q^3 holds as the trace and

$$\int_{\mathcal{C}_Q} \left(\sum_{i,j=1}^n a_{ij} v_{x_j} w_{x_i} - \sum_{i=1}^n \psi^i w_{x_i} + f w \right) dx + \sum_{k=1}^2 \int_{\Sigma_Q^k} \sum_{i=1}^n \psi^i \nu_i w d\mathcal{H}^{n-1} = 0 \quad (3.49)$$

for any function $w \in H^1(\mathcal{C}_Q)$ with $w = 0$ on Σ_Q^3 . First we assume $\psi^i \in C^{1,\alpha}(\overline{\mathcal{C}_Q})$ for $i = 1, \dots, n$. Then, denoting $\tilde{f} = f + \sum_{i=1}^n \psi^i_{x_i}$, we have $\tilde{f} \in C(\overline{\mathcal{C}_Q})$. Then we find a weak solution v of (3.49) as a minimizer of

$$I[v] = \int_{\mathcal{C}_Q} \left(\frac{1}{2} \sum_{i,j=1}^n a_{ij} v_{x_i} v_{x_j} + \tilde{f} v \right) dx$$

on the set $\{v \in H^1(\mathcal{C}_Q) : v = \tilde{h} \text{ on } \Sigma_Q^3\}$. In general case, assuming that $\psi^i \in C^{0,\alpha}(\overline{\mathcal{C}_Q})$ and using $w = v - h$ in (3.49), and using ellipticity, we obtain

$$\|v\|_{H^1(\mathcal{C}_{(-Q,Q)})} \leq C(\|f\|_{L^2(\mathcal{C}_{(-Q,Q)})} + \|\psi\|_{C^0(\mathcal{C}_{(-Q,Q)})} + \|h\|_{H^1(\mathcal{C}_{(-Q,Q)})}), \quad (3.50)$$

where C depends only on $n, \Lambda, Q, \lambda, \kappa$. Then approximating $\psi = (\psi^1, \dots, \psi^n)$ by $\psi_l \in C^{1,\alpha}(\overline{\mathcal{C}_Q})$ such that $\psi_l \rightarrow \psi$ in $C^{0,\alpha}(\overline{\mathcal{C}_Q})$, denoting by v_l a weak solution of (3.49) with ψ_l instead of ψ , and using (3.50) and the linearity of the problem, we find that $\{v_l\}_{l=1}^\infty$ is a Cauchy sequence in $H^1(\mathcal{C}_Q)$, and its limit is a weak solution of (3.49) with $\psi \in C^{0,\alpha}(\overline{\mathcal{C}_Q})$.

Now let $v \in H^1(\mathcal{C}_Q)$ be a weak solution of (3.49) with f as in Proposition, and

$$\psi^i = - \sum_{j=1}^n a_{ij}(\tilde{u}_1 + G)_{x_j}, \quad \tilde{h} = h - u_1 - G,$$

where h is as in Proposition. Note that $\psi^i \in C^{0,\alpha}(\overline{\mathcal{C}_Q})$ and $\tilde{h} \in H^1(\mathcal{C}_{(-Q,Q)})$, thus such v exists. Then function $u = u_1 + G + v$ is a weak solution of (3.1)–(3.3) $\square$

Next we prove some regularity properties of weak solutions near Σ_Q^3 , more specifically near the wedge $W := \partial\Lambda \times \{x_n = Q\}$. For $a < b$, denote $\mathcal{C}_{(a,b)} = \Lambda \times (a, b)$ and $\Sigma_{(a,b)}^2 = \partial\Lambda \times (a, b)$.

Proposition 3.3. *Let $\lambda > 0$ and $b > 0$ be constants. Let the coefficients $a_{ij}(x)$ satisfy the ellipticity condition (3.35) and let $\|a_{ij}\|_{0,\alpha,\Lambda \times (Q-b,Q)} \leq \lambda^{-1}$. Let $u \in H^1(\mathcal{C}_{(Q-b,Q)})$ be a weak solution of the conormal problem*

$$\sum_{i,j=1}^n (a_{ij}u_{x_j})_{x_i} = 0 \quad \text{in } \mathcal{C}_{(Q-b,Q)}, \quad \sum_{i,j=1}^n a_{ij}u_{x_j}\nu_i = 0 \quad \text{on } \Sigma_{(Q-b,Q)}^2, \quad (3.51)$$

$$u = 0 \quad \text{on } \Lambda \times \{x_n = Q\} \quad (3.52)$$

in the sense that (3.52) holds as the trace and

$$\int_{\mathcal{C}_{(Q-b,Q)}} \sum_{i,j=1}^n a_{ij}u_{x_j}w_{x_i} dx = 0 \quad (3.53)$$

for any function $w \in H^1(\mathcal{C}_{(Q-b,Q)})$ with $w = 0$ on $\partial\mathcal{C}_{(Q-b,Q)} \setminus \Sigma_{(Q-b,Q)}^2$. Then there exists $\beta \in (0, 1)$ and $C > 0$, depending only on the data (i.e., on $n, \Lambda, b, \alpha, \lambda, \kappa$) such that $u \in C^{1,\alpha}(\overline{\mathcal{C}_{(Q-b/2,Q)}} \setminus W)$ and

$$|Du(x)| \leq C\|u\|_{L^2(\Lambda \times (Q-b,Q))}(\text{dist}(x, W))^{\beta-1} \quad \text{for any } x \in \mathcal{C}_{(Q-b/2,Q)}, \quad (3.54)$$

where $W := \partial\Lambda \times \{x_n = Q\}$.

Proof. The regularity $u \in C^{1,\alpha}(\overline{\mathcal{C}_{(Q-b/2,Q)}} \setminus W)$ follows from the standard interior estimates for linear elliptic equations and the estimates for the Dirichlet and conormal problems. Thus it suffices to prove (3.54).

Denote

$$\Sigma_{(Q-b,Q)}^D := \partial\mathcal{C}_{(Q-b,Q)} \setminus \Sigma_{(Q-b,Q)}^2.$$

We first note that any weak solution u of (3.51) (without requiring (3.52)) satisfies the following local estimates: For any $B_{2R} = B_{2R}(y) \subset \mathbf{R}^n$ with $R \in (0, b/10)$, denote $M = \sup_{B_{2R} \cap \Sigma_{(Q-b,Q)}^D} u^+$ and $m = \inf_{B_{2R} \cap \Sigma_{(Q-b,Q)}^D} u$, where in the case $B_{2R} \cap \Sigma_{(Q-b,Q)}^D = \emptyset$ we set $m = -\infty$ and $M = \infty$. Define the functions u_M^+ and u_m^- in $\mathcal{C}_{(-\infty, \infty)}$ by

$$u_M^+(x) := \begin{cases} \sup(u(x), M), & x \in \mathcal{C}_{(Q-b,Q)}, \\ M, & x \in \mathcal{C}_{(-\infty, \infty)} \setminus \mathcal{C}_{(Q-b,Q)}, \end{cases} \quad u_m^-(x) := \begin{cases} \inf(u(x), m), & x \in \mathcal{C}_{(Q-b,Q)}, \\ m, & x \in \mathcal{C}_{(-\infty, \infty)} \setminus \mathcal{C}_{(Q-b,Q)}. \end{cases}$$

Then, for any $p > 1$,

$$\sup_{B_R(y) \cap \mathcal{C}_{(-\infty, \infty)}} u_M^+ \leq CR^{-n/p} \|u_M^+\|_{L^p(B_{2R}(y) \cap \mathcal{C}_{(-\infty, \infty)})}.$$

If u is nonnegative in $B_{4R}(y) \cap \mathcal{C}_{(Q-b,Q)}$, then, for any $1 \leq p \leq n/(n-2)$, the weak Harnack inequality holds:

$$\inf_{B_R(y) \cap \mathcal{C}_{(-\infty, \infty)}} u_m^- \geq CR^{-n/p} \|u_m^-\|_{L^p(B_{2R}(y) \cap \mathcal{C}_{(-\infty, \infty)})},$$

where the constant C depends only on Λ, b, λ , and p . The proof follows the proofs of [20, Thm 8.17, 8.18, 8.25, 8.26] since, in the weak formulation (3.53), we can use the same test functions as in [20, Thm 8.17, 8.18, 8.25, 8.26] and obtain the same expressions in the result, with the only

difference that we integrate over $\mathcal{C}_{(-\infty, \infty)}$ instead of $\mathbf{R}^n$. Also, in our case, the test functions may be nonzero on the boundary part $\Sigma_{(Q-b, Q)}^2$, and thus we use the Sobolev inequalities for the functions that vanish only on a part of the boundary (cf. [34, Thm. 6.37]).

Once the weak Harnack inequality is obtained, we follow the proofs of [20, Thm 8.22, 8.27] to obtain the existence of $\beta \in (0, \alpha)$ and $C > 0$ depending only on the data such that, if u is a weak solution of (3.51)–(3.52), then

$$|u(x)| \leq C \|u\|_{L^\infty(\mathcal{C}_{(Q-3b/4, Q)})} |Q - x_n|^\beta \quad \text{for any } x \in \mathcal{C}_{(Q-b/2, Q)}.$$

We also obtain $\|u\|_{L^\infty(\mathcal{C}_{(Q-3b/4, Q)})} \leq C \|u\|_{L^2(\Lambda \times (Q-b, Q))}$ by applying the local estimates to $\pm u$ with $p = 2$ and using (3.52). Then, using $0 \leq Q - x_n \leq \text{dist}(x, W)$ for $x \in \mathcal{C}_{(Q-b/2, Q)}$, we have

$$|u(x)| \leq C \|u\|_{L^2(\Lambda \times (Q-b, Q))} (\text{dist}(x, W))^\beta \quad \text{for all } x \in \mathcal{C}_{(Q-b/2, Q)}. \quad (3.55)$$

Now we follow the argument of Step 5 of the proof of Lemma 3.1. For $x \in \mathbf{R}^n$ and $\rho > 0$, denote $\Omega_\rho(x) := \mathcal{C}_{(Q-b, Q)} \cap B_\rho(x)$. Let $x \in \Lambda \times \{x_n = Q\}$ and $\rho = \min(b, \text{dist}(x, W))/10$. Then $B_\rho(x) \cap \partial \mathcal{C}_{(Q-b, Q)} \subset \Lambda \times \{x_n = Q\}$. Thus, applying the standard estimates for the Dirichlet problem [20, Thm 8.32, 8.33], properly scaled, we obtain

$$\|u\|_{0,0,\Omega_{\rho/2}(x)} + \rho \|Du\|_{0,0,\Omega_{\rho/2}(x)} + \rho^{1+\alpha} [Du]_{0,\alpha,\Omega_{\rho/2}(x)} \leq C \rho^{-n/2} \|u\|_{L^2(\Omega_\rho(x))}, \quad (3.56)$$

where C depends only on the data. Now let $x \in \Sigma_{(Q-b/2, Q)}^2$ and $\rho = \min(b, \text{dist}(x, W))/10$. Then $B_\rho(x) \cap \partial \mathcal{C}_{(Q-b, Q)} \subset \Sigma_{(Q-b, Q)}^2$. Thus the proof of (3.40) yields (3.56) in the present case. Finally, if $x \in \mathcal{C}_{(Q-b/2, Q)}$, $\rho = \min(b, \text{dist}(x, W))/40$, and $B_\rho(x) \subset \mathcal{C}_{(Q-b, Q)}$, then the interior estimates [20, Thm 8.32] also yield (3.56). Note that, if $z \in \overline{\mathcal{C}_{(Q-b/2, Q)}} \setminus W$, $0 < r \leq \min(b, \text{dist}(x, W))/10$, and $y \in \Omega_r(z)$, then $\frac{1}{2} \text{dist}(z, W) \leq \text{dist}(y, W) \leq 2 \text{dist}(z, W)$. Then (3.55) and (3.56) imply

$$\|Du\|_{0,0,\Omega_{\rho/2}(x)} \leq C \|u\|_{L^2(\Lambda \times (Q-b, Q))} (\text{dist}(x, W))^{\beta-1} \quad (3.57)$$

for all choices of (x, ρ) discussed above. Moreover, the sets $\Omega_{\rho/2}(x)$, with (x, ρ) specified above, cover $\mathcal{C}_{(Q-b/2, Q)}$. Then we use again that $\frac{1}{2} \text{dist}(x, W) \leq \text{dist}(y, W) \leq 2 \text{dist}(x, W)$ for all $y \in \Omega_{\rho/2}(x)$ to have (3.54). $\square$

4. FREE BOUNDARY PROBLEMS IN THE INFINITE NOZZLE

In order to solve Problem (FB), we first reformulate it into a free boundary problem for the subsonic part of the solution, since Problem (FB) is originally hyperbolic-elliptic mixed.

We first modify equation (1.1) to make it uniformly elliptic so that it coincides with the original equation in the range of $D\varphi$ in the subsonic region Ω^+ for φ satisfying (2.31) with sufficiently small σ . The details of the truncation procedure are in [4, Section 4.2].

Let $\varepsilon = (c_* - q_0^+)/2$. Then there exist $\tilde{\rho} \in C^{1,1}([0, \infty))$ and $c_j > 0, j = 1, 2, 3$, depending only on q_0^+ and γ , such that

$$\tilde{\rho}(q^2) = \rho(q^2) \quad \text{if } 0 \leq q < c_* - \varepsilon, \quad (4.1)$$

$$\tilde{\rho}(q^2) = c_0 + \frac{c_1}{q} \quad \text{if } q > c_* - \varepsilon, \quad (4.2)$$

$$0 < c_0 \leq (\tilde{\rho}(q^2)q)' \leq c_2 \quad \text{for } q \in (0, \infty). \quad (4.3)$$

Then the equation

$$\tilde{\mathcal{L}}\varphi := \text{div}(\tilde{\rho}(|D\varphi|^2)D\varphi) = 0 \quad (4.4)$$

is uniformly elliptic, whose ellipticity constants depend only on q_0^+ and γ .

To formulate the problem into a free boundary problem, the main point is to replace the pointwise gradient condition $\{|D\varphi(x)| < c_*\}$ defining $\Omega^+(\varphi)$ by a condition stated in terms of φ so that our problem is formulated into the framework of free boundary problems.

To do that, we first need to construct a local $C^{1,\alpha}$ supersonic solution φ^- in the domain $\Omega_1 := \Omega \cap \{-1 < x_n < 1\}$ of the initial-boundary value problem (2.18)–(2.19) for the nonlinear hyperbolic equation (1.1) such that

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_1} \leq C_1 \sigma \quad (4.5)$$

when σ is sufficiently small. This is ensured by the local existence theorem, Proposition A.1 in Appendix. By the standard extension argument (see [5] and Section 7.2), we can extend φ^- to the whole infinite nozzle Ω such that

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega} \leq C_0 \sigma, \quad \varphi^-|_{\partial_l \Omega} = 0. \quad (4.6)$$

Then the following heuristic observation motivates our formulation: By (2.13) and $q_0^- > q_0^+$, we have $\Omega^+(\varphi_0) = \{x \in \Omega : \varphi_0(x) < \varphi_0^-(x)\}$. Since φ^- is a small $C^{1,\alpha}$ perturbation of φ_0^- and $q_0^- > q_0^+$, then we expect that φ^+ is close to φ_0^+ in $C^{1,\alpha}(\overline{\Omega^+(\varphi)})$ so that we can expect that $\Omega^+(\varphi) = \{x \in \Omega : \varphi(x) < \varphi^-(x)\}$.

We also perform the corresponding truncation of the free boundary condition (2.4):

$$\tilde{\rho}(|D\varphi|^2)\varphi_\nu = \rho(|D\varphi^-|^2)D\varphi^- \cdot \nu \quad \text{on } S. \quad (4.7)$$

On the right-hand side of (4.7), we use the original function ρ since $\rho \neq \tilde{\rho}$ on the range of $|D\varphi^-|^2$. Note that (4.7), with the right-hand side considered as a known function, is the conormal boundary condition for the uniformly elliptic equation (4.4).

We first solve the following free boundary problem (TFB), which is a truncated version of Problem (FB), in which the gradient condition that determines Ω^+ in Problem (FB) is also replaced by the condition $\varphi \leq \varphi^-$ in Ω with the definition $\Omega^+ := \{\varphi < \varphi^-\} \cap \Omega$.

Problem (TFB): Truncated Free Boundary Problem. Given a supersonic upstream flow $\varphi^- \in C^{1,\alpha}(\Omega_1)$ of (1.1) satisfying (4.5) and (2.18)–(2.19) for small $\sigma > 0$ and some constant C_0 , find $\varphi \in C(\Omega)$ such that

(i) φ satisfies

$$\varphi \leq \varphi^- \quad \text{in } \Omega \quad (4.8)$$

and conditions (2.18)–(2.19) on the boundary and (2.20) at infinity;

(ii) $\varphi \in C^{1,\alpha}(\overline{\Omega^+}) \cap C^2(\Omega^+)$ is a solution of (4.4) in $\Omega^+ := \{\varphi < \varphi^-\} \cap \Omega$, the non-coincidence set;

(iii) the free boundary $S = \partial\Omega^+ \cap \Omega$ is given by the equation $x_n = f(x')$ so that $\Omega^+ = \{x_n > f(x')\} \cap \Omega$ with $f \in C^{1,\alpha}$;

(iv) the free boundary condition (4.7) holds on S .

In the next sections we first solve Problem (TFB) by an iteration procedure using uniform estimates at infinity and the regularity estimates near the nonsmooth boundary. Finally, we use an estimate for $|D\varphi|$ to conclude that the solution of the truncated problem, Problem (TFB), is actually a solution of Problem (FB) and thus Problem (TN).

5. ITERATION PROCEDURE AND UNIFORM ESTIMATES

We now introduce an iteration procedure to construct approximate solutions of Problem (TFB) in the domain Ω and make uniform estimates of the solutions on the unbounded domain with nonsmooth boundary.

5.1. Iteration set. Let $M \geq 1$. We set

$$\mathcal{K}_M := \left\{ \psi \in C^{1,\alpha}(\overline{\Omega}) : \|\psi - q^+ x_n\|_{1,\alpha,\Omega} \leq M\sigma \right\}, \quad (5.1)$$

where q^+ is defined by equation (2.26) (see Remark 2.4) and $\alpha \in (0, 1)$. By definition, $\mathcal{K}_M$ is convex. Now we will show that $\mathcal{K}_M$ is compact in the weighted Hölder space $C_{(-1)}^{1,\alpha/2}(\overline{\Omega})$ as defined below.

For $\alpha \in (0, 1)$, $m \in \mathbf{N}$, and $k \in \mathbf{R}$, we define the following spaces:

$$C_{(k)}^{m,\alpha}(\overline{\Omega}) := \{u \in C^{m,\alpha}(\overline{\Omega}) : \|u\|_{m,\alpha,\Omega}^{(k)} < \infty\}, \quad (5.2)$$

where $\|u\|_{m;\alpha;\Omega}^{(k)} = \|u\|_{m;0;\Omega}^{(k)} + [u]_{m;\alpha;\Omega}^{(k)}$ are the norms with

$$\begin{aligned} \|u\|_{m;0;\Omega}^{(k)} &= \sum_{0 \leq j \leq m} \left(\sum_{|\beta|=j} \sup_{x \in \Omega} (\delta_x^k |D^\beta u(x)|) \right), \\ [u]_{m;\alpha;\Omega}^{(k)} &= \sum_{|\beta|=m} \sup_{x,y \in \Omega, x \neq y} \left(\delta_{x,y}^k \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \right), \end{aligned}$$

$\delta_x = |x_n| + 1$ for $x = (x', x_n) \in \mathcal{D}$, and $\delta_{x,y} = \min(\delta_x, \delta_y)$ for $x, y \in \mathcal{D}$.

Lemma 5.1. *If $\alpha > \beta > 0$, then $\mathcal{K}_M$ is a compact subset of $C_{(-1)}^{1,\beta}(\overline{\Omega})$.*

Proof. Let $\psi_j \in \mathcal{K}_M$ for $j = 1, 2, \dots$. By a standard argument, we can extract a subsequence (still denoted by) ψ_j , which converges in $C^{1,\beta}$ on every compact subset of $\overline{\Omega}$ to the limit $\psi \in C^{1,\alpha}(\overline{\Omega})$. Then $\|\psi - q^+ x_n\|_{1,\alpha,\Omega} \leq M\sigma$ and thus $\psi \in \mathcal{K}_M$. It remains to show that $\|\psi_j - \psi\|_{1,\beta,\Omega}^{(-1)} \rightarrow 0$ as $j \rightarrow \infty$.

Fix $0 < \varepsilon < 1$. Then $\|\psi_j\|_{1,\alpha,\Omega \cap \{x_n > 1/\varepsilon\}}^{(-1)} \leq (M\sigma + q^+)\varepsilon$, and the same estimate holds for ψ . Also, since $\psi_j \rightarrow \psi$ in $C^{1,\beta}(\overline{\Omega} \cap \{x_n \leq 2/\varepsilon\})$, there exists j_0 such that, for $j > j_0$, $\|\psi_j - \psi\|_{1,\beta,\Omega \cap \{x_n \leq 2/\varepsilon\}}^{(-1)} \leq \varepsilon$. Then, for $j > j_0$, we have $\|\psi_j - \psi\|_{1,\beta,\Omega}^{(-1)} \leq C\varepsilon$, and the assertion is proved. $\square$

5.2. Construction of the iteration scheme. Let $\psi \in \mathcal{K}_M$. Since $q_0^- > q_0^+$, it follows that, if

$$\sigma \leq \frac{q_0^- - q_0^+}{C(M+1)} \quad (5.3)$$

with large C depending only on n , then (4.5) implies

$$(\varphi^- - \psi)_{x_n}(x) \geq \frac{q_0^- - q_0^+}{2} > 0 \quad \text{in } \Omega \quad (5.4)$$

and $\varphi^- > \psi$ on $\{x_n > 1\}$. Then the set $\Omega^+(\psi) := \{\psi < \varphi^-\} \cap \Omega$ has the form:

$$\Omega^+(\psi) = \{x_n > f(x')\} \cap \Omega \quad \text{with} \quad \|f\|_{1,\alpha,\mathbf{R}^{n-1}} \leq CM\sigma \quad (5.5)$$

for C depending only on $q_0^- - q_0^+$. The inward unit normal on $S_\psi := \{x_n = f(x')\} \cap \Omega$ to Ω^+ is

$$\nu_\psi(x) = \frac{D\varphi^-(x) - D\psi(x)}{|D\varphi^-(x) - D\psi(x)|} \quad \text{for } x \in S_\psi. \quad (5.6)$$

By the definition of $\mathcal{K}_M$ and (5.3), formula (5.6) also defines $\nu_\psi(x)$ on Ω_1 and

$$\|\nu_\psi - e_n\|_{0,\alpha,\Omega_1} \leq CM\sigma \quad \text{with } C = C(q_0^+, q_0^-). \quad (5.7)$$

Motivated by (4.7), we define the function

$$G_\psi(x) := \rho(|D\varphi^-(x)|^2) D\varphi^-(x) \cdot \nu_\psi(x) \quad \text{on } \Omega_1. \quad (5.8)$$

Then we consider the following problem in the domain $\Omega^+(\psi)$:

$$\operatorname{div}(\tilde{\rho}(|D\varphi|^2)D\varphi) = 0 \quad \text{in } \Omega^+(\psi), \quad (5.9)$$

$$\tilde{\rho}(|D\varphi|^2)\varphi_\nu = G_\psi(x) \quad \text{on } S_\psi := \{x_n = f(x')\}, \quad (5.10)$$

$$\varphi_\nu = 0 \quad \text{on } \partial_t \Omega^+ := \partial \Omega^+(\psi) \cap \partial_t \Omega, \quad (5.11)$$

$$\lim_{R \rightarrow \infty} \|\varphi - q^+ x_n\|_{L^\infty(\Omega^+(\psi) \cap \{x_n > R\})} = 0, \quad (5.12)$$

show that there exists a unique solution $\varphi(x)$, and extend the solution to the whole domain Ω so that $\varphi \in \mathcal{K}_M$.

We rewrite problem (5.9)–(5.12) in terms of the function $u(x) = \varphi(x) - q^+x_n$. We first note that the boundary condition (5.11) is equivalent to

$$\tilde{\rho}(|D\varphi|^2)\varphi_\nu = 0 \quad \text{on } \partial_l\Omega^+.$$

Now $u(x)$ is a solution of the following problem:

$$\operatorname{div} A(Du) = 0 \quad \text{in } \Omega^+(\psi), \quad (5.13)$$

$$A(Du) \cdot \nu = g_\psi(x) \quad \text{on } S_\psi, \quad (5.14)$$

$$A(Du) \cdot \nu = -\tilde{\rho}((q^+)^2)q^+\nu \cdot e_n \quad \text{on } \partial_l\Omega^+, \quad (5.15)$$

$$\lim_{R \rightarrow \infty} \|u\|_{L^\infty(\Omega^+(\psi) \cap \{x_n > R\})} = 0, \quad (5.16)$$

where

$$A(P) = \tilde{\rho}(|P + q^+e_n|^2)(P + q^+e_n) - \tilde{\rho}((q^+)^2)q^+e_n \quad \text{for } P \in \mathbf{R}^n, \quad (5.17)$$

$$g_\psi(x) = G_\psi(x) - \rho((q^+)^2)q^+\nu \cdot e_n. \quad (5.18)$$

Thus, $u(x)$ satisfies the uniformly elliptic equation with the same ellipticity constants as in (4.4):

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n A_{P_j}^i(P)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \text{for any } P, \xi \in \mathbf{R}^n. \quad (5.19)$$

Moreover, from the definition of $A(P)$ and (4.1)–(4.2), $A(P)$ satisfies

$$A(0) = 0, \quad (1 + |P|)|D_P A_{P_j}^i(P)| \leq C. \quad (5.20)$$

Finally, we state a linear problem corresponding to (5.13)–(5.16). Namely, we use (5.20) to find that, for $i = 1, \dots, n$,

$$A^i(Du(x)) = \sum_{j=1}^n \tilde{a}_{ij}(x)u_{x_j}(x), \quad \tilde{a}_{ij}(x) = \int_0^1 A_{p_j}^i(tDu(x))dt.$$

We replace $u \equiv \varphi - q^+x_n$ in the definition of the coefficients $\tilde{a}_{ij}$ by $\psi - q^+x_n$ for $\psi \in \mathcal{K}_M$ to define

$$a_{ij}^{(\psi)}(x) = a_{ij}^{(\psi)}(x) = \int_0^1 A_{p_j}^i(t(D\psi(x) - q^+e_n))dt \quad \text{for } x \in \Omega, \quad i, j = 1, \dots, n. \quad (5.21)$$

The ellipticity (5.19) of $A(P)$ implies that the coefficients $a_{ij}^{(\psi)}(x)$ satisfy the ellipticity condition (3.35) for any $x \in \Omega$.

Also, from (5.17) and (5.21),

$$\begin{aligned} a_{ij}^{(\psi)}(x) &= \int_0^1 \left\{ \tilde{\rho}(|tD\psi(x) + (1-t)q^+e_n|^2)\delta_i^j \right. \\ &\quad \left. + 2\tilde{\rho}'(|tD\psi(x) + (1-t)q^+e_n|^2)(t\psi_{x_i}(x) + (1-t)q^+\delta_i^n)(t\psi_{x_j}(x) + (1-t)q^+\delta_j^n) \right\} dt \end{aligned} \quad (5.22)$$

for $x \in \Omega$, where $\delta_i^j = 1$ if $i = j$ and $\delta_i^j = 0$ if $i \neq j$. In particular, we have $a_{ij} = a_{ji}$.

We note that, for $\check{\psi}_0 = q^+x_n$, the corresponding coefficients $\check{a}_{ij}$ defined by (5.22) are constants and satisfy

$$\check{a}_{ij} = \kappa_i \delta_i^j \quad \text{for } i, j = 1, \dots, n, \quad (5.23)$$

where κ_i are $\kappa_i = \tilde{\rho}((q^+)^2)$ for $i = 1, \dots, n-1$, and $\kappa_n = \phi'(q^+)$ for $\phi(s) = \tilde{\rho}(s^2)s$. By (4.1)–(4.3), we have

$$\lambda \leq \kappa_i \leq \lambda^{-1} \quad \text{for } i = 1, \dots, n.$$

Now, for $\psi \in \mathcal{K}_M$, we use (5.22) and (4.1)–(4.3) to obtain

$$\|a_{ij}^{(\psi)} - \check{a}_{ij}\|_{0,\alpha,\Omega} \leq CM\sigma. \quad (5.24)$$

Thus, we formulate the following conormal fixed boundary elliptic problem:

$$\sum_{i,j=1}^n (a_{ij}^{(\psi)} u_{x_j})_{x_i} = 0 \quad \text{in } \Omega^+(\psi), \quad (5.25)$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} \nu_i = g_\psi(x) \quad \text{on } S_\psi, \quad (5.26)$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} \nu_i = -\tilde{\rho}((q^+)^2) q^+ \nu \cdot e_n \quad \text{on } \partial_t \Omega^+(\psi), \quad (5.27)$$

$$\lim_{R \rightarrow \infty} \|u\|_{C(\Omega^+(\psi) \cap \{x_n > R\})} = 0. \quad (5.28)$$

Since the coefficients are only C^α , we can expect to find only a weak solution $u \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ of (5.25)–(5.28) in the sense that $u(x)$ satisfies (5.28) and, for any $w \in C_c^1(\mathbf{R}^n)$,

$$\begin{aligned} \int_{\Omega^+(\psi)} \sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} w_{x_i} dx + \int_{S_\psi} g_\psi w d\mathcal{H}^{n-1} \\ - \int_{\partial_t \Omega^+(\psi)} \tilde{\rho}((q^+)^2) q^+ \nu \cdot e_n w d\mathcal{H}^{n-1} = 0. \end{aligned} \quad (5.29)$$

We will determine the iteration map $J(\psi) = \varphi$ by solving (5.25)–(5.28) for u , extending u from $\Omega^+(\psi)$ to Ω so that $u + q^+ x_n \in \mathcal{K}_M$, and defining $\varphi = u + q^+ x_n$. A fixed point of this map is obviously a solution of Problem (TFB) in Ω . In Sections 6–7, we prove the existence of such a map J , as well as its fixed point.

6. FIXED BOUNDARY PROBLEMS IN A BOUNDED DOMAIN $\Omega_R^+(\psi)$

In order to find a solution of (5.25)–(5.28) in the unbounded domain $\Omega^+(\psi)$, we first solve the corresponding problem in the bounded domain

$$\Omega_R^+(\psi) := \Omega^+(\psi) \cap \{x_n < R\}, \quad R > L,$$

and then pass to the limit as $R \rightarrow \infty$. Thus we consider the following problem:

$$\sum_{i,j=1}^n (a_{ij}^{(\psi)} u_{x_j})_{x_i} = 0 \quad \text{in } \Omega_R^+(\psi), \quad (6.1)$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} \nu_i = g_\psi(x) \quad \text{on } S_\psi, \quad (6.2)$$

$$\sum_{i,j=1}^n a_{ij}^{(\psi)} u_{x_j} \nu_i = -\tilde{\rho}((q^+)^2) q^+ \nu \cdot e_n \quad \text{on } \partial_t \Omega_R^+(\psi), \quad (6.3)$$

$$u = 0 \quad \text{on } \partial \Omega_R^+(\psi) \cap \{x_n = R\}, \quad (6.4)$$

where $a_{ij}^{(\psi)}$ are defined by (5.21) and $\partial_t \Omega_R^+(\psi) = \partial_t \Omega^+ \cap \partial \Omega_R^+(\psi)$.

We first note several properties of g_ψ , which will be used later.

Lemma 6.1. *There exists $C > 0$, depending only on the data and independent of M and R , such that*

$$\|g_\psi\|_{0,\alpha,\Omega_1} \leq C\sigma. \quad (6.5)$$

This is proved in [5, (4.48), page 334]. Another property of g_ψ is

Lemma 6.2.

$$\int_{S_f} g_\psi d\mathcal{H}^{n-1} = \tilde{\rho}((q^+)^2)q^+ \int_{\partial_t \Omega_R^+} \nu \cdot e_n d\mathcal{H}^{n-1}.$$

Proof. Note that

$$0 = \int_{\Omega_R} \Delta(x_n) = - \int_{\Lambda \times \{x_n=R\}} e_n \cdot \nu dx' - \int_{S_f} e_n \cdot \nu d\mathcal{H}^{n-1} - \int_{\partial_t \Omega_R^+} e_n \cdot \nu d\mathcal{H}^{n-1},$$

where ν is the inward unit normal. Since $\nu = -e_n$ on $\Lambda \times \{x_n = R\}$, then, from the last equality, we have

$$\int_{S_f} e_n \cdot \nu d\mathcal{H}^{n-1} + \int_{\partial_t \Omega_R^+} e_n \cdot \nu d\mathcal{H}^{n-1} = |\Lambda|. \quad (6.6)$$

Also, since G_ψ is defined by (5.8) and $\varphi^- \in C^{1,\alpha}(\overline{\Omega \setminus \Omega_R^+(\psi)})$ is a weak solution of (1.1) in $\Omega \setminus \overline{\Omega_R^+(\psi)}$ and satisfies the boundary condition (2.19), we find

$$\int_{S_f} G_\psi d\mathcal{H}^{n-1} = \int_{\partial_o \Omega} \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu d\mathcal{H}^{n-1}. \quad (6.7)$$

Then we have

$$\begin{aligned} & \int_{S_f} g_\psi d\mathcal{H}^{n-1} - \tilde{\rho}((q^+)^2)q^+ \int_{\partial_t \Omega_R^+} \nu \cdot e_n d\mathcal{H}^{n-1} \\ &= \int_{S_f} G_\psi d\mathcal{H}^{n-1} - \tilde{\rho}((q^+)^2)q^+ \left(\int_{S_f} \nu \cdot e_n d\mathcal{H}^{n-1} + \int_{\partial_t \Omega_R^+} \nu \cdot e_n d\mathcal{H}^{n-1} \right) \\ &= \int_{S_f} G_\psi d\mathcal{H}^{n-1} - \tilde{\rho}((q^+)^2)q^+ |\Lambda| = 0, \end{aligned}$$

where we used (6.7) and the fact that q^+ satisfies (2.26)–(2.27). $\square$

We now fix $\psi \in \mathcal{K}_M$ and write Ω_R^+ for $\Omega_R^+(\psi)$ and a_{ij} for $a_{ij}^{(\psi)}$ to simplify the presentation. Also, C denotes a universal constant, which may be different at each occurrence, and depends only on the data and is independent of M and R . Furthermore, there exists $\sigma_0 > 0$ depending only on M and the data such that, when $\sigma \leq \sigma_0$, $S_\psi \subset \{-1/10 < x_n < 1/10\}$ by (5.5); we always assume $\sigma \leq \sigma_0$ below.

Lemma 6.3. *Let $\sigma \in (0, \sigma_0)$ be sufficiently small, depending only on the data and M . Let $u \in H^1(\Omega_R^+)$, $R > L$, be a weak solution of (6.1)–(6.4) in the sense that*

- (i) $u = 0$ on $\partial\Omega_R^+ \cap \{x_n = R\}$ as the trace;
- (ii) For any $w \in H^1(\Omega_R^+)$ satisfying $w = 0$ on $\partial\Omega_R^+ \cap \{x_n = R\}$ in the sense of traces,

$$\int_{\Omega_R^+} \sum_{i,j=1}^n a_{ij} u_{x_j} w_{x_i} dx + \int_{S_f} g_\psi w d\mathcal{H}^{n-1} - \int_{\partial_t \Omega_R^+} \tilde{\rho}((q^+)^2)q^+ \nu \cdot e_n w d\mathcal{H}^{n-1} = 0. \quad (6.8)$$

Then

$$\|Du\|_{L^2(\Omega_R^+)} \leq C\sigma. \quad (6.9)$$

Proof. We use the approach in the proof of [5, Lemma 4.2]. We first choose $w = u$ in (6.8) to obtain

$$\int_{\Omega_R^+} \sum_{i,j=1}^n a_{ij} u_{x_j} u_{x_i} dx = - \int_{S_f} g_\psi u d\mathcal{H}^{n-1} + \int_{\partial_t \Omega_R^+} \tilde{\rho}((q^+)^2)q^+ \nu \cdot e_n u d\mathcal{H}^{n-1}.$$

Now, if Q is a constant which will be chosen below, we use Lemma 6.2 to obtain

$$\int_{\Omega_R^+} \sum_{i,j=1}^n a_{ij} u_{x_j} u_{x_i} dx = - \int_{S_f} g_\psi (u - Q) d\mathcal{H}^{n-1} - \int_{\partial_t \Omega_L^+} \tilde{\rho}((q^+)^2)q^+ \nu \cdot e_n (u - Q) d\mathcal{H}^{n-1},$$

where we used that $\nu \cdot e_n = 0$ on $\partial_l \Omega_R^+ \setminus \partial_l \Omega_L^+$ by (2.14) and (2.17).

Choosing $Q = (u)_L := \frac{1}{|\Omega_L^+(\psi)|} \int_{\Omega_L^+(\psi)} u(x) dx$ and using the L^2 estimate of the boundary traces of functions in the Sobolev space $H^1(\Omega_L^+(\psi))$, (6.5), and $|\nu \cdot e_n| \leq C\sigma$ on $\partial_l \Omega_L^+$ by (2.16), we find

$$\begin{aligned} \int_{\Omega_R^+} \sum_{i,j=1}^n a_{ij} u_{x_j} u_{x_i} dx &\leq \left(\int_{S_f} g_\psi^2 d\mathcal{H}^{n-1} \right)^{1/2} \left(\int_{S_f} (u - (u)_L)^2 d\mathcal{H}^{n-1} \right)^{1/2} \\ &\quad + \tilde{\rho}((q^+)^2) q^+ \left(\int_{\partial_l \Omega_L^+} |\nu \cdot e_n|^2 d\mathcal{H}^{n-1} \right)^{1/2} \left(\int_{\partial_l \Omega_L^+} (u - (u)_L)^2 d\mathcal{H}^{n-1} \right)^{1/2} \\ &\leq C\sigma \left(\int_{\Omega_L^+(\psi)} ((u - (u)_L)^2 + |Du|^2) dx \right)^{1/2} \\ &\leq C\sigma \left(\int_{\Omega_L^+(\psi)} |Du|^2 dx \right)^{1/2}, \end{aligned}$$

where, in the last estimate, we used the Poincaré inequality in the domain $\Omega_L^+(\psi)$, and thus the constant C depends only on n, L , the norms in (2.15)–(2.16), and $\|f_\psi\|_{1,\alpha,\Lambda}$. On the other hand, it follows from (5.5) that if $\sigma < 1/M$, then $\|f_\psi\|_{1,\alpha,\Lambda} \leq C$ with $C > 0$ independent of M, R , and $\psi \in \mathcal{K}_M$. Now, using (3.35), we have

$$\int_{\Omega_R^+} |Du|^2 dx \leq \frac{1}{\lambda} \int_{\Omega_R^+} \sum_{i,j=1}^n a_{ij} u_{x_j} u_{x_i} dx \leq \frac{C}{\lambda} \sigma \left(\int_{\Omega_R^+} |Du|^2 dx \right)^{1/2}.$$

This completes the proof. $\square$

By [35], the weak solution u of (6.1)–(6.4) is in $C^{1,\alpha}(\overline{\Omega_R^+} \setminus (\overline{S_\psi} \cup \{x_n = R\}))$, since $\partial_l \Omega \cap \partial \Omega_R^+$ is $C^{1,\alpha}$.

Lemma 6.4. *Let σ and u be as in Lemma 6.3. Then, for any $x_n \in (L, R)$, there exists $x' \in \overline{\Lambda}$ such that $u(x', x_n) = 0$.*

Proof. We follow the scheme of the proof [5, Lemma 4.3, Step 2]. Fix $z \in (L, R)$. Suppose that there is no $x' \in \overline{\Lambda}$ such that $u(x', z) = 0$. Then we can assume $u(x', z) > 0$ for all $x' \in \overline{\Lambda}$, since the case $u(x', z) < 0$ for all $x' \in \overline{\Lambda}$ can be handled similarly. From the compactness of Λ and continuity of u , there exists a constant $\kappa > 0$ such that

$$u(x', z) \geq \kappa > 0 \quad \text{for all } x' \in \overline{\Lambda}.$$

Consider the domain $\mathcal{D} := \Lambda \times (z, R)$. Then, using Proposition 3.3, we get $u \in C^{1,\alpha}(\overline{\mathcal{D}} \setminus W)$, with $W = \partial_l \Omega \cap \{x_n = R\}$. Moreover, u is a weak solution of

$$\begin{aligned} \sum_{i,j=1}^n (a_{ij} u_{x_j})_{x_i} &= 0 \quad \text{in } \mathcal{D}, \\ \sum_{i,j=1}^n a_{ij} u_{x_j} \nu_i &= 0 \quad \text{on } \partial_l \mathcal{D} = \partial \mathcal{D} \cap \partial_l \Omega \cap \{x_n < R\}, \\ u &\geq \kappa > 0 \quad \text{on } \partial \mathcal{D} \cap \{x_n = z\}, \\ u &= 0 \quad \text{on } \partial \mathcal{D} \cap \{x_n = R\}. \end{aligned}$$

Thus, by the maximum principle, $u \geq 0$ in $\mathcal{D}$. In particular,

$$u_\nu = -u_{x_n} \geq 0 \quad \text{on } (\partial \mathcal{D} \cap \{x_n = R\}) \setminus W. \quad (6.10)$$

By the strong maximum principle [20, Theorem 8.19], $u > 0$ in the interior of $\mathcal{D}$. Also, denoting $\mathcal{D}_\delta = \{x \in \mathcal{D} : \text{dist}(x, \partial_l \Omega) > \delta\}$, where $\delta > 0$ is fixed to be small so that $\mathcal{D}_\delta \neq \emptyset$, we have

$$a_{ij} \in C^{0,\alpha}(\overline{\mathcal{D}_{\delta/2}}), \quad u \in C^{1,\alpha}(\overline{\mathcal{D}_{\delta/2}}).$$

Thus, by [5, Lemma A.1],

$$u_\nu = -u_{x_n} > 0 \quad \text{on } \overline{\mathcal{D}_\delta} \cap \{x_n = R\}. \quad (6.11)$$

Since $u = 0$ on $\Lambda \times \{x_n = R\}$ and $a_{nn} \geq \lambda > 0$ from the ellipticity, we conclude from (6.10)–(6.11) that

$$\sum_{i,j=1}^n a_{ij} u_{x_j} \nu_i = -a_{nn} u_{x_n} \begin{cases} \geq 0 & \text{on } (\partial \mathcal{D} \cap \{x_n = R\}) \setminus W, \\ > 0 & \text{on } \overline{\mathcal{D}_\delta} \cap \{x_n = R\}. \end{cases}$$

Since a_{nn} and u_{x_n} are continuous on $\overline{\mathcal{D}_\delta} \cap \{x_n = R\}$, we conclude

$$\int_{\Omega \cap \{x_n = R\}} \sum_{i,j=1}^n a_{ij} u_{x_j} \nu_i d\mathcal{H}^{n-1} > 0. \quad (6.12)$$

On the other hand, we show that

$$\int_{\Omega \cap \{x_n = R\}} \sum_{i,j=1}^n a_{ij} u_{x_j} \nu_i d\mathcal{H}^{n-1} = 0, \quad (6.13)$$

thus contradicting (6.12). To prove (6.13), we choose $\rho \in (L, R)$, and use in (6.8) the test function $w(x) = \zeta((\rho - x_n)/\varepsilon)$, where $\varepsilon > 0$, $\zeta \in C^\infty(\mathbf{R})$ with $\zeta \geq 0$ on $\mathbf{R}$, $\zeta \equiv 1$ on $(1, \infty)$ and $\zeta \equiv 0$ on $(-\infty, 1/2)$. Note that $w \in C^\infty(\overline{\Omega_R})$ and $w \equiv 0$ on $\{x_n = R\}$. Thus we use this w in (6.8) and $u \in C^{1,\alpha}(\overline{\mathcal{D}} \setminus W)$ and send $\varepsilon \rightarrow 0+$ to obtain

$$-\int_{\Omega \cap \{x_n = \rho\}} \sum_{i,j=1}^n a_{nj} u_{x_j} d\mathcal{H}^{n-1} + \int_{S_f} g_\psi d\mathcal{H}^{n-1} - \int_{\partial_l \Omega_\rho^+} \tilde{\rho}((q^+)^2) q^+ \nu \cdot e_n d\mathcal{H}^{n-1} = 0. \quad (6.14)$$

Since $\rho > L$ and $\nu \cdot e_n = 0$ on $\partial_l \Omega \cap \{x_n > L\}$, we find from (6.14) and Lemma 6.2 that

$$\int_{\Omega \cap \{x_n = \rho\}} \sum_{i,j=1}^n a_{nj} u_{x_j} d\mathcal{H}^{n-1} = 0. \quad (6.15)$$

From $u \in C^{1,\alpha}(\overline{\mathcal{D}} \setminus W)$ and $a_{ij} \in C^\alpha(\overline{\Omega})$, we have $\sum_{i,j=1}^n (a_{nj} u_{x_j})(x', \rho) \rightarrow \sum_{i,j=1}^n (a_{nj} u_{x_j})(x', R)$ as $\rho \rightarrow R-$ for every $x' \in \Lambda$. Moreover, from (3.54) of Proposition 3.3, there exist $C > 0$ and $\beta \in (0, 1)$ such that, for each $\rho \in ((R+L)/2, R)$ and $x' \in \Lambda$, we have

$$|Du(x', \rho)| \leq C(\text{dist}((x', \rho), W))^{\beta-1} \leq C(\text{dist}(x', \partial \Lambda))^{\beta-1},$$

where the last inequality holds since $\text{dist}(x, W) \geq \text{dist}(x', \partial \Lambda)$ for any $x = (x', x_n) \in \Omega_R$. Since $(\text{dist}(\cdot, \partial \Lambda))^{\beta-1} \in L^1(\Lambda)$, we use the dominated convergence theorem to send to the limit as $\rho \rightarrow R-$ in (6.15) and use that $\nu = -e_n$ on $\partial \mathcal{D} \cap \{x_n = R\}$ to obtain (6.13).

Thus, we conclude a contradiction with (6.12), which completes the proof. $\square$

From (2.14)–(2.17), there exist $r > 0$ and $C > 0$ with the following properties: For any $x_0 \in \partial_l \Omega \cap \{x_n > -1/2\}$, there exist an orthonormal coordinate system $(y_1, \dots, y_n)$ with $y_n = x_n$ and a function $\zeta : \mathbf{R}^{n-1} \rightarrow \mathbf{R}$ such that

$$\Omega \cap B_{10r}(x_0) = \{y_1 > \zeta(y_2, \dots, y_n)\} \cap B_{10r}(x_0), \quad \|\zeta\|_{1,\alpha,\mathbf{R}^{n-1}} \leq C, \quad D\zeta(x_0) = 0. \quad (6.16)$$

Proposition 6.5. *Let $u(x)$ be as in Lemma 6.3. If σ is sufficiently small, depending only on the data and M , then, for any $x^0 = (x_1^0, \dots, x_n^0) \in \overline{\Omega_{R-10r}^+}$ with $L+1 \leq x_n^0 < R-10r$,*

$$\|Du\|_{0,\alpha,B_r(x_0) \cap \Omega_R^+} \leq C \|Du\|_{L^2(B_{4r}(x_0) \cap \Omega_R^+)}. \quad (6.17)$$

Proof. We use the fact that $u - K$ also satisfies equation (6.1) and the conormal boundary conditions (6.2)–(6.3) for any constant $K \in \mathbf{R}$.

When x_0 satisfies $B_{2r}(x_0) \subset \Omega_R^+$, we apply [26, Theorem 3.13] to $u - K$ and obtain

$$\|Du\|_{0,\alpha,B_r(x_0)\cap\Omega_R^+} \leq C\|u - K\|_{L^2(B_{4r}(x_0)\cap\Omega_R^+)}. \quad (6.18)$$

When $x_0 \in \partial_l\Omega^+$ with $L + 1 \leq x_n^0 < R - 10r$, then, since the right-hand side of (6.3) vanishes on $\partial_l\Omega^+ \cap \{x_n > L\}$, we use (5.24) and (a linear version of) the estimates for the conormal derivative problem [35], applied to $u - K$, to obtain

$$\|Du\|_{0,\alpha,B_r(x_0)\cap\Omega_R^+} \leq C\|u - K\|_{C^0(B_{2r}(x_0)\cap\Omega_R^+)}. \quad (6.19)$$

Now we use the standard estimates [20, Theorem 8.17] for the equations of divergence form (extended to the case of local estimates near the boundary for the conormal problem, see e.g. [37, Chapter 4, Section 10] where these modifications are indicated in the parabolic case; note that the Lipschitz regularity of the boundary in our case allows to use the weak form of the conormal problem) to obtain

$$\|u - K\|_{C^0(B_{2r}(x_0)\cap\Omega_R^+)} \leq C\|u - K\|_{L^2(B_{4r}(x_0)\cap\Omega_R^+)}.$$

That is, (6.18) is satisfied for the case $x_0 \in \partial_l\Omega_R^+$.

Now choosing

$$K = \frac{1}{|B_{4r}(x_0) \cap \Omega_R^+|} \int_{B_{4r}(x_0) \cap \Omega_R^+} u \, dx,$$

using the Poincaré inequality, and noting that the constant in the Poincaré inequality can be chosen independent of $x_0 \in \Omega_R^+$ by (6.16), we complete the proof. $\square$

Proposition 6.6. *Let σ and $u(x)$ be as in Lemma 6.3. Then*

$$\|u\|_{C^0(\overline{\Omega_R^+})} \leq C\sigma.$$

Proof. Lemma 6.3 and Proposition 6.5 imply that

$$|Du| \leq C\sigma \quad \text{on } \Omega_R^+ \cap \{L + 1 < x_n < R - 10r\}.$$

Combining this with Lemma 6.4, we have

$$|u| \leq C\sigma \quad \text{on } \Omega_R^+ \cap \{x_n \in (L + 1, R - 10r)\}.$$

To extend this bound to the domains $\Omega_R^+ \cap \{R - 10r < x_n < R\} = \Lambda \times (R - 10r, R)$ and Ω_{L+1}^+ , we note that these domains are of the fixed size and structure (the cylinder $\Lambda \times (a, b)$ and its small perturbation, respectively). Note that, on the boundary parts $\{x_n = L + 1\} \subset \partial\Omega_{L+1}^+$ and $\{x_n = R - 10r\} \subset \partial(\Lambda \times (R - 10r, R))$, we have $|u| \leq C\sigma$, as we proved above. Moreover, the right-hand sides of (6.2)–(6.3) are estimated in L^∞ by $C\sigma$ from (2.16) and (6.5), respectively. Thus we can use the standard estimates [20, Theorem 8.15] for the equations of divergence form (extended to the case when we have the conormal boundary conditions on a part of the boundary, see e.g. [37, Chapter 4, Section 10]). $\square$

Proposition 6.7. *If σ is sufficiently small depending only on the data and M , there exists a unique weak solution $u \in H^1(\Omega_R^+)$ of (6.1)–(6.4). Furthermore, $u \in C^{1,\alpha}(\overline{\Omega_{R-10r}^+})$ satisfies (6.9) and*

$$\|u\|_{1,\alpha,\Omega_{R-10r}^+} \leq C\sigma. \quad (6.20)$$

Proof. Since the map Ψ in (2.14) satisfies (2.16), then, if σ is sufficiently small, there exists the inverse map $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ of Ψ . The map Φ satisfies

$$\|\Phi - Id\|_{2,\alpha,\mathbf{R}^n} \leq C\sigma. \quad (6.21)$$

Then, from (2.14),

$$\Phi(\partial_l\Omega) \subset \partial\Lambda \times \mathbf{R}. \quad (6.22)$$

Note that, if σ is small, then, by (2.16), (4.5), and (5.4), we conclude that the set $\Phi(\Omega^+(\psi)) = \{\psi \circ \Psi < \varphi^- \circ \Psi\} \cap (\Lambda \times \mathbf{R})$ has the form

$$\Phi(\Omega^+(\psi)) = \{x_n > \tilde{f}(x')\} \cap (\Lambda \times \mathbf{R}) \quad \text{with} \quad \|\tilde{f}\|_{1,\alpha,\mathbf{R}^{n-1}} \leq CM\sigma. \quad (6.23)$$

Now we consider a map

$$\tilde{\Phi} = \Phi^1 \circ \Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n \quad \text{for} \quad \Phi^1(x', x_n) = (x', x_n - \eta(x_n)\tilde{f}(x')), \quad (6.24)$$

where $\eta \in C^\infty(\mathbf{R})$ is nonincreasing and satisfies $\eta \equiv 1$ on $(-\infty, 1/8)$ and $\eta \equiv 0$ on $(1/4, \infty)$. From this definition, we have

$$\|\tilde{\Phi} - Id\|_{1,\alpha,\mathbf{R}^n} \leq CM\sigma.$$

Thus, for small σ , the map $\tilde{\Phi}$ is invertible and

$$\|\tilde{\Phi}^{-1} - Id\|_{1,\alpha,\mathbf{R}^n} \leq CM\sigma.$$

In particular, for small σ ,

$$\|\tilde{\Phi} - Id\|_{1,\alpha,\mathbf{R}^n} + \|\tilde{\Phi}^{-1} - Id\|_{1,\alpha,\mathbf{R}^n} \leq 1/10. \quad (6.25)$$

Moreover, if σ is small, then

$$\Lambda \times (0, \rho) = \tilde{\Phi}(\Omega_R^+ \cap \{x_n < \rho\}) \quad \text{for} \quad \rho > L/4. \quad (6.26)$$

Define the function $v(y) := u(x) = u(\tilde{\Phi}^{-1}(y))$ for $y \in \mathcal{C}_R := \Lambda \times (0, R)$. Then u is a weak solution of the conormal problem (6.1)–(6.4) if and only if $v \in H^1(\mathcal{C}_R)$ is a weak solution of the corresponding problem of the form (3.1)–(3.3) in $\mathcal{C}_R$. To obtain the expressions of the coefficients a_{ij} and the right-hand sides f, g_1 , and g_2 of problem (3.1)–(3.2) for v , in terms of the data of problem (6.1)–(6.3) and the map $\tilde{\Phi}$, we make the change of variables $y = \tilde{\Phi}(x)$ in (6.8). Then we have

$$\|a_{ij} - \kappa_i \delta_i^j\|_{0,\alpha,\mathcal{C}_R} \leq CM\sigma \quad \text{for} \quad i, j = 1, \dots, n, \quad (6.27)$$

with κ_i defined by (5.23), where we used (5.24) and (6.25). In this case, the right-hand side in (3.1) is $f = 0$; to estimate g_1 and g_2 in (3.2), we use (2.16), (6.5), and (6.25) to obtain

$$\|g_k\|_{0,\alpha,\Sigma_k} \leq C\sigma, \quad k = 1, 2. \quad (6.28)$$

Also, from (6.4) and (6.26), we see that $h = 0$ in (3.3) for v .

Now the existence a solution $v(y)$ follows from Proposition 3.2 if $\sigma > 0$ is sufficiently small depending only on the data and L , so that the right-hand side of (6.27) is smaller than κ in Proposition 3.2.

Thus $u(x) := v(\tilde{\Phi}(x))$ is a weak solution of (6.1)–(6.4).

Since a_{ij} satisfy ellipticity condition, uniqueness of weak solution u of problem (6.1)–(6.4) follows from the uniqueness argument in the proof of Theorem 3.2.

Estimate (6.9) for $u(x)$ follows from Lemma 6.3.

It remains to prove (6.20). For any $x \in \overline{\Omega_R^+} \cap \{L/2 \leq x_n \leq R - 10r\}$, the standard interior estimates [20, Theorem 8.32] and the estimates for oblique derivative problems [35] imply

$$\|u\|_{1,\alpha,B_r(x_0) \cap \Omega_R^+} \leq C(\|u\|_{C^0(B_{4r}(x_0) \cap \Omega_R^+)} + \|\tilde{\rho}((q^+)^2)q^+ \nu \cdot e_n\|_{0,\alpha,B_r(x_0) \cap \partial\Omega_R^+}) \leq C\sigma, \quad (6.29)$$

where we used (2.16) and Proposition 6.6. Thus it suffices to obtain an estimate in $\Omega_R^+ \cap \{x_n \leq L\}$. By (6.25)–(6.26), it suffices to show that

$$\|v\|_{1,\alpha,\Lambda \times (0,L)} \leq C\sigma, \quad (6.30)$$

since the similar estimate then holds for $\|u\|_{1,\alpha,\Omega_R^+ \cap \{x_n \leq L\}}$.

To show (6.30), we note that using (6.25)–(6.26) and Proposition 6.6 yields

$$\|v\|_{L^2(\Lambda \times (0,2L))} \leq 2\|u\|_{L^2(\Omega_R^+ \cap \{x_n < 2L\})} \leq C\sigma.$$

Now estimate (3.8), applied to v in the cylinder $\Lambda \times (0, 2L)$, with the use of (6.28), implies (6.30), if σ is small depending only on M and the data. $\square$

7. EXISTENCE OF SOLUTIONS OF THE FREE BOUNDARY PROBLEM

We now show the existence of solutions of the free boundary problem. In this section, C denotes a universal constant, which may be different at each occurrence and depends only on the data.

7.1. Solutions of the fixed boundary problem in the infinite nozzle. We first have

Proposition 7.1. *If σ is sufficiently small depending only on the data and M , then there exists a unique weak solution $u \in C^{1,\alpha}(\overline{\Omega^+}(\psi))$ of (5.25)–(5.28). This solution satisfies*

$$\|Du\|_{L^2(\Omega^+(\psi))} \leq C\sigma, \quad (7.1)$$

$$\|u\|_{1,\alpha,\Omega^+(\psi)} \leq C\sigma, \quad (7.2)$$

$$\lim_{\rho \rightarrow \infty} \|u\|_{1,\alpha,\Omega \cap \{x_n > \rho\}} = 0. \quad (7.3)$$

Moreover, we have

- (i) $u = \lim_{R \rightarrow \infty} u_R$, where u_R is the unique solution of (6.1)–(6.4) for each $R > 0$ and the convergence is in $C^{1,\beta}$ on any compact subset of $\overline{\Omega^+}(\psi)$ for any $0 \leq \beta < \alpha$;
- (ii) For every $R \geq L$, there exists $x' \in \overline{\Lambda}$ such that $u(x', R) = 0$.

Proof. We assume that σ is sufficiently small to satisfy the conditions stated in Sections 5–6.

We first show that any weak solution $u \in C^{1,\alpha}(\overline{\Omega^+}(\psi))$ of (5.25)–(5.28) satisfies (7.3). From (5.28), for any $\varepsilon > 0$, there exists $\rho > L + 2r$ such that

$$\|u\|_{C(\Omega \cap \{x_n > \rho\})} \leq \varepsilon.$$

Note that (2.17) implies that the right-hand side of (5.27) vanishes on $\partial_t \Omega^+(\psi) \cap \{x_n > \rho\}$. Then the standard estimates [20, Theorem 6.2, Lemma 6.29] imply that, for $x = (x', x_n)$ with $x_n \geq \rho > L + 2r$,

$$\|u\|_{1,\alpha,B_r(x) \cap \Omega} \leq C\|u\|_{0,0,B_{2r}(x) \cap \Omega} \leq C\varepsilon.$$

Thus, (7.3) follows.

Now we prove the uniqueness of weak solutions of (5.25)–(5.28) in $C^{1,\alpha}(\overline{\Omega^+}(\psi))$. Let $u, v \in C^{1,\alpha}(\overline{\Omega^+}(\psi))$ be two solutions of (5.25)–(5.28). Then, for any $R > L + \text{diam}(\Lambda)$, define

$$\zeta(x) := \eta_R(x)(u(x) - v(x)),$$

where $\eta_R(x) = \eta(|x|/R)$ and $\eta \in C_c^\infty(\overline{\mathbf{R}_+})$ is a nonnegative function such that $\eta \equiv 1$ on $[0, 1]$. Then $\zeta \in C_c^1(\overline{\Omega^+})$. Using ζ as a test function in (5.29) for u and v respectively and subtracting them, we have

$$0 = \int_{\Omega^+} \sum_{i,j=1}^n a_{ij}^{(\psi)} ((u_{x_i} - v_{x_i})(u_{x_j} - v_{x_j})\eta_R + (u - v)(u_{x_j} - v_{x_j})\partial_{x_i}\eta_R) dx.$$

Thus,

$$\int_{\Omega^+} \sum_{i,j=1}^n a_{ij}^{(\psi)} (u_{x_i} - v_{x_i})(u_{x_j} - v_{x_j})\eta_R dx \leq \frac{C}{R} \int_{\Omega^+ \cap (B_{2R}(0) \setminus B_R(0))} (|Du| + |Dv|)(|u| + |v|) dx.$$

Note that $|\Omega^+ \cap (B_{2R}(0) \setminus B_R(0))| \leq 2R|\Lambda|$. Thus, using the ellipticity of $a_{ij}^{(\psi)}$, we find

$$\lambda \int_{\Omega^+ \cap B_R(0)} |Du - Dv|^2 dx \leq C(\|u\|_{1,0,\Omega \setminus B_R(0)} + \|v\|_{1,0,\Omega \setminus B_R(0)})$$

for large R . Now we send $R \rightarrow \infty$ and use (7.3) to obtain

$$\lambda \int_{\Omega_R^+} |Du - Dv|^2 dx = 0.$$

Thus, $u - v = \text{const.}$, which yields $u = v$ by using (7.3).

It remains to prove the existence of a weak solution of (5.25)–(5.28) satisfying (7.1)–(7.2) and properties (i)–(ii) of Proposition 7.1.

The existence of a unique solution $u_R \in C^{1,\alpha}(\overline{\Omega_{R-10r}^+}) \cap H^1(\Omega_R^+)$ of (6.1)–(6.4) for any $R > L$ follows from Proposition 6.7.

Now, let $R_j \rightarrow \infty$. Using estimates (6.9) and (6.20) for u_R , which hold by Proposition 6.7, we can extract from $\{u_{R_j}\}$ a subsequence converging in $C^{1,\alpha/2}$ on compact subsets of $\overline{\Omega}$ to a function $u \in C^{1,\alpha}(\overline{\Omega})$ that satisfies (7.1). Indeed, this can be achieved by extracting a subsequence converging in $C^{1,\alpha/2}(\Omega_{L+1}^+)$, a further subsequence converging in $C^{1,\alpha/2}(\Omega_{L+2}^+)$, etc., and by using the diagonal procedure. Obviously, u is a weak solution of (5.25)–(5.27), since we can pass to the limit in (5.29) for any fixed $w \in C_c^1(\overline{\Omega^+})$.

Moreover, this solution u satisfies property (ii): Indeed, for fixed $R_0 \geq L$, $R_j \geq R_0$ for sufficiently large j . For such j , by Lemma 6.4, there exists $x'_j \in \overline{\Lambda}$ such that $u_{R_j}(x'_j, R_0) = 0$. Then there exists a subsequence of $\{x'_j\}$ converging to some point $x' \in \overline{\Lambda}$. Since $u_{R_j}(\cdot, R_0) \rightarrow u(\cdot, R_0)$ uniformly in $\overline{\Lambda}$, we have $u(x', R_0) = 0$. Thus, the solution u satisfies properties (i)–(ii).

Also, since each u_R satisfies (6.9) and (6.20) with C independent of R , then (7.1)–(7.2) hold.

Now we prove that u satisfies (5.28). From (7.1), for any $\varepsilon > 0$, there exists $\rho > L$ such that

$$\|Du\|_{L^2(\Omega^+ \cap \{x_n > \rho\})} \leq \varepsilon.$$

Then, by Proposition 6.5 (which holds for u as u_R),

$$\|Du\|_{0,\alpha,\Omega^+ \cap \{x_n > \rho+1\}} \leq C\varepsilon.$$

From this estimate and property (ii) proved above, we have

$$\|u\|_{C^0(\Omega^+ \cap \{x_n > \rho+1\})} \leq C\varepsilon.$$

Now (5.28) is proved. $\square$

7.2. Iteration map. We first define an extension operator $\mathcal{P}_\psi$, similar to [4, Proposition 4.5] and [5, Proposition 5.1], to obtain

Proposition 7.2. *Let $\sigma > 0$ be sufficiently small depending only on the data and M , and φ^- satisfy (4.5). Then, for any $\psi \in \mathcal{K}_M$, there exists an extension operator $\mathcal{P}_\psi : C^{1,\alpha}(\overline{\Omega^+(\psi)}) \rightarrow C^{1,\alpha}(\overline{\Omega})$ satisfying the following two properties:*

- (i) *There exists C depending only on n, γ, q_0^+ , and α (but independent of M, R, σ , and ψ) such that, for the solution $u \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ of problem (5.25)–(5.28) and $\varphi = u + q^+x_n$,*

$$\|\mathcal{P}_\psi \varphi - q^+x_n\|_{1,\alpha,\Omega} \leq C\sigma; \quad (7.4)$$

- (ii) *Let $\beta \in (0, \alpha)$. If a sequence $\psi_k \in \mathcal{K}_M$ converges to $\psi \in C^{1,\beta}(\overline{\Omega})$ in the norm $\|\cdot\|_{1,\beta,\Omega}^{(-1)}$ defined in (5.2), then $\psi \in \mathcal{K}_M$; Furthermore, if $u_k \in C^{1,\alpha}(\overline{\Omega^+(\psi_k)})$ and $u \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$ are the solutions of problems (5.25)–(5.28) for ψ_k and ψ , respectively, and if $\varphi_k = u_k + q^+x_n$ and $\varphi = u + q^+x_n$, then $\mathcal{P}_{\psi_k} \varphi_k \rightarrow \mathcal{P}_\psi \varphi$ in the norm $\|\cdot\|_{1,\beta,\Omega}^{(-1)}$.*

Proof. Let $\psi \in \mathcal{K}_M$. We first define an appropriate extension operator $\mathcal{P}_\psi : C^{1,\alpha}(\overline{\Omega^+(\psi)}) \rightarrow C^{1,\alpha}(\overline{\Omega})$.

Let $\varphi \in C^{1,\alpha}(\overline{\Omega^+(\psi_j)})$ and $u = \varphi - q^+x_n$. Then $v := u \circ \tilde{\Phi}^{-1}$ satisfies $v \in C^{1,\alpha}(\overline{\mathcal{C}_{(0,\infty)}})$ with $\mathcal{C}_{(a,b)} = \Lambda \times (a, b)$, where $\tilde{\Phi}$ is defined by (6.24).

We first define the extension operator $\mathcal{E}_1 : C^{1,\beta}(\overline{\mathcal{C}_{(0,\infty)}}) \rightarrow C^{1,\beta}(\overline{\mathcal{C}_{(-2,\infty)}})$ for any $\beta \in (0, 1)$. Let $v \in C^{1,\beta}(\overline{\mathcal{C}_{(0,\infty)}})$. Define $\mathcal{E}_1 v = v$ in $\mathcal{C}_{(0,\infty)}$. For $(x', x_n) \in \mathcal{C}_{(-1,0)}$, define

$$\mathcal{E}_1 v(x', x_n) = \sum_{i=1}^2 c_i v(x', -\frac{x_n}{i}),$$

where $c_1 = -3$ and $c_2 = 4$, which are determined by $\sum_{i=1}^2 c_i \left(-\frac{1}{i}\right)^m = 1$ for $m = 0, 1$. Note that, for sufficiently small σ , we have $\Omega \subset \tilde{\Phi}(\mathcal{C}_{(-2,\infty)})$ by (6.25). Thus, $(\mathcal{E}_1 v) \circ \tilde{\Phi} \in C^{1,\alpha}(\bar{\Omega})$.

Finally, we define

$$\mathcal{P}_\psi(\varphi)(x) = (\mathcal{E}_1 v) \circ \tilde{\Phi}(x) + q^+ x_n.$$

Obviously, $\mathcal{P}_\psi(\varphi) \in C^{1,\alpha}(\bar{\Omega})$. Also, since $\tilde{\Phi}(\Omega^+(\psi)) = \mathcal{C}_{(0,\infty)}$, we have

$$\mathcal{P}_\psi(\varphi)(x) = v \circ \tilde{\Phi}(x) + q^+ x_n = u(x) + q^+ x_n = \varphi(x) \quad \text{for } x \in \Omega^+(\psi),$$

i.e., $\mathcal{P}_\psi : C^{1,\alpha}(\overline{\Omega^+(\psi)}) \rightarrow C^{1,\alpha}(\bar{\Omega})$ is an extension operator.

Now, using (6.25), we obtain the following estimate as in the proof of [4, Proposition 4.5]:

$$\|\mathcal{P}_\psi \varphi - q^+ x_n\|_{1,\alpha,\Omega} \leq C \|\varphi - q^+ x_n\|_{1,\alpha,\Omega^+(\psi)}. \quad (7.5)$$

Then, if u and φ are those defined in assertion (i), we have (7.2) for $u = \varphi - q^+ x_n$ by Proposition 7.1, and thus (7.5) implies (7.4). Assertion (i) is proved.

Now we prove assertion (ii). First, by Lemma 5.1, $\psi \in \mathcal{K}_M$.

By (7.4) applied to ψ_j and by Lemma 5.1, there exists a subsequence (still denoted by) $\varphi_j(x)$ such that

$$\mathcal{P}_{\psi_j} \varphi_j \rightarrow \tilde{\varphi} \quad \text{in } C_{(-1)}^{1,\beta}(\bar{\Omega}) \quad (7.6)$$

for some $\tilde{\varphi} \in C^{1,\alpha}(\bar{\Omega})$.

Denote $\tilde{\psi}_k := \psi_k \circ \Psi$ and $\tilde{\psi} := \psi \circ \Psi$. Then, by (2.16), $\tilde{\psi}_k \rightarrow \tilde{\psi}$ in $C^{1,\beta}(\bar{\Lambda} \times [-1, 1])$. Denote by $\tilde{f}_k(x')$ the function from (6.23) for $\psi_k(x)$. Then, by (2.16), (4.5), and (5.4), $\tilde{f}_k \rightarrow \tilde{f}$ in $C^{1,\beta}(\bar{\Lambda})$. Now let $\tilde{\Phi}_k$ and $\tilde{\Phi}$ be the maps from (6.24) for $\tilde{f}_k$ and $\tilde{f}$, respectively. Then $\tilde{\Phi}_k$ and $\tilde{\Phi}$ and their inverse maps satisfy (6.25), and $\tilde{\Phi}_k \rightarrow \tilde{\Phi}$ and $\tilde{\Phi}_k^{-1} \rightarrow \tilde{\Phi}^{-1}$ in $C^{1,\beta}$ on compact subsets of $\mathbf{R}^n$. Note $v_k := u_k \circ \tilde{\Phi}_k^{-1} \in C^{1,\alpha}(\bar{\Lambda} \times [0, \infty))$. Recall that u_k is a weak solution of (5.25)–(5.28) in the domain $\Omega^+(\psi_k)$, with coefficients $a_{ij}^{(\psi_k)}$ defined by (5.22) for ψ_k and corresponding right-hand sides (5.18) and (5.27) for ψ_k . Then, by Proposition 7.1, each u_k satisfies (7.1)–(7.2). Therefore,

$$\|Dv_k\|_{L^2(\Lambda \times (0, \infty))} \leq C\sigma, \quad (7.7)$$

$$\|v_k\|_{1,\alpha,\Lambda \times (0, \infty)} \leq C\sigma \quad \text{for } k = 1, \dots \quad (7.8)$$

Moreover, v_k is a weak solution of a conormal problem

$$\sum_{i,j=1}^n (b_{ij}^{(\psi_k)} \partial_{x_j} v_k)_{x_i} = 0 \quad \text{in } \Lambda \times (0, \infty), \quad (7.9)$$

$$\sum_{i,j=1}^n b_{ij}^{(\psi_k)} \partial_{x_j} v_k \nu_i = \tilde{g}_{\psi_k}(x) \quad \text{on } \Lambda \times \{0\}, \quad (7.10)$$

$$\sum_{i,j=1}^n b_{ij}^{(\psi_k)} \partial_{x_j} v_k \nu_i = h_{\psi_k}(x) \quad \text{on } \partial\Lambda \times (0, \infty), \quad (7.11)$$

where $b_{ij}^{(\psi_k)}$, $\tilde{g}_{\psi_k}$, and h_{ψ_k} are defined by the coefficients and right-hand sides of (5.25)–(5.27) for ψ_k and by $\tilde{\Phi}_k^{-1}$. In particular, since $a_{ij}^{(\psi_k)} \rightarrow a_{ij}^{(\psi)}$ in $C^{1,\beta}$ on compact subsets of $\bar{\Omega}$ as $k \rightarrow \infty$, which follows from (5.22), and since $\tilde{\Phi}_k^{-1} \rightarrow \tilde{\Phi}^{-1}$ in $C^{1,\beta}$ on compact subsets of $\mathbf{R}^n$, it is easy to show that, for any compact $K \subset \bar{\Lambda} \times [0, \infty)$,

$$b_{ij}^{(\psi_k)} \rightarrow b_{ij}^{(\psi)} \quad \text{in } C^{1,\beta}(K) \quad \text{and} \quad (\tilde{g}_{\psi_k}, h_{\psi_k}) \rightarrow (\tilde{g}_\psi, h_\psi) \quad \text{in } C^\beta(K),$$

where $b_{ij}^{(\psi)}$, $\tilde{g}_\psi$, and h_ψ are the coefficients and right-hand sides of (7.9)–(7.11) for ψ . By (7.8), there exists a subsequence v_{k_m} which converges in $C^{1,\beta}$ on the compact subsets of $\bar{\Lambda} \times [0, \infty)$ to

some $v \in C^{1,\beta}(\bar{\Lambda} \times [0, \infty))$. Then, choosing $w \in C_c^1(\mathbf{R}^n)$ in the weak formulation (5.29) for the conormal problem (7.9)–(7.11):

$$\int_{\Lambda \times (0, \infty)} \sum_{i,j=1}^n b_{ij}^{(\psi_k)} v_{x_k} w_{x_i} dx + \int_{\Lambda \times \{0\}} \tilde{g}_{\psi_k} w dx' - \int_{\partial \Lambda \times (0, \infty)} h_{\psi_k} w d\mathcal{H}^{n-1} = 0,$$

we pass to the limit for the subsequence $\{k_m\}$ and conclude that v is a weak solution of problem (7.9)–(7.11) with coefficients $b_{ij}^{(\psi)}$ and right-hand sides $\tilde{g}_\psi$ and h_ψ . Note also that v satisfies (7.7) since each v_{k_m} satisfies (7.7). Transforming back, we conclude that $\tilde{u} := v \circ \tilde{\Phi}$ is a weak solution of problem (5.25)–(5.27) for ψ and satisfies (7.1). Now the proof of Proposition 7.1 implies that $\tilde{u}$ satisfies (5.28). Then the uniqueness assertion of Proposition 7.1 implies $u = \tilde{u}$ on $\Omega^+(\psi)$. Thus,

$$\mathcal{P}_\psi(\varphi) = (\mathcal{E}_1 v) \circ \tilde{\Phi} + q^+ x_n. \quad (7.12)$$

Finally,

$$\mathcal{P}_{\psi_{k_m}}(\varphi_{k_m}) = (\mathcal{E}_1 v_{k_m}) \circ \tilde{\Phi}_{k_m} + q^+ x_n \rightarrow (\mathcal{E}_1 v) \circ \tilde{\Phi} + q^+ x_n \quad \text{in } C^{1,\beta}(K)$$

for any compact $K \subset \bar{\Omega}$, since $(\tilde{\Phi}_k, v_{k_m}) \rightarrow (\tilde{\Phi}, v)$ in $C^{1,\beta}(K)$ and $\mathcal{E}_1 : C^{1,\beta}(\overline{\mathcal{C}_{(0,\infty)}}) \rightarrow C^{1,\beta}(\overline{\mathcal{C}_{(-2,\infty)}})$ is continuous. From this, using (7.6) and (7.12), we obtain that $\mathcal{P}_{\psi_{k_m}}(\varphi_{k_m}) \rightarrow \mathcal{P}_\psi(\varphi)$ in $C^{1,\beta}_{(-1)}(\bar{\Omega})$. Moreover, by the same argument, from any subsequence of $\mathcal{P}_{\psi_k}(\varphi_k)$, we can extract a further subsequence converging in $C^{1,\beta}_{(-1)}(\bar{\Omega})$ to the same limit $\mathcal{P}_\psi(\varphi)$. Thus the whole sequence $\mathcal{P}_{\psi_k}(\varphi_k)$ converges to this limit. Proposition 7.2 is proved. $\square$

Now we can define the iteration map $J : \mathcal{K}_M \rightarrow C^{1,\alpha}(\bar{\Omega})$ by

$$J\psi := \mathcal{P}_\psi \varphi \quad (7.13)$$

with $\varphi(x) = u(x) + q^+ x_n$, where $u \in C^{1,\alpha}(\bar{\Omega}^+(\psi))$ is the unique solution of problem (5.25)–(5.28) with $\psi \in \mathcal{K}_M$. By Proposition 7.2(ii), J is continuous in $C^{1,\beta}(\bar{\Omega})$ for any $\beta \in (0, \alpha)$.

7.3. Existence of solutions of the free boundary problem. Now we can prove the existence of solutions of the free boundary problem. We denote by φ both the function φ in $\Omega^+(\psi)$ and its extension $\mathcal{P}_\psi \varphi$.

Choose M to be the constant C in (7.4). Then, for $\psi \in \mathcal{K}_M$, we have from Proposition 7.2(i) that $\varphi := J\psi \in \mathcal{K}_M$ if $\sigma > 0$ is sufficiently small and depends only on n, γ, q^+ , and Ω , since $M > 0$ is now fixed. Thus, (7.13) defines the iteration map $J : \mathcal{K}_M \rightarrow \mathcal{K}_M$ and, from Proposition 7.2(ii), J is continuous on $\mathcal{K}_M$ in $C^{1,\alpha/2}_{(-1)}(\bar{\Omega})$.

In order to find a solution of Problem (TFB), we seek a fixed point of the map J . We use the Schauder fixed point theorem (cf. [20, Theorem 11.1]) in the following setting:

Let $\sigma > 0$ satisfy the conditions of Proposition 7.2. By Lemma 5.1, the set $\mathcal{K}_M$ is a compact convex subset of $C^{1,\alpha/2}_{(-1)}(\bar{\Omega})$. We have shown that $J(\mathcal{K}_M) \subset \mathcal{K}_M$ and J is continuous in $C^{1,\alpha/2}_{(-1)}(\bar{\Omega})$. Then, by the Schauder fixed point theorem, J has a fixed point $\varphi \in \mathcal{K}_M$.

If φ is such a fixed point, then

$$\tilde{\varphi}(x) := \min(\varphi^-(x), \varphi(x))$$

is a classical solution of Problem (TFB) and S_φ is its free boundary.

It follows that $\tilde{\varphi}$ is a solution of Problem (FB), thus Problem (TN), provided that σ is small enough so that (7.4) implies that $|D\varphi(x)| < c_* - \varepsilon$, where $\varepsilon = (c_* - q_0^+)/2$. Indeed, then (4.1) implies that φ lies in the untruncated region for equation (4.4). Also, $|D\tilde{\varphi}| < c_* - \varepsilon$ on $\Omega^+(\tilde{\varphi}) := \{\tilde{\varphi}(x) < \varphi^-(x)\}$ and $|D\tilde{\varphi}| > c_*$ on $\Omega \setminus \Omega^+$ since $\tilde{\varphi} = \varphi$ on $\Omega^+(\tilde{\varphi})$ and $\tilde{\varphi} = \varphi^-$ on $\Omega \setminus \Omega^+$.

8. UNIQUENESS AND STABILITY

We now prove the uniqueness of the velocity at infinity in Problem (TN) and, when $\varphi^- \in C^{2,\alpha}$, prove the uniqueness and stability of solutions of Problem (TN). We first prove the following lemma.

Lemma 8.1. *Let φ^- satisfy (4.5). Let $\varphi \in C^{0,1}(\Omega)$ be a solution of Problem (TN) with $\omega \in (0, c_*)$ satisfying (2.24). Then there exists $\sigma_0 > 0$ depending only on n, α, γ , and q_0^+ such that, if $\sigma \in (0, \sigma_0)$, φ satisfies (2.23), (2.30), and*

$$\|D\varphi - \omega e_n\|_{C^0(\Omega \cap \{x_n > R\})} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (8.1)$$

Proof. The proof that φ satisfies (2.23) and (2.30) is the same as in [5, Lemma 7.1].

Now we prove (8.1). Setting $w(x) := \varphi(x) - \omega x_n$, noting that $\nu \cdot e_n = 0$ on $\partial_l \Omega^+ \cap \{x_n > L\}$ by (2.17), and using (2.23) and (2.30), we see that w satisfies

$$\begin{aligned} \sum_{i,j=1}^n (a_{ij} w_{x_j})_{x_i} &= 0 & \text{in } \Omega \cap \{x_n > L\}, \\ \sum_{i,j=1}^n a_{ij} w_{x_j} \nu_i &= 0 & \text{on } \partial_l \Omega^+ \cap \{x_n > L\}, \\ \lim_{R \rightarrow \infty} \|w\|_{C(\Omega \cap \{x_n > R\})} &= 0, \end{aligned}$$

where

$$\begin{aligned} a_{ij}(x) &= \int_0^1 \left\{ \tilde{\rho} (|tD\varphi(x) + (1-t)\omega e_n|^2) \delta_i^j \right. \\ &\quad \left. + 2\tilde{\rho}' (|tD\varphi(x) + (1-t)\omega e_n|^2) (t\varphi_{x_i}(x) + (1-t)\omega \delta_i^n) (t\varphi_{x_j}(x) + (1-t)\omega \delta_j^n) \right\} dt. \end{aligned}$$

By (2.24), the equation is elliptic if σ is small. Then the proof of (8.1) is similar to the proof of (7.3) in Proposition 7.1. $\square$

Now we show that ω in Problem (TN) is uniquely determined by φ^- .

Proposition 8.2. *There exists σ_0 depending only on the data as in Lemma 8.1 such that, if φ^- satisfies (4.5) and $\varphi \in C^{0,1}(\Omega)$ is a solution of Problem (TN) with $\omega \in (0, c_*)$ satisfying (2.24), then $\omega = q^+$ in (2.20), with q^+ defined by (2.26) and $\varphi(x)$ satisfies (2.28).*

This can be achieved by following the proof of [5, Proposition 7.2] and using Lemma 8.1 instead of [5, Lemma 7.1].

It remains to prove the uniqueness of solutions of Problem (TN) satisfying (2.24) when φ^- satisfies (2.32).

Proposition 8.3. *There exists σ_0 depending only on the data as in Lemma 8.1 such that, if φ^- satisfies (2.32), then the solution $\varphi \in C^{0,1}(\Omega)$ of Problem (TN) with $\omega = q^+$ satisfying (2.24) is unique.*

The rest of this section is to prove Proposition 8.3 by using a version of the partial hodograph transform developed in [6].

8.1. Extension of φ^- to the domain Ω . Since φ^- satisfies (2.19) and (2.32) in the domain $\Omega_1 := \Omega \cap \{x_n < 1\}$, then we can extend φ^- to Ω so that the extension (still denoted by) φ^- is in $C^{2,\alpha}(\Omega)$, coincides with the original φ^- in $\Omega \cap \{x_n < 7/8\}$, and

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega} \leq C(n, \alpha)\sigma, \quad (8.2)$$

$$\text{supp}(\varphi^- - \varphi_0^-) \subset \Omega \cap \{x_n < L\}, \quad (8.3)$$

$$\partial_\nu \varphi^- = 0 \quad \text{on } \partial_l \Omega, \quad (8.4)$$

where $\varphi_0^-(x) = q_0^- x_n$. To construct such an extension, we first note that we can modify the mapping Ψ and replace it by a mapping $\Psi_1 \in C^{2,\alpha}(\mathbf{R}^n, \mathbf{R}^n)$ such that Ψ_1 satisfies (2.14), $\|\Psi_1 - Id\|_{2,\alpha,\mathbf{R}^n} \leq C\sigma$, (2.17), and

$$D\Psi_1(x)\nu_{\partial\mathcal{C}}(x) = \zeta(x)\nu_{\partial_l\Omega}(\Psi_1(x)) \quad \text{for any } x \in \partial\mathcal{C},$$

where $\mathcal{C} = \Lambda \times (-\infty, \infty)$, $\nu_{\partial\mathcal{C}}(\cdot)$ and $\nu_{\partial_l\Omega}(\cdot)$ are the inward unit normals to $\partial\mathcal{C}$ and $\partial_l\Omega$ respectively, and $\|\zeta - 1\|_{2,\alpha,\partial\mathcal{C}} \leq C\sigma$. To construct such Ψ_1 , we first define the corresponding $C^{2,\alpha}$ mapping $\tilde{\Psi}_1$ between sufficiently small neighborhoods (depending only on Λ and σ) of $\partial\mathcal{C}$ and $\partial_l\Omega$ by

$$\tilde{\Psi}_1(z + s\nu_{\partial\mathcal{C}}(z)) = \Psi(z) + s\nu_{\partial_l\Omega}(\Psi(z))$$

for $z \in \partial\mathcal{C}$ and $s \in (0, \varepsilon)$, where $\varepsilon > 0$ depends only on Λ and L . Then we let

$$\Psi_1(x) = \eta_\varepsilon(d_{\partial_l\Omega}(x))\tilde{\Psi}_1(x) + (1 - \eta_\varepsilon(d_{\partial_l\Omega}(x)))\Psi(x),$$

where $d_{\partial_l\Omega}(\cdot)$ is the distance function to $\partial_l\Omega$ and $\eta_\varepsilon : \mathbf{R} \rightarrow \mathbf{R}$ is a smooth function satisfying $0 \leq \eta_\varepsilon \leq 1$ on $\mathbf{R}$, $\eta_\varepsilon \equiv 1$ on $[-\infty, 7\varepsilon/8)$, and $\eta_\varepsilon \equiv 0$ on (ε, ∞) . Now we define $\tilde{\varphi}^-(x) = \varphi^-(\Psi_1(x))$ on $\Psi_1^{-1}(\Omega_1)$. Then, if σ is small, $\tilde{\varphi}^-$ is defined at least on $\Lambda \times (-7/8, 7/8)$ and satisfies $D\tilde{\varphi}^- \cdot \nu_{\mathcal{C}} = 0$ on $\partial\Lambda \times (-7/8, 7/8)$. Now we define

$$\hat{\varphi}^-(x) = \eta_1(x_n)\tilde{\varphi}^-(x) + (1 - \eta_1(x_n))\varphi_0^-(x) \quad \text{on } \Lambda \times (-7/8, \infty),$$

and then the function $\varphi^-(x) = \hat{\varphi}^-(\Psi_1^{-1}(x))$ is a $C^{2,\alpha}$ extension of φ^- satisfying (8.2)–(8.4).

Denote

$$g = \operatorname{div}(\rho(|D\varphi^-|^2)D\varphi^-) \quad \text{in } \Omega. \quad (8.5)$$

Since φ^- and φ_0^- satisfy (1.1) in Ω_1 and Ω , respectively, then we find from (8.2)–(8.3) that g satisfies

$$\begin{aligned} g &\in C^\alpha(\bar{\Omega}), \quad \|g\|_{0,\alpha,\Omega} \leq C\sigma, \\ g &\equiv 0 \quad \text{on } (\Omega \cap \{x_n < 7/8\}) \cup (\Omega \cap \{x_n > 1\}). \end{aligned} \quad (8.6)$$

From now on, we use the extended function $\varphi^- = \varphi^-(x)$ and C may denote a different constant at each occurrence, depending only on the data, unless otherwise is specified.

8.2. Partial hodograph transform. Let φ be a solution of Problem (TN) with $\omega = q^+$ and satisfy (2.24). Then $\Omega^+ := \{|D\varphi| < c_*\}$ has the form (2.23) and (2.30). Define a function u in Ω^+ by

$$u(x) := \varphi^-(x) - \varphi(x).$$

Then (2.31) and (8.2) imply

$$\|u - (q_0^- - q^+)x_n\|_{1,\alpha,\Omega^+} \leq C\sigma. \quad (8.7)$$

In particular, if σ is sufficiently small, we use (2.29) to obtain

$$0 < (q_0^- - q_0^+)/2 \leq u_{x_n}(x) \leq 2(q_0^- - q_0^+) \quad \text{for any } x \in \Omega^+. \quad (8.8)$$

Furthermore, (2.28) and (8.3) imply

$$\|u - (q_0^- - q^+)x_n\|_{C^1(\Omega \cap \{x_n > R\})} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (8.9)$$

Since φ is a solution of Problem (TN), φ satisfies (1.1) in Ω^+ and (2.4) on S . Then, since $S \subset \Omega_1$ and φ^- satisfies (8.4) on $\partial_l\Omega$, and also using (8.5), we see that u is a solution of the following problem:

$$\begin{aligned} \operatorname{div}(A(x, Du)) &= -g && \text{in } \Omega^+, \\ A(x, Du) \cdot \nu &= 0 && \text{on } S, \\ A(x, Du) \cdot \nu &= 0 && \text{on } \partial_l\Omega^+, \end{aligned} \quad (8.10)$$

where, for $x \in \Omega^+$ and $P \in \mathbf{R}^n$,

$$A(x, P) = \rho(|D\varphi^-(x) - P|^2)(D\varphi^-(x) - P) - \rho(|D\varphi^-(x)|^2)D\varphi^-(x). \quad (8.11)$$

Note that u satisfies the weak form of problem (8.10):

$$\int_{\Omega^+} (A(x, Du) \cdot D\eta - g\eta) dx = 0 \quad \text{for any } \eta \in C_0^1(\mathbf{R}^n). \quad (8.12)$$

Since $\varphi = \varphi^-$ on S , it follows that

$$u = 0 \quad \text{on } S. \quad (8.13)$$

Now we make a change of variables $z = \Phi(x)$ to rewrite problem (8.12) into a cylindric domain $\mathcal{C}^+ := \Phi(\Omega^+)$, where Ψ and its inverse Φ are the maps in (2.14), (2.16), and (6.21). It follows that, if σ is sufficiently small, then

$$\mathcal{C}^+ = (\Lambda \times \mathbf{R}) \cap \{z_n > \tilde{f}(z')\}, \quad \|\tilde{f}\|_{1,\alpha,\mathbf{R}^{n-1}} \leq CM\sigma, \quad (8.14)$$

and $\partial\mathcal{C}^+ = \overline{\partial_t\mathcal{C}^+} \cup \tilde{S}$, where

$$\begin{aligned} \partial_t\mathcal{C}^+ &:= (\partial\Lambda \times \mathbf{R}) \cap \{z_n > \tilde{f}(z')\} = \Phi(\partial_t\Omega^+), \\ \tilde{S} &:= (\Lambda \times \mathbf{R}) \cap \{z_n = \tilde{f}(z')\} = \Phi(S). \end{aligned}$$

Also, $\Omega^+ = \Psi(\mathcal{C}^+)$. Consider the function

$$\tilde{u}(z) = u(\Psi(z)) \quad \text{for } z \in \mathcal{C}^+.$$

Then (2.16)–(2.17) and (8.7) imply

$$\|\tilde{u} - (q_0^- - q^+)x_n\|_{1,\alpha,\mathcal{C}^+} \leq C\sigma. \quad (8.15)$$

In particular, $\tilde{u}$ satisfies (8.8) if σ is small. By (8.9) and (2.17),

$$\|\tilde{u} - (q_0^- - q^+)x_n\|_{C^1(\Lambda \times (R, \infty))} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (8.16)$$

From (8.13), $\tilde{u}$ satisfies

$$\tilde{u} = 0 \quad \text{on } \tilde{S}. \quad (8.17)$$

Moreover, $\tilde{u}$ is a weak solution of a fixed boundary problem of the form (8.10) in the domain $\mathcal{C}^+$ with modified functions $\tilde{A}(x, P)$ and $\tilde{g}(x)$ satisfying $\tilde{A} \in C^{1,\alpha}(\bar{\Lambda} \times (-1, \infty) \times \mathbf{R}^n)$ and $\tilde{g} \in C^\alpha(\bar{\Lambda} \times (-1, \infty))$. Precisely, $\tilde{u}$ satisfies

$$\int_{\mathcal{C}^+} (\tilde{A}(x, D\tilde{u}) \cdot D\eta - \tilde{g}\eta) dx = 0 \quad \text{for any } \eta \in C_0^1(\mathbf{R}^n). \quad (8.18)$$

This is obtained by using the test function $\eta \circ \Psi$, making the change of variables $x \rightarrow \Psi(x)$ in (8.12), and obtaining the expressions for $\tilde{A}$ in $\mathcal{C}^+ \times \mathbf{R}^n$ and $\tilde{g}$ in $\mathcal{C}^+$ in terms of A , g , and the map Ψ . We do not write the explicit expressions of $\tilde{A}$ and $\tilde{g}$, but only point out some properties of $\tilde{A}$ and $\tilde{g}$, which can be obtained readily by using these expressions. First, we emphasize that $\tilde{A}$ and $\tilde{g}$ are given in terms of the original functions A , g , and the map Ψ , which are independent of the solution φ of Problem (TN). Next, using (2.17), we see that, for any $R > 0$, there exists C_R such that

$$\|A - \tilde{A}\|_{1,\alpha,\mathcal{C}^+ \times B_R} \leq C_R \|\Psi\|_{C^{2,\alpha}(\mathbf{R}^n)}. \quad (8.19)$$

Furthermore, from (2.17), it follows that $A = \tilde{A}$ for $x_n > L$, i.e.,

$$A(x, P) = \tilde{A}(x, P) \quad \text{for any } x \in \Lambda \times (L, \infty), P \in \mathbf{R}^n. \quad (8.20)$$

Similarly, from (2.16)–(2.17) and (8.6), we conclude that $\tilde{g}$ satisfies

$$\begin{aligned} \tilde{g} &\in C^\alpha(\bar{\Lambda} \times [-1, \infty)), \quad \|\tilde{g}\|_{0,\alpha,\Lambda \times (-1, \infty)} \leq C\sigma, \\ \tilde{g} &\equiv 0 \quad \text{on } (\Omega \cap \{x_n < 1/2\}) \cup (\Omega \cap \{x_n > L\}) \end{aligned} \quad (8.21)$$

if $L > 1$ and σ is sufficiently small. Define

$$\tilde{F}(x', x_n) = \int_0^{x_n} \tilde{g}(x', s) ds \quad \text{for } (x', x_n) \in \Lambda \times (-1, \infty). \quad (8.22)$$

Then, from (8.21)–(8.22), we have

$$\tilde{F}, \tilde{F}_{x_n} \in C^\alpha(\bar{\Lambda} \times [-1, \infty)), \quad (8.23)$$

$$\|\tilde{F}\|_{0,\alpha,\Lambda \times (-1,\infty)} \leq C\sigma, \quad \|\tilde{F}_{x_n}\|_{0,\alpha,\Lambda \times (-1,\infty)} \leq C\sigma, \quad (8.24)$$

$$\tilde{F} \equiv 0 \quad \text{in } \Lambda \times (-1, 1/2), \quad (8.25)$$

$$\tilde{F} \text{ is independent of } x_n \in \Lambda \times (L, \infty). \quad (8.26)$$

Note that, from (8.14), for sufficiently small σ , the free boundary $\tilde{S}$ lies within the domain $\Lambda \times (-1, 1/2)$. Then, by (8.25), the function $\tilde{F}$ vanishes on $\tilde{S}$. Also, using (8.14) and (8.23), we can rewrite (8.18) as

$$\int_{\mathcal{C}^+} (\tilde{A}(x, D\tilde{u}) + \tilde{F}(x)e_n) \cdot D\eta \, dx = 0 \quad \text{for any } \eta \in C_0^1(\mathbf{R}^n). \quad (8.27)$$

Now we make the partial hodograph transform. Define a mapping $\mathcal{F} : \bar{\mathcal{C}}^+ \rightarrow \mathbf{R}^n$ by

$$(x', x_n) \rightarrow (y', y_n) = (x', \tilde{u}(x', x_n)).$$

The nondegeneracy property (8.8) of $\tilde{u}$ implies that the mapping $\mathcal{F}$ is one-to-one on $\bar{\mathcal{C}}^+$ and, from (8.8) for $\tilde{u}$, (8.14), and (8.17)

$$\mathcal{F}(\mathcal{C}^+) = \Lambda \times (0, \infty), \quad \mathcal{F}(\tilde{S}) = \Lambda \times \{0\},$$

i.e., the free boundary $\tilde{S}$ is mapped to the fixed boundary $\Lambda \times \{0\}$. Also, by (8.8) for $\tilde{u}$, there exists a function $v \in C^{1,\alpha}(\bar{\Lambda} \times (0, \infty))$ such that, for $(x', x_n) \in \bar{\mathcal{C}}^+$ and $y_n \geq 0$,

$$\tilde{u}(x', x_n) = y_n \quad \text{if and only if} \quad v(x', y_n) = x_n. \quad (8.28)$$

Thus,

$$\mathcal{F}^{-1}(y', y_n) = (y', v(y', y_n)).$$

Differentiating the identity $\tilde{u}(x', v(x', y_n)) = y_n$, which holds for any $(x', y_n) \in \bar{\Lambda} \times (0, \infty)$, we find

$$v_{y_n} > 0 \quad \text{in } \Lambda \times (0, \infty)$$

and

$$D_{x'} u = -\frac{1}{v_{y_n}} D_{y'} v, \quad u_{x_n} = \frac{1}{v_{y_n}}, \quad (8.29)$$

where the left-hand and right-hand sides are taken at the points (x', x_n) and $\mathcal{F}(x', x_n)$, respectively. In particular, property (8.8) of $\tilde{u}$ implies

$$0 < \frac{1}{2(q_0^- - q_0^+)} \leq v_{y_n}(y) \leq \frac{2}{q_0^- - q_0^+} \quad \text{for any } y \in \mathcal{C}^+. \quad (8.30)$$

From this and (8.15), we obtain

$$\|v - v_0\|_{1,\alpha,\Lambda \times (0,\infty)} \leq C\sigma, \quad (8.31)$$

where $v_0(y) = y_n/(q_0^- - q_0^+)$. From (8.16) and (8.30),

$$\|v - v_0\|_{C^1(\Lambda \times (R,\infty))} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (8.32)$$

Now, since $\tilde{u}(x)$ is a solution of the conormal problem (8.27) in $\mathcal{C}^+$, then $v(y)$ is a solution of the corresponding problem in $\Lambda \times (0, \infty)$. In order to show that this problem has also a conormal structure, we make the change of variables $x \rightarrow y = \mathcal{F}(x)$ in (8.27). In order to do that, we especially need to change the variables in the test function η . For that, we note that the function $\psi(y) := \eta \circ \mathcal{F}^{-1}(y) = \eta(y', v(y', y_n))$ satisfies $\psi \in C^1(\bar{\Lambda} \times (0, \infty))$ and, by (8.30), if $\eta \equiv 0$ on $\mathbf{R}^n \setminus B_R$, then $\psi \equiv 0$ on $(\Lambda \times (0, \infty)) \setminus B_{R_1}$ for some R_1 , i.e., $\psi = \tilde{\psi}|_{\bar{\Lambda} \times (0, \infty)}$ for some $\tilde{\psi} \in C_0^1(\mathbf{R}^n)$. Similarly, for any $\psi \in C_0^1(\mathbf{R}^n)$, there exists $\eta \in C_0^1(\mathbf{R}^n)$ such that $\psi|_{\bar{\mathbf{R}}_+^n} = \eta \circ \mathcal{F}^{-1}$

and $\eta(x) = \psi \circ \mathcal{F}(x)$ for $x \in \overline{\mathcal{C}^+}$. We differentiate the identity $\eta(x) = \psi(x', u(x', x_n))$ and use (8.29) to obtain

$$D_{x'}\eta = D_{y'}\psi - \frac{\psi_{y_n}}{v_{y_n}}D_{y'}v, \quad \eta_{y_n} = \frac{\psi_{y_n}}{v_{y_n}}. \quad (8.33)$$

Now, in (8.27), we make the change of variables $x \rightarrow y = \mathcal{F}(x)$, use (8.29) and (8.33), note that the Jacobian of $\mathcal{F}^{-1}$ is $J\mathcal{F}^{-1}(y) = v_{y_n}(y)$, and write $\tilde{A}(x', x_n, p', p_n)$ for $\tilde{A}(x, P)$ and $\tilde{F}(x', x_n)$ for $\tilde{F}(x)$ to obtain

$$\int_{\Lambda \times (0, \infty)} \left(B(y', v, Dv) + \tilde{F}(y', v)e_n \right) \cdot D\psi dy = 0 \quad \text{for any } \psi \in C_0^1(\mathbf{R}^n), \quad (8.34)$$

where, for $y' \in \Lambda$, $z \in \mathbf{R}$, $P = (p', p_n) \in \mathbf{R}^{n-1} \times \mathbf{R}_+$,

$$B^i(y', z, P) = \tilde{A}^i(y', z, -\frac{p'}{p_n}, \frac{1}{p_n})p_n \quad \text{for } i = 1, \dots, n-1, \quad (8.35)$$

$$B^n(y', z, P) = \tilde{A}^n(y', z, -\frac{p'}{p_n}, \frac{1}{p_n}) - \sum_{i=1}^{n-1} \tilde{A}^i(y', z, -\frac{p'}{p_n}, \frac{1}{p_n})p_i.$$

Thus, $v(y)$ is a weak solution of the corresponding conormal problem in $\Lambda \times (0, \infty)$.

Note that the functions $B(y', z, P)$ and $\tilde{F}(y', z)$ and the domain $\Lambda \times (0, \infty)$ in (8.34) are independent of the solution φ of Problem (TN).

Conversely, let $v(y)$ is a solution of (8.34) satisfying (8.31)–(8.32) with sufficiently small σ depending only on the data so that (8.30) holds. Then the corresponding solution φ of Problem (TN), satisfying (2.24), can be determined. Indeed, from (8.30),

$$\mathcal{C}^+ := \{(x', v(x', y_n)) : x' \in \Lambda, y_n > 0\} \quad (8.36)$$

has the form (8.14), and a function $\tilde{u} \in C^{1,\alpha}(\overline{\mathcal{C}^+})$ can be defined such that (8.28) holds. Then (8.27) holds. Moreover, (8.31) implies (8.15). Next, we conclude that $\Omega^+ := \Psi(\mathcal{C}^+)$ has the form (2.23) and (2.30), and $u(x) := \tilde{u}(\Phi(x))$ satisfies (8.12). Moreover, (8.15) implies (8.7). If σ is sufficiently small, then (8.7) and (8.15) imply (8.8) for u and $\tilde{u}$, respectively. Now, (8.32) implies (8.16) and further (8.9). Then it follows that φ , defined by $\varphi = \varphi^- - u$ in Ω^+ and $\varphi = \varphi^-$ in $\Omega \setminus \Omega^+$, is a weak solution of Problem (TN) satisfying (2.24).

Thus, the uniqueness of solutions of the conormal problem (8.34) satisfying (8.31)–(8.32) implies the uniqueness of solutions of Problem (TN) satisfying (2.24).

We also note that, since φ satisfies equation (1.1) which is uniformly elliptic in Ω^+ and the function ρ in (1.1) is smooth in the range of $|D\varphi|^2$, and since φ satisfies (2.19) on $\partial_t\Omega^+$ and $\partial_t\Omega$ is a $C^{2,\alpha}$ boundary, then the standard regularity results for elliptic equations and oblique derivative problems imply $\varphi \in C^{2,\alpha}(\overline{\Omega^+} \setminus S)$. Then it follows that $v \in C^{2,\alpha}(\overline{\Lambda} \times (0, \infty))$.

8.3. Uniqueness of weak solutions of the conormal problem (8.34) satisfying (8.31)–(8.32). In the argument of this section, the constant C depends only on the data, unless otherwise is specified.

Suppose that v_1 and v_2 are two solutions of (8.34) in $\Lambda \times (0, \infty)$ satisfying (8.31)–(8.32). We assume that σ is small so that (8.30) holds for v_1 and v_2 . Consider the function $w = v_2 - v_1$. Then w satisfies

$$\|w\|_{1,\alpha,\Lambda \times (0,\infty)} \leq 2C\sigma \quad (8.37)$$

and

$$\|w\|_{1,0,\Lambda \times (R,\infty)} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \quad (8.38)$$

Moreover, $w \in C^{2,\alpha}(\bar{\Lambda} \times (0, \infty))$ since v_1 and v_2 have this regularity. Also, subtracting equations (8.34) for v_1 and v_2 , we obtain

$$\int_{\Lambda \times (0, \infty)} \left(\sum_{i,j=1}^n a_{ij}(y) w_{y_j} \varphi_{y_i} + \sum_{i=1}^n b_i(y) w \varphi_{y_i} \right) dy = 0 \quad (8.39)$$

for any $\varphi \in C_c^1(\mathbf{R}^n)$, where a_{ij} and b_i are defined by

$$\begin{aligned} a_{ij}(y) &= \int_0^1 B_{p_j}^i(y', (1-t)v_1(y) + tv_2(y), (1-t)Dv_1(y) + tDv_2(y)) dt, \\ b_i(y) &= \int_0^1 \left(B_z^i(y', (1-t)v_1(y) + tv_2(y), (1-t)Dv_1(y) + tDv_2(y)) \right. \\ &\quad \left. + \delta_i^n \partial_n \tilde{F}(y', (1-t)v_1(y) + tv_2(y)) \right) dt. \end{aligned} \quad (8.40)$$

Note that $a_{ij}, b_i \in C^\alpha(\bar{\Lambda} \times [0, \infty))$.

Let (a_{ij}^0, b_i^0) be the coefficients (a_{ij}, b_i) for the background solution: $\Psi \equiv Id$, $\varphi^-(x) = q_0^- x_n$, $v_1(y) = v_2(y) = v_0 \equiv y_n / (q_0^- - q_0^+)$. Then, using (2.11), (8.11), and (8.35), we find

$$a_{ij}^0 = \kappa_i \delta_i^j, \quad b_i^0 = 0 \quad \text{for } i, j = 1, \dots, n, \quad (8.41)$$

where κ_i are the following constants: $\kappa_i = \tilde{\rho}((q_0^+)^2)$ for $i = 1, \dots, n-1$, and $\kappa_n = \Phi'(q_0^+) / (q_0^- - q_0^+)$ for $\Phi(s)$ defined by (2.5). Then, since $q_0^+ \in (0, c_*)$, we obtain from (2.5) and (2.7) that

$$\kappa_i > 0 \quad \text{for } i = 1, \dots, n.$$

From (8.41) and using (2.16), (8.2), (8.11), (8.19), (8.24), (8.30)–(8.31), and (8.35), we have

$$\|a_{ij} - \kappa_i \delta_i^j\|_{0,\alpha,\Lambda \times (0, \infty)} + \|b_i\|_{0,\alpha,\Lambda \times (0, \infty)} \leq C\sigma \quad \text{for } i, j = 1, \dots, n. \quad (8.42)$$

Moreover, using (8.3), (8.20), and (8.26), we see that

$$b^i \equiv 0 \quad \text{on } \{y_n > L_1\} \quad \text{for } i = 1, \dots, n, \quad (8.43)$$

where $L_1 = 2L / (q_0^- - q_0^+)$.

Note that, by (8.42) with sufficiently small σ depending only on the data, the coefficients $a_{ij}(y)$ satisfy the ellipticity condition (3.35) with constant $\tilde{\lambda} > 0$ depending only on the data for any $y \in \Lambda \times (0, \infty)$.

Now we prove that $w \equiv 0$. We will do this in three steps.

Step 1. We show that there exist $R_1 \geq 2L_1$ and a nonempty open set $\Lambda' \subset \Lambda$ such that $w(x', R_1) \leq 0$ for all $x' \in \Lambda'$.

Denote

$$g := \sum_{i=1}^n a_{ni} w_{x_i} \equiv \sum_{i,j=1}^n a_{ij} w_{x_j} (e_n)_i \quad \text{on } \Lambda \times \{2L_1\}. \quad (8.44)$$

From (8.37) and (8.42), we have

$$\|g\|_{0,\alpha,\Lambda} \leq C\sigma. \quad (8.45)$$

Now we use (8.43) and $w \in C^2(\bar{\Lambda} \times (0, \infty))$ to conclude that w , restricted to $\Lambda \times (2L_1, \infty)$, is a C^2 solution of the conormal problem

$$\begin{aligned} \sum_{i,j=1}^n (a_{ij} w_{x_j})_{x_i} &= 0 \quad \text{in } \Lambda \times (2L_1, \infty), & \sum_{i,j=1}^n a_{ij} w_{x_j} \nu_i &= 0 \quad \text{on } \partial\Lambda \times (2L_1, \infty), \\ \sum_{i,j=1}^n a_{ij} w_{x_j} \nu_i &= g \quad \text{on } \Lambda \times \{2L_1\}. \end{aligned} \quad (8.46)$$

The uniqueness argument in the proof of Proposition 7.1 can show that w is the unique weak solution in $C^{1,\alpha}(\bar{\Lambda} \times [2L_1, \infty))$ of problem (8.46) satisfying (8.38).

Moreover,

$$\int_{\Lambda \times \{2L_1\}} g \, dx' = 0. \quad (8.47)$$

Indeed, we use in (8.39) the test function $\varphi(x) = \eta(x)\zeta((2L_1 - x_n)/\varepsilon)$, where $\varepsilon > 0$, $\eta \in C_0^\infty(\mathbf{R}^n)$ with $\eta \equiv 1$ on $\Lambda \times (0, 4L_1)$, and $\zeta \in C^\infty(\mathbf{R})$ with $\zeta \geq 0$ on $\mathbf{R}$, $\zeta \equiv 1$ on $(1, \infty)$ and $\zeta \equiv 0$ on $(-\infty, 1/2)$, and then we send $\varepsilon \rightarrow 0+$ using that $w \in C^2(\bar{\Lambda} \times (0, \infty))$ to get (8.47).

Let $R > 2L_1$. Consider the conormal problem

$$\begin{aligned} \sum_{i,j=1}^n (a_{ij}w_{x_j}^{(R)})_{x_i} &= 0 \quad \text{in } \Lambda \times (2L_1, \infty), \\ \sum_{i,j=1}^n a_{ij}w_{x_j}^{(R)}\nu_i &= 0 \quad \text{on } \partial\Lambda \times (2L_1, \infty), \quad \sum_{i,j=1}^n a_{ij}w_{x_j}^{(R)}\nu_i = g \quad \text{on } \Lambda \times \{2L_1\}, \\ w^{(R)} &= 0 \quad \text{on } \Lambda \times \{R\}. \end{aligned} \quad (8.48)$$

Now we can follow the argument of Section 6. Let $w^{(R)} \in H^1(\Lambda \times (2L_1, R))$ be a weak solution of problem (8.48). Using (8.45) and (8.47) instead of (6.5) and (6.2) respectively, we follow the proofs of Lemmas 6.3–6.4 to show that $w^{(R)}$ satisfies

$$\|Dw^{(R)}\|_{L^2(\Lambda \times (2L_1, R))} \leq C\sigma$$

with C independent of R and that, for every $x_n \in (2L_1, R)$, there exists $x' \in \bar{\Lambda}$ such that $w^{(R)}(x', x_n) = 0$. Then we follow the proofs of Propositions 6.5–6.7 to show that if $\sigma > 0$ is small depending only on the data, there exists a unique weak solution $w^{(R)} \in H^1(\Lambda \times (2L_1, R))$ of problem (8.48), and it satisfies

$$\|w^{(R)}\|_{1,\alpha,\Lambda \times (2L_1, R-10r)} \leq C\sigma,$$

with C independent of R . Now we follow the proof of Proposition 7.1 to show that there exists a sequence $R_j \rightarrow \infty$ such that $w^{(R_j)}$ converge in $C^{1,\alpha/2}$ on any compact subset of $\bar{\Lambda} \times [2L_1, \infty)$ to a solution $\hat{w} \in C^{1,\alpha}(\bar{\Lambda} \times [2L_1, \infty))$ of problem (8.46) satisfying (8.38). Moreover, for any $x_n \in [2L_1, \infty)$ there exists $x' \in \bar{\Lambda}$ such that $\hat{w}(x', x_n) = 0$. From the uniqueness of a weak solution in $C^{1,\alpha}(\bar{\Lambda} \times [2L_1, \infty))$ of problem (8.46) satisfying (8.38), we conclude that $\hat{w} = w$. Thus there exists $x'_0 \in \bar{\Lambda}$ such that $w(x'_0, 3L_1) = 0$.

If w is constant on $\Lambda \times (2L_1, \infty)$, then $w \equiv 0$ on $\Lambda \times (2L_1, \infty)$ by (8.38), and the assertion of Step 1 is proved. Thus we assume that w is not constant on $\Lambda \times (2L_1, \infty)$. Also, w is bounded on $\Lambda \times (2L_1, \infty)$ by (8.37).

Now, we use that $w \in C^{2,\alpha}(\bar{\Lambda} \times [L_1, \infty))$ and satisfies $\sum_{i,j=1}^n (a_{ij}w_{x_j})_{x_i} = 0$ in $\Lambda \times (L_1, \infty)$ and $\sum_{i,j=1}^n a_{ij}w_{x_j}\nu_i = 0$ on $\partial\Lambda \times (L_1, \infty)$. We also note that, since $v_1, v_2 \in C^{2,\alpha}(\bar{\Lambda} \times [L_1, \infty))$, we have $a_{ij} \in C^{1,\alpha}(\bar{\Lambda} \times [L_1, \infty))$ so that the equation in (8.46) rewritten in the non-divergence form has continuous coefficients in $\bar{\Lambda} \times [L_1, \infty)$. Then it follows that $\inf_{x \in \bar{\Lambda} \times [2L_1, \infty)} w(x)$ cannot be achieved at any $x \in \Lambda \times [2L_1, \infty)$ by the strong maximum principle and at any $x \in \partial\Lambda \times [2L_1, \infty)$ since Hopf's lemma contradicts the conormal boundary condition at such point. Thus, from $w(x'_0, 3L_1) = 0$, we conclude that, if $w \geq 0$ on $\Lambda \times (2L_1, \infty)$, then $w \equiv 0$ on $\Lambda \times (2L_1, \infty)$.

Thus, if w is not identically zero on $\Lambda \times (2L_1, \infty)$, then there exists $x^* \in \bar{\Lambda} \times [2L_1, \infty)$ such that $w(x^*) < 0$. Since w is continuous, the assertion of Step 1 follows.

Step 2. We use that $w \leq 0$ on $\Lambda' \times \{R_1\}$ with $R_1 \geq 2L_1$ by Step 1 to prove that $w \leq 0$ on $\Lambda \times (0, 2L_1)$.

Choose the following test function φ in (8.39): For fixed $\varepsilon > 0$ and $R > 0$,

$$\varphi = \frac{(w - m_R)^+}{(w - m_R)^+ + \varepsilon}, \quad \text{where } m_R := \max \left(0, \sup_{y \in \Lambda \times (R, \infty)} w(y) \right) < \infty$$

by (8.37). Then $\varphi \in C^{0,1}(\bar{\Lambda} \times [0, \infty))$ and $\varphi \equiv 0$ on $\Lambda \times [R, \infty)$. Thus we can substitute this φ into (8.39) to obtain

$$\int_{\Lambda \times (0, R)} \left(\frac{\varepsilon a_{ij} w_{y_i}^+ ((w - m_R)^+)^{y_j}}{((w - m_R)^+ + \varepsilon)^2} + \frac{\varepsilon b_j w^+ ((w - m_R)^+)^{y_j}}{((w - m_R)^+ + \varepsilon)^2} \right) dy = 0.$$

Then using the ellipticity of (8.39) with constant $\tilde{\lambda} > 0$, (8.37), and (8.42)–(8.43) yields

$$\begin{aligned} \tilde{\lambda} \int_{\Lambda \times (0, R)} |D \log(1 + (w - m_R)^+ / \varepsilon)|^2 dy &= \tilde{\lambda} \int_{\Lambda \times (0, R)} \frac{|D(w - m_R)^+|^2}{((w - m_R)^+ + \varepsilon)^2} dy \\ &\leq - \int_{\Lambda \times (0, R)} b_j \frac{w^+ ((w - m_R)^+)^{y_j}}{((w - m_R)^+ + \varepsilon)^2} dy \\ &= - \int_{\Lambda \times (0, L_1)} b_j \left(\frac{(w - m_R)^+ ((w - m_R)^+)^{y_j}}{((w - m_R)^+ + \varepsilon)^2} + m_R \frac{((w - m_R)^+)^{y_j}}{((w - m_R)^+ + \varepsilon)^2} \right) dy \\ &\leq C \int_{\Lambda \times (0, L_1)} \left(|D \log(1 + (w - m_R)^+ / \varepsilon)| + \frac{m_R}{\varepsilon^2} \right) dy \\ &\leq C \left(\int_{\Lambda \times (0, L_1)} |D \log(1 + (w - m_R)^+ / \varepsilon)|^2 dy \right)^{1/2} + C \frac{m_R}{\varepsilon^2}. \end{aligned}$$

Thus, for $R > L_1$,

$$\int_{\Lambda \times (0, R)} |D \log(1 + (w - m_R)^+ / \varepsilon)|^2 dy \leq C + C \frac{m_R}{\varepsilon^2}.$$

Letting $R \rightarrow \infty$ and noting that $\lim_{R \rightarrow \infty} m_R = 0$ by (8.38), we obtain

$$\int_{\Lambda \times (0, \infty)} |D \log(1 + w^+ / \varepsilon)|^2 dy \leq C.$$

Since $\log(1 + w^+ / \varepsilon) = 0$ on $\Lambda' \times \{x_n = R_1\}$ where $\Lambda' \subset \Lambda$ is nonempty and open, it follows from the Poincaré inequality that

$$\int_{\Lambda \times (0, R_1)} |\log(1 + w^+ / \varepsilon)|^2 dy \leq C,$$

where C depends only on the data, Λ' , and R_1 , but is independent of ε . Letting $\varepsilon \rightarrow 0$, we conclude $w^+ \equiv 0$ in $\Lambda \times (0, R_1)$, which implies $w \leq 0$ in $\Lambda \times (0, 2L_1)$, since $R_1 \geq 2L_1$.

Step 3. Similarly, we prove $w \geq 0$ in $\Lambda \times (0, 2L_1)$, thus $w \equiv 0$ in $\Lambda \times (0, 2L_1)$. Also, $w \in C^1(\bar{\Lambda} \times [0, \infty))$. Then $g \equiv 0$ by (8.44). Now $w \equiv 0$ in $\Lambda \times (2L_1, \infty)$ by uniqueness of a weak solution in $C^{1,\alpha}(\bar{\Lambda} \times [2L_1, \infty))$ of problem (8.46) satisfying (8.38). Thus $w \equiv 0$ in $\Lambda \times (0, \infty)$, and the uniqueness of weak solutions of the conormal problem (8.34) satisfying (8.31)–(8.32) follows.

8.4. Stability. As a consequence of the uniqueness, nondegeneracy, and regularity of solutions of the free boundary problem, we have the following stability theorem.

Theorem 8.1. *There exist a constant $\sigma_0 > 0$ and a nonnegative nondecreasing function $\mathcal{G} \in C([0, \infty))$ with $\mathcal{G}(0) = 0$, depending only on $n, \alpha, \gamma, C_0, \Lambda, L$, and q_0^+ , such that, if*

- (i) $0 < \kappa < \sigma < \sigma_0$;
- (ii) $\Psi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies (2.16)–(2.17), and $\hat{\Psi} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies (2.17) and

$$\|\Psi - \hat{\Psi}\|_{3,\alpha,\Lambda \times (-\frac{1}{2}, \frac{1}{2})} \leq \kappa; \tag{8.49}$$

- (iii) $\varphi^-(x)$ satisfies (2.32) and $\hat{\varphi}^-(x)$ satisfies

$$\|\varphi^- \circ \Psi^{-1} - \hat{\varphi}^- \circ \hat{\Psi}^{-1}\|_{2,\alpha,\Lambda \times (-\frac{1}{2}, \frac{1}{2})} \leq \kappa, \tag{8.50}$$

then the unique solutions $\varphi(x)$ and $\hat{\varphi}(x)$ of Problem (TN) for the nozzles defined by the mappings Ψ and $\hat{\Psi}$ in (2.14) and supersonic upstream flows $\varphi^-(x)$ and $\hat{\varphi}^-(x)$, respectively, satisfy

$$\|\tilde{f}_\varphi - \tilde{f}_{\hat{\varphi}}\|_{1,\alpha,\Lambda} \leq \mathcal{G}(\kappa), \quad (8.51)$$

where $\tilde{f}_\varphi(x')$ and $\tilde{f}_{\hat{\varphi}}(x')$ are the free boundary functions of the cylindric domains $\mathcal{C}^+(\varphi) := \Phi(\Omega^+(\varphi))$ and $\mathcal{C}^+(\hat{\varphi}(x)) := \hat{\Phi}(\Omega^+(\hat{\varphi}))$ in (8.14), respectively, where $\Phi = \Psi^{-1}$ and $\hat{\Phi} = \hat{\Psi}^{-1}$.

Its proof repeats the one of [4, Theorem 6.1]. See also the remarks on the proof of the stability part of [5, Theorem 2.3]. In the present case, as an additional step, the mapping from Ω to $\mathcal{C}$ is involved.

APPENDIX A. EXISTENCE OF SMOOTH SUPERSONIC UPSTREAM FLOWS IN THE NOZZLE

In this paper, we require a local $C^{1,\alpha}$ supersonic solution φ^- in Sections 3–7 and $C^{2,\alpha}$ supersonic solution φ^- in Section 8 in the domain $\Omega_1 := \Omega \cap \{-1 < x_n \leq 1\}$ of the initial-boundary value problem (2.18)–(2.19) for the nonlinear wave equation (1.1) such that

$$\|\varphi^- - \varphi_0^-\|_{k,\alpha,\Omega_1} \leq C_0\sigma, \quad k = 1 \text{ or } 2, \quad (A.1)$$

for some $C_0 > 0$ independent of σ , when σ is sufficiently small. This is a direct corollary of the standard local theory of smooth solutions of the initial-boundary value problem for the nonlinear wave equations.

Proposition A.1. *Let the nozzle boundary satisfy (2.15)–(2.16). Let the Cauchy data (φ_e^-, ψ_e^-) on $\partial_o\Omega$ satisfy*

$$\|\varphi_e^- - q_0^- x_n\|_{H^{s+k}} + \|\psi_e^- - q_0^-\|_{H^{s+k-1}} \leq \sigma, \quad k = 1 \text{ or } 2, \quad (A.2)$$

for some integer $s > n/2 + 1$ and the compatibility conditions on the intersection $\overline{\partial_l\Omega} \cap \overline{\partial_o\Omega}$ up to the $(s+k)^{th}$ -order. Then there exist $\sigma_0 > 0$ and $C_0 > 0$ such that there is a unique classical solution $\varphi^- \in C^{k+1}(\overline{\Omega_1})$ of problem (2.18)–(2.19) for the nonlinear wave equation (1.1), respectively for $k = 1$ or 2 , such that, when $\sigma \leq \sigma_0$,

$$\|\varphi^- - \varphi_0^-\|_{k+1,0,\Omega_1} \leq C_0\sigma. \quad (A.3)$$

If, in addition, conditions (2.15)–(2.16) and (A.2), as well as the compatibility conditions, hold for higher-order derivatives, then the solution φ^- has the corresponding higher regularity.

As in Section 8, we set

$$w(x) := \varphi^-(x) - q_0^- x_n.$$

Then w satisfies the following initial-boundary value problem:

$$\operatorname{div}(A(Dw)) = 0, \quad (A.4)$$

$$(w, w_{x_n})|_{x_n=-1} = (\varphi_e^- + q_0^-, \psi_e^- - q_0^-), \quad (A.5)$$

$$w_\nu|_{\partial_l\Omega} = -q_0^- \nu_n, \quad (A.6)$$

as well as the compatibility conditions up to the $(s+k)^{th}$ -order, where

$$A(P) = \rho(|P + q_0^- e_n|^2)(P + q_0^- e_n) - \rho((q_0^-)^2)q_0^- e_n.$$

Since $q_0^- > c_*$, then (A.4) is strictly hyperbolic, nonlinear wave equations with respect to the x_n -direction (whose role is as the time direction) for a small perturbation of the trivial solution $u = 0$. By the standard energy estimates and the iteration procedure (cf. [27, 28, 30] and the references cited therein), for small $\sigma > 0$, there exists a unique solution

$$w(x', x_n) \in \cap_{j=0}^{s+k} C^j([-1, 1]; H^{s+1-j}(\bar{\Omega}))$$

such that

$$\sum_{j=0}^{s+k} \|w\|_{C^j([-1, 1]; H^{s+1-j}(\bar{\Omega}))} \leq C\sigma,$$

for $k = 1$ or 2 , respectively. Since $s > n/2 + 1$, the Sobolev embedding theorem implies (A.3).

Acknowledgments. Gui-Qiang Chen's research was supported in part by the National Science Foundation under Grants DMS-0244473, DMS-0204225, DMS-0204455, and INT-9987378, and an Alexandre von Humboldt Foundation Fellowship. Mikhail Feldman's research was supported in part by the National Science Foundation under Grants DMS-0200644 and DMS-0074037.

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