Existence and Stability of Multidimensional Transonic Shocks for the Euler Equations for Steady Potential Fluids in Unbounded Domains

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1 Introduction

We are concerned with the existence and stability of multidimensional transonic shocks in inviscid compressible fluid dynamics. In this paper, we focus on inviscid steady potential fluid flows, which are governed by the Euler equations consisting of the conservation law of mass and the Bernoulli law for velocity. Then the Euler equations for the velocity potential $\varphi : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ can be formulated into the following nonlinear second order elliptic-hyperbolic equations of mixed type:

$$\operatorname{div}\left(\rho(|D\varphi|^2)D\varphi\right) = 0, \tag{1}$$

where the density function $\rho(q^2)$ has the form:

$$\rho(q^2) = \left(1 - \frac{\gamma-1}{2}q^2\right)^{\frac{1}{\gamma-1}}, \tag{2}$$

with the adiabatic exponent $\gamma > 1$.

The second order nonlinear equation (1) is elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0, \tag{3}$$

and is hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \tag{4}$$

The elliptic regions of the equation (1) correspond to the subsonic flow, and the hyperbolic regions to the supersonic flow. Shocks are discontinuities in the velocity $D\varphi$. We study weak solutions of (1) with transonic (hyperbolic-elliptic) shocks.

Shiffman [30], Bers [4], and Finn-Gilbarg [17] studied subsonic (elliptic) solutions of (1) outside an obstacle; also see Dong [16]. Alt-Caffarelli-Friedman [3] studied free boundary problems for subsonic flows.

Morawetz in [26] showed that the flows of (1) past an obstacle may contain transonic shocks in general. One exception is that, when the obstacle forms a wedge with small angle, the uniform outflow with supersonic speed may produce a non-transonic shock (hyperbolic-hyperbolic shock); see [13, 22, 23, 29, 31]. Transonic shocks also arise in many other situations of physical importance. For example, when a plane shock hits a wedge, a reflected flow may form a transonic shock; see Glimm-Majda [19] and Morawetz [27]. Steady transonic shocks are also very useful for solving some unsteady important problems; see Chen-Glimm [11] and Chen-Wang [12].

Canic-Keyfitz-Lieberman [8] studied perturbations of steady transonic shocks between two uniform flows for the two-dimensional transonic small-disturbance (TSD) equation, which governs the behavior of the first non-trivial term in the geometric optics expansion to (1) near a certain physical point. The TSD model can be written as a second order nonlinear equation of mixed type with coefficients depending only upon the unknown function itself, but independent of its gradient as in (1). This feature is essential for the approach in [8] (also see [9]).

Majda [24] studied the existence and stability of multidimensional shock fronts, locally in time, for the Euler equations for compressible fluids (also see Métivier [25]).

In this paper, we discuss several recent results on the existence and stability of multidimensional transonic shocks for (1) in unbounded domains and present a nonlinear method for establishing these results. For the detailed proofs, see Chen-Feldman [10].

2 Transonic Shocks

In this section, we first introduce multidimensional transonic shocks for (1).

A function $\varphi \in W^{1,\infty}(\Omega)$ is a weak solution of (1) in an unbounded domain Ω if

(i) $|D\varphi(x)| \le 1/\sqrt{\theta}$ a.e.

(ii) For any $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega \rho(|D\varphi|^2) D\varphi \cdot D\zeta \, dx = 0. \tag{5}$$

We are interested in weak solutions with shocks. Let Ω^+ and $\Omega^- = \Omega \setminus \overline{\Omega^+}$ be nonempty open subsets of Ω, and $S = \partial\Omega^+ \cap \Omega$. Let $\varphi \in W^{1,\infty}(\Omega)$ be a weak solution of (1) and be in $C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ so that $D\varphi$ experiences a jump across S that is an $(n-1)$-dimensional smooth surface. Then the condition that $\varphi(x)$ is a weak solution of (1) implies the continuity of $\varphi(x)$ across S:

$$\varphi^+ = \varphi^- \qquad \text{on } S \tag{6}$$

and the following Rankine-Hugoniot condition on S:

$$\Big[\rho(|D\varphi|^2) D\varphi \cdot \nu\Big]_S = 0, \tag{7}$$

where ν is the unit normal to S in the direction of Ω^+, and the bracket denotes the difference between the values of the function along S on the $\Omega^\pm$ sides, respectively. We can also write (7) as

$$\rho(|D\varphi^+|^2) D\varphi^+ \cdot \nu = \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu \qquad \text{on} \quad S. \tag{8}$$

Let $K > 0$. Then the function

$$\Phi_K(p) := \left(K - \theta p^2\right)^{\frac{1}{2\theta}} p, \tag{9}$$

defined for $p \in \left[0, \sqrt{K/\theta}\right]$, satisfies

$$\lim_{p \to 0+} \Phi_K(p) = \lim_{p \to \sqrt{K/\theta}-} \Phi_K(p) = 0, \tag{10}$$

$$\Phi_K(p) > 0 \quad \text{for } p \in \left(0, \sqrt{K/\theta}\right), \tag{11}$$

$$0 < \Phi_K'(p) < K^{\frac{1}{2\theta}} \text{ on } (0, p_{sonic}^K), \text{ and } \Phi_K'(p) < 0 \quad \text{on} \quad \left(p_{sonic}^K, \sqrt{K/\theta}\right), \tag{12}$$

$$\Phi_K''(p) < 0 \text{ on } (0, p_{sonic}^K], \tag{13}$$

where

$$p_{sonic}^{K} = \sqrt{K/(\theta+1)}. \tag{14}$$

Suppose that a weak solution $\varphi(x)$ in Ω and nonempty open $\Omega^{\pm} \subset \Omega$ satisfy

$$\begin{aligned} &\varphi(x) \in C^2(\Omega^{\pm}) \cap C^1(\overline{\Omega^{\pm}}), \qquad |D\varphi(x)| < p_{sonic}^{1} = \frac{1}{\sqrt{\theta+1}} \ \text{ in } \ \Omega^{+}, \\ &|D\varphi(x)| > p_{sonic}^{1} \quad \text{in} \ \ \Omega^{-} = \Omega \setminus \overline{\Omega^{+}}, \qquad D\varphi^{\pm}\cdot\nu > 0 \qquad \text{along} \ \ S = \partial\Omega^{+} \cap \Omega, \end{aligned} \tag{15}$$

besides (6) and (7). Then $\varphi(x)$ is a *transonic shock solution* with *transonic shock* S dividing Ω into the *subsonic region* Ω^{+} and the *supersonic region* Ω^{-}. Solution $\varphi(x)$ satisfies the physical entropy condition (see Courant-Friedrichs [14]; also see Lax [21]):

$$\rho(|D\varphi_{-}|^2) < \rho(|D\varphi_{+}|^2) \qquad \text{along } S, \tag{16}$$

that is, the density ρ increases across a shock in the flow direction. Note that the equation (1) is elliptic in the subsonic region and hyperbolic in the supersonic region.

In order to deal with multidimensional transonic shocks in an unbounded domain Ω, we define the following weighted Hölder semi-norms and norms in a domain $\mathcal{D} \subset \mathbf{R}^n$: Let $x \to \delta_x$ be a given non-negative function defined on $\mathcal{D}$, which will be specified in each case we consider below. Let $\delta_{x,y} := \min(\delta_x, \delta_y)$ for $x, y \in \mathcal{D}$. For $k \in \mathbf{R}$, $\alpha \in (0,1)$, and $m \in \mathbf{N}$ (the set of nonnegative integers), we define

$$\begin{aligned} &[u]^{(k)}_{m;0;\mathcal{D}} = \sum_{|\beta|=m} \sup_{x\in\mathcal{D}} \left(\delta_x^{m+k} |D^{\beta}u(x)|\right), \\ &[u]^{(k)}_{m;\alpha;\mathcal{D}} = \sum_{|\beta|=m} \sup_{x,y\in\mathcal{D}, x\neq y} \left(\delta_{x,y}^{m+\alpha+k} \frac{|D^{\beta}u(x) - D^{\beta}u(y)|}{|x-y|^{\alpha}}\right), \\ &\|u\|^{(k)}_{m;0;\mathcal{D}} = \sum_{j=0}^{m} [u]^{(k)}_{j;0;\mathcal{D}}, \qquad \|u\|^{(k)}_{m;\alpha;\mathcal{D}} = \|u\|^{(k)}_{m;0;\mathcal{D}} + [u]^{(k)}_{m;\alpha;\mathcal{D}}, \end{aligned} \tag{17}$$

where $D^{\beta} = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$, $\beta = (\beta_1, \ldots, \beta_n)$ is a multi-index with $\beta_j \geq 0, \beta_j \in \mathbf{N}$, and $|\beta| = \beta_1 + \cdots + \beta_n$.

We denote by $\|u\|_{m;\alpha;\Omega}$ the (non-weighted) Hölder norms [18, (4.6)] in a domain Ω, i.e., the norms defined as above with $\delta_x = \delta_{x,y} = 1$.

3 Multidimensional Transonic Shocks near Flat Shocks

Let (x', x_n) be the coordinates in $\mathbf{R}^n$, where $x_n \in \mathbf{R}$ and $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$.

Let $q_0^{-} \in \left(p_{sonic}^{1}, 1/\sqrt{\theta}\right)$ and $\varphi_0^{-}(x) := q_0^{-} x_n$. Then $\varphi_0^{-}(x)$ is a supersonic solution in Ω. According to (10)–(12), there exists a unique $q_0^{+} \in (0, p_{sonic}^{1})$ such that

$$\left(1 - \theta(q_0^{+})^2\right)^{\frac{1}{2\theta}} q_0^{+} = \left(1 - \theta(q_0^{-})^2\right)^{\frac{1}{2\theta}} q_0^{-}. \tag{18}$$

Thus, the function

$$\varphi_0(x) = \begin{cases} q_0^{-} x_n, & x \in \Omega_0^{-} := \Omega \cap \{x \mid x_n < 0\}, \\ q_0^{+} x_n, & x \in \Omega_0^{+} := \Omega \cap \{x \mid x_n > 0\} \end{cases} \tag{19}$$

is a plane transonic shock solution in Ω, Ω_0^{+} and Ω_0^{-} are its subsonic and supersonic regions, respectively, and $S = \{x_n = 0\}$ is a transonic shock. Defining $\varphi_0^{\pm}(x) := q_0^{\pm} x_n$ in Ω, we have

$$\varphi_0(x) = \min(\varphi_0^{+}(x), \varphi_0^{-}(x)) \qquad \text{in } \Omega. \tag{20}$$

3.1 Transonic Shocks near Flat Shocks in Infinite Cylinders

In this section, we focus on an infinite cylinder $(0,a)^{n-1}\times(-\infty,\infty)$. We use the norms in (17) with the weight function
$$\delta_x = 1+|x_n|.$$

Since it is not necessary to require that the supersonic perturbation $\varphi^-(x)$ be defined in the whole infinite cylinder, we introduce a finite subcylinder $\Omega_1 := (0,a)^{n-1}\times(-1,1)$ and focus on the cylinder domain $\Omega := (0,a)^{n-1}\times(-1,\infty)$ without loss of generality. Then our multidimensional transonic shock problem can be formulated into the following form:

Problem A. Given a supersonic weak solution $\varphi^-(x)$ of (1) in Ω_1, which is a $C^{1,\alpha}$ perturbation of $\varphi_0^-(x)$, for some $\alpha\in(0,1)$:
$$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_1} \le \sigma, \tag{21}$$
with $\sigma>0$ small, and
$$\varphi_\nu^- = 0 \qquad \text{on} \quad \partial(0,a)^{n-1}\times[-1,1], \tag{22}$$
find a transonic shock solution $\varphi(x)$ in Ω such that $\Omega^-\subset\Omega_1$,
$$\varphi=\varphi^- \qquad \text{in} \ \ \Omega^-,$$
where Ω^- is the supersonic region (15) of $\varphi(x)$, and
$$\varphi=\varphi^-,\ \varphi_{x_n}=\varphi^-_{x_n} \qquad \text{on } (0,a)^{n-1}\times\{-1\}, \tag{23}$$
$$\varphi_\nu = 0 \qquad \text{on } \partial(0,a)^{n-1}\times[-1,\infty), \tag{24}$$
$$\varphi(\cdot,x_n)-\omega x_n\to 0 \qquad \text{uniformly on } (0,a)^{n-1} \text{ as } x_n\to\infty, \tag{25}$$
for some constant $\omega\in(0,p^1_{sonic})$ which is not prescribed.

Then we have

Theorem 1. *Let $q_0^+\in(0,p^1_{sonic})$ and $q_0^-\in\left(p^1_{sonic},1/\sqrt{\theta}\right)$ satisfy (18), and let $\varphi_0(x)$ be the plane transonic shock solution (19). Then there exist $\sigma_0>0$, C_0 and C depending only on n, a, α, γ, and q_0^+ such that, for every $\sigma\in(0,\sigma_0)$ and any supersonic solution $\varphi^-(x)$ of (1) satisfying the conditions stated in Problem A, there exists a solution $\varphi(x)$ of Problem A satisfying*
$$\|\varphi - q_0^+x_n\|^{(-1)}_{1;\alpha;\Omega^+(\varphi)} \le C_0\sigma. \tag{26}$$
Moreover, $\varphi(x)$ satisfies the following properties:

(i) *The constant ω in (25) must be q^+:*
$$\omega = q^+, \tag{27}$$
where q^+ is the unique solution in the interval $(0,p^1_{sonic})$ of the equation
$$\rho((q^+)^2)q^+ = Q^+$$
with
$$Q^+ = \frac{1}{a^{n-1}}\int_{(0,a)^{n-1}} \rho(|D\varphi^-(x',-1)|^2)\varphi^-_{x_n}(x',-1)dx'. \tag{28}$$
The q^+ satisfies
$$|q^+ - q_0^+| \le C\sigma. \tag{29}$$

(ii) *The subsonic region* $\Omega^+(\varphi) := \{x \in \Omega \;\; : \;\; |D\varphi(x)| < p^1_{sonic}\}$ *is of the form:*

$$\Omega^+(\varphi) = \{x_n > f(x')\} \cap \Omega \quad with \quad f \in C^{1,\alpha}([0,a]^{n-1}),\ Df|_{\partial(0,a)^{n-1}} = 0, \tag{30}$$

where f *satisfies*

$$\|f\|_{1,\alpha,[0,a]^{n-1}} \le C\sigma. \tag{31}$$

(iii) *For every* $k = 1, 2, \ldots.$

$$\|\varphi - q^+x_n\|^{(k)}_{1;\alpha;\Omega^+(\varphi)} \le C_k\sigma, \tag{32}$$

where C_k *depends only on* k, n, a, γ, *and* q_0^+. *That is, when* $x_n \to \infty$, $\varphi(x)$ *uniformly converges to the linear function* q^+x_n *with respect to* $x' \in (0,a)^{n-1}$ *at a rate faster than any algebraic order.*

If $\varphi^- \in C^{2,\alpha}(\overline{\Omega_1})$ *in addition to the previous assumptions, then* $\varphi \in C^{2,\alpha}(\overline{\Omega^+(\varphi)})$ *and* $f \in C^{2,\alpha}([0,a]^{n-1})$, *and*

$$\|D^2 f\|_{0,\alpha,[0,a]^{n-1}} \le C(n, a, \alpha, \gamma, q_0^+, \sigma, \|\varphi^-\|_{2,\alpha,\Omega_1}) < \infty, \tag{33}$$

$$\|D^2\varphi\|^{(k)}_{0;\alpha;\Omega^+(\varphi)} \le C(n, a, \alpha, \gamma, q_0^+, \sigma, k, \|\varphi^-\|_{2,\alpha,\Omega_1}) < \infty, \tag{34}$$

for $k = 1, 2, 3, \ldots.$

Furthermore, if

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega_1} \le \sigma, \tag{35}$$

then the solution φ *of Problem A, satisfying (26), is unique.*

Remark 1. Theorem 2.1 indicates that, given any supersonic solution $\varphi^-(x)$ in the upstream region, the necessary and sufficient condition for the existence of a transonic shock solution in an infinite cylinder Ω is that the uniform velocity state $\omega\nu_0, \nu_0 = (0, \ldots, 0, 1)$, at infinity in the downstream direction must be $q^+\nu_0$ determined uniquely by $\varphi^-(x)$. In other words, given a supersonic solution $\varphi^-(x)$ in the upstream region, there is no transonic shock solution that contains a transonic shock dividing Ω into the subsonic region and the supersonic region, provided that one expects a different velocity state at infinity from $q^+\nu_0$ in the downstream direction. This means that, for this problem, one can apriori prescribe the uniformity condition of the flow, but can not prescribe a velocity state, at infinity in the downstream direction in general, which otherwise the problem is overdetermined.

Furthermore, as a consequence of the uniqueness, non-degeneracy, and regularity of solutions of the free boundary problem, we have the following stability theorem.

Theorem 2. *There exist a nonnegative function* $\Psi \in C([0,\infty))$ *satisfying* $\Psi(0) = 0$ *and* $\sigma_0 > 0$ *depending only upon* n, a, α, γ, *and* q_0^+ *such that, if* $\sigma < \sigma_0$, $\varphi^-(x)$ *satisfies* (35), *and* $\hat{\varphi}^-(x)$ *satisfies*

$$\|\varphi^- - \hat{\varphi}^-\|_{2,\alpha,\Omega_1} \le \kappa, \tag{36}$$

with $\kappa < \sigma$, *then the unique solutions* $\varphi(x)$ *and* $\hat{\varphi}(x)$ *of Problem A for* $\varphi^-(x)$ *and* $\hat{\varphi}^-(x)$, *respectively, satisfy*

$$\|f_\varphi - f_{\hat{\varphi}}\|_{2,\alpha,(0,a)^{n-1}} \le \Psi(\kappa), \tag{37}$$

where $f_\varphi(x')$ *and* $f_{\hat{\varphi}}(x')$ *are the free boundary functions in* (30) *with* $\varphi(x)$ *and* $\hat{\varphi}(x)$, *respectively.*

3.2 Transonic Shocks near Flat Shocks in $\mathbf{R}^n$

We continue to study perturbations of the uniform transonic shock solution (19). In this section we focus on the whole space: $\mathbf{R}^n$ with $n \geq 3$. But, as in previous section, since it is enough to specify the supersonic perturbation φ^- only in a neighborhood of the nonperturbed shock surface $\{x_n = 0\}$, we introduce domains $\Omega := \mathbf{R}^{n-1} \times (-1, \infty)$ and $\Omega_2 := \mathbf{R}^{n-1} \times (-1, 1)$. Note that we expect the subsonic region Ω^+ to be close to the half-space $\Omega_0^+ = \{x_n > 0\}$.

We use the norms in (17) with the weight function

$$\delta_x = 1 + |x|,$$

and consider the following problem:

Problem B. Given a supersonic solution $\varphi^-(x)$ of (1) in $\mathbf{R}^n$ such that, for some $\alpha > 0$,

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_2}^{(n-1)} < \sigma, \tag{38}$$

with $\sigma > 0$ small, find a transonic shock solution $\varphi(x)$ in Ω such that $\Omega^- \subset \Omega_2$ and $\varphi = \varphi^-$ in Ω^-, where Ω^- is the supersonic region (15) of $\varphi(x)$.

Remark 2. Note that, since $n \geq 3$, our assumptions imply that the perturbation is not only small in $C^{1,\alpha}$, but also "localized", that is, the perturbation has an appropriate polynomial decay at infinity.

Then we have the following theorem.

Theorem 3. *Let $q^+ \in (0, p^1_{sonic})$ and $q^- \in \left(p^1_{sonic}, 1/\sqrt{\theta}\right)$ satisfy (18), and let $\varphi_0(x)$ be the transonic shock solution (19). Then there exist positive constants σ_0, C_1, and C_2 depending only on n, γ, and q^+ such that, for every $\sigma \leq \sigma_0$ and any supersonic solution $\varphi^-(x)$ of (1) satisfying the conditions stated in Problem B, there exists a solution $\varphi(x)$ of Problem B such that*

$$\|\varphi - \varphi_0^+\|_{1,\alpha,\Omega^+}^{(n-2)} \leq C_1\sigma. \tag{39}$$

Moreover, $\Omega^+ = \{x_n > f(x')\}$, and $f : \mathbf{R}^{n-1} \to \mathbf{R}$ satisfies

$$\|f\|_{1,\alpha,\mathbf{R}^{n-1}}^{(n-2)} \leq C_2\sigma, \tag{40}$$

that is, the shock surface $S = \{(x', x_n) \in \mathbf{R}^n \; : \; x_n = f(x'), x' \in \mathbf{R}^{n-1}\}$ is $C^{1,\alpha}$ and converges at infinity, with an appropriate polynomial rate, to the hyperplane $S_0 = \{x_n = 0\}$.

If $\varphi^- \in C^{2,\alpha}(\Omega_2)$, in addition to the previous assumptions, then $\varphi \in C^{2,\alpha}(\overline{\Omega^+(\varphi)})$ and $f \in C^{2,\alpha}(\mathbf{R}^{n-1})$. In particular, if $\|\varphi^-\|_{2,\alpha,\mathbf{R}^n}^{(n-1)} < \infty$, then

$$\|D^2 f\|_{0,\alpha,\mathbf{R}^{n-1}}^{(n)} \leq C(n, \alpha, \gamma, q_0^+, \sigma, \|\varphi^-\|_{2,\alpha,\Omega_2}^{(n-1)}) < \infty, \tag{41}$$

$$\|D^2 \varphi\|_{0;\alpha;\Omega^+(\varphi)}^{(n)} \leq C(n, \alpha, \gamma, q_0^+, \sigma, \|\varphi^-\|_{2,\alpha,\Omega_2}^{(n-1)}) < \infty. \tag{42}$$

Furthermore, if

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega_2}^{(n-1)} < \sigma, \tag{43}$$

then the solution φ of Problem B, satisfying (39), is unique.

We also establish a similar stability theorem to Theorem 2 for the multidimensional transonic shocks near flat shocks in $\mathbf{R}^n$.

4 Multidimensional Transonic Shocks near Spherical Transonic Shocks

We first construct spherical transonic shocks in $\mathbf{R}^n, n \geq 3$. Choosing any

$$0 < R_1 < R_0 < \infty,$$

we consider the domain $\Omega = \{x \in \mathbf{R}^n \mid R_1 < |x| < \infty\}$ and show that there exists a weak solution $\varphi_0 \in W^{1,\infty}(\Omega)$ of (1) in the sense of (5) with $\varphi_0(x) = w(|x|)$ for some $w : \mathbf{R} \to \mathbf{R}$ such that $\varphi_0 \in C^\infty(\overline{\Omega_0^\pm})$, and

$$\Omega_0^- = \{x \in \mathbf{R}^n \mid R_1 < |x| < R_0\} \quad \text{and} \quad \Omega_0^+ = \{x \in \mathbf{R}^n \mid \ R_0 < |x| < \infty\}$$

are respectively supersonic and subsonic regions of $\varphi_0(x)$, that is,

$$|D\varphi_0| > p^1_{sonic} \quad \text{in} \quad \Omega_0^-, \qquad |D\varphi_0| < p^1_{sonic} \quad \text{in} \quad \Omega_0^+, \tag{44}$$

and $|D\varphi_0|$ has a jump across $S_0 := \{|x| = R_0\}$.

It is easy to see that a function $\varphi_0(x) = w(|x|)$ satisfies (5) in Ω if and only if, for any $\zeta \in C_0^\infty((R_1, \infty))$,

$$\int_{R_1}^\infty \rho(|w'(r)|^2) w'(r) \zeta'(r) r^{n-1} dr = 0.$$

It follows that $w(|x|)$ is a smooth solution of (1) if

$$\big(\rho(|w'(r)|^2) w'(r) r^{n-1}\big)' = 0.$$

Thus $\rho(|w'(r)|^2) w'(r) = \frac{c}{r^{n-1}}$, which can be written as

$$\Phi_1(w'(r)) = \frac{c}{r^{n-1}}, \tag{45}$$

where $\Phi_1(\cdot)$ is defined by (9). From (10)–(13), it follows that there exist smooth functions

$$\begin{aligned} &\Phi_+^{-1} : (0, \Phi_1(p^1_{sonic})) \to (0, p^1_{sonic}), \\ &\Phi_-^{-1} : (0, \Phi_1(p^1_{sonic})) \to (p^1_{sonic}, \sqrt{1/\theta}), \end{aligned}$$

which are the inverse functions of $\Phi_1(\cdot)$ in the sense that $\Phi_1(\Phi_\pm^{-1}(\tau)) = \tau$, for any $\tau \in (0, \Phi_1(p^1_{sonic}))$, such that $(\Phi_+^{-1})'(\tau) > 0$ and $(\Phi_-^{-1})'(\tau) < 0$ for $\tau \in (0, \Phi_1(p^1_{sonic}))$. Thus, in order to satisfy (44) and (45), we have to choose $c > 0$ such that

$$0 < \frac{c}{R_0^{n-1}} < \frac{c}{R_1^{n-1}} < \Phi_1(p^1_{sonic}),$$

and set

$$w'(r) = \begin{cases} \Phi_-^{-1}\left(\frac{c}{r^{n-1}}\right) & \text{for } R_1 < r < R_0, \\ \Phi_+^{-1}\left(\frac{c}{r^{n-1}}\right) & \text{for } R_0 < r < \infty. \end{cases} \tag{46}$$

Thus, we obtain a weak solution $\varphi_0(x) = w(|x|)$ of (1) in Ω, satisfying (44), by setting

$$w(r) = \begin{cases} -\int_r^{R_0} \Phi_-^{-1}\left(\frac{c}{\tau^{n-1}}\right) d\tau < 0 & \text{for } R_1 < r < R_0, \\ \int_{R_0}^r \Phi_+^{-1}\left(\frac{c}{\tau^{n-1}}\right) d\tau > 0 & \text{for } R_0 < r < \infty. \end{cases}$$

Note that, since $\Phi'(0+) > 0$, the function $w(r)$ is bounded on (R_1, ∞) when $n \geq 3$.

We can express the function $\varphi_0(x)$ as

$$\varphi_0(x) = \min(\varphi_0^+(x), \varphi_0^-(x)), \tag{47}$$

where $\varphi_0^\pm \in C^\infty(\overline{\Omega})$ are defined as

$$\varphi_0^\pm(x) = w_\pm(|x|), \qquad w_\pm(r) = \int_{R_0}^r \Phi_\pm^{-1}\left(\frac{c}{\tau^{n-1}}\right) d\tau \quad \text{for } r > R_1. \tag{48}$$

The solution $\varphi_0(x)$ contains the spherical transonic shock and satisfies the entropy condition (16) across the transonic shock from the hyperbolic phase to the elliptic phase, which is the flow direction.

We now state our results on the existence and stability of multidimensional transonic shocks that are close to the solution $\varphi_0(x)$ for $n \geq 3$. We use the Hölder norms in (17) with the weight function:

$$\delta_x = 1 + |x|.$$

Since it does not require to define the perturbation on the whole Ω, we introduce a bounded domain

$$\Omega_3 := \{R_1 < |x| < R_0 + 1\}.$$

Problem C. Given a supersonic solution $\varphi^-(x)$ of (1) in Ω_3, which is a $C^{1,\alpha}$ perturbation of $\varphi_0^-(x)$, for some $\alpha > 0$:

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_3} \leq \sigma, \tag{49}$$

with $\sigma > 0$ small, find a transonic shock solution $\varphi(x)$ in Ω such that $\Omega^- \subset \Omega_3$ and $\varphi(x) = \varphi^-(x)$ in Ω^-, where Ω^- is the supersonic region (15) of $\varphi(x)$, and

$$\varphi = \varphi^-, \quad \varphi_\nu = \varphi_\nu^- \qquad \text{on} \qquad \{|x| = R_1\}. \tag{50}$$

Theorem 4. *There exist positive constants σ_0, C_1, and C_2, depending only on n, γ, c, and Ω, such that, for every $\sigma \leq \sigma_0$ and any supersonic solution $\varphi^-(x)$ of (1) satisfying the conditions stated in Problem C, there exists a solution $\varphi(x)$ of Problem C satisfying*

$$\|\varphi - \varphi_0^+\|_{1,\alpha,\Omega^+}^{(n-2)} \leq C_1\sigma. \tag{51}$$

Moreover, the shock surface $S = \{x \in \mathbf{R}^n \mid x = f(\frac{x}{|x|})\} \cap \Omega$ is C^α, that is, $\Omega^+ = \{|x| > f(\frac{x}{|x|})\} \cap \Omega$, with $f : S^{n-1} \to \mathbf{R}$ and

$$\|f - R_0\|_{0,\alpha,S^{n-1}} \leq C_2\sigma.$$

If $\varphi^- \in C^{2,\alpha}(\Omega_3)$, in addition to the previous assumptions, then $\varphi \in C^{2,\alpha}(\overline{\Omega^+(\varphi)})$ and $f \in C^{2,\alpha}(S^{n-1})$ satisfy

$$\|D^2 f\|_{0,\alpha,S^{n-1}} \leq C(n, \alpha, \gamma, q_0^+, \sigma, \|\varphi^-\|_{2,\alpha,\Omega_3}) < \infty, \tag{52}$$

$$\|D^2\varphi\|_{0;\alpha;\Omega^+(\varphi)}^{(n)} \leq C(n, \alpha, \gamma, q_0^+, \sigma, \|\varphi^-\|_{2,\alpha,\Omega_3}) < \infty. \tag{53}$$

Furthermore, if

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega_3} \leq \sigma, \tag{54}$$

then the solution φ of Problem B, satisfying (51), is unique.

We also establish a similar stability theorem to Theorem 2 for the multidimensional transonic shocks near spherical shocks in $\mathbf{R}^n$.

5 Methods of the Proofs: Free Boundary Problems

The transonic shock problems can be formulated into a one-phase free boundary problem for a nonlinear elliptic equation: Given $\varphi^- \in C^{1,\alpha}(\overline{\Omega})$, find a function $\varphi(x)$ that is continuous in Ω and satisfies

$$\varphi \leq \varphi^- \qquad \text{in } \bar{\Omega}, \tag{55}$$

the equation (1) and the ellipticity condition (3) in the non-coincidence set

$$\Omega^+ = \{\varphi < \varphi^-\},$$

the free boundary condition (8) on the boundary $S = \partial\Omega^+ \cap \Omega$, as well as the prescribed conditions on the fixed boundary $\partial\Omega$ and at infinity. These conditions are different in different cases (Problems A, B, C), say, conditions (23)–(25) for Problem A.

The free boundary is the location of the shock, and the free boundary condition (8) is the Rankine-Hugoniot condition (7). Note that the condition (55) is motivated by the similar properties (20) and (47) of non-perturbed shocks; and, locally on the shock, (55) is equivalent to the entropy condition (16). The condition (55) transforms the transonic shock problem, in which the subsonic region Ω^+ is determined by the gradient condition $|D\varphi(x)| < p^1_{sonic}$, into a free boundary problem, in which Ω^+ is the non-coincidence set.

In order to solve this free boundary problem, we first modify the equation (1) to make it uniformly elliptic and, correspondingly, modify the free boundary condition (8). Then we solve this modified free boundary problem. Since $\varphi^-(x)$ is a small $C^{1,\alpha}$ perturbation of $\varphi_0^-(x)$, we show that the solution $\varphi(x)$ of the free boundary problem is a small $C^{1,\alpha}$ perturbation of the given subsonic shock solution $\varphi_0^+(x)$ in Ω^+. In particular, the gradient estimate implies that $\varphi(x)$ in fact satisfies the original free boundary problem, hence the transonic shock problem, Problem A (Problem B and Problem C, respectively).

The modified free boundary problem does not directly fit into the variational framework of Alt-Caffarelli [1] and Alt-Caffarelli-Friedman [2] and the regularization framework of Berestycki-Caffarelli-Nirenberg [5]. Also, the nonlinearity of (56) makes it difficult to apply the Harnack inequality approach of Caffarelli [7]. In particular, a boundary comparison principle for positive solutions of nonlinear elliptic equations in Lipschitz domains is unavailable yet for the equations that are not homogeneous with respect to D^2u, Du, and u, which however is our case.

The approach we develop is an iteration scheme based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound. Our iteration process is as follows: Suppose the domain Ω_k^+ is given so that $S_k := \partial\Omega_k^+ \setminus \partial\Omega$ is $C^{1,\alpha}$. We solve the oblique derivative problem in Ω_k^+ obtained by rewriting the (modified) equation (1) and free boundary condition (8) in terms of the function $u := \varphi - \varphi_0^+$. Then the problem has the following form:

$$\begin{array}{ll} \operatorname{div} A(x, Du) = F(x) & \text{in } \ \Omega_k^+ := \{u > 0\}, \\ A(x, Du) \cdot \nu = G(x, \nu) & \text{on } \ S := \partial\Omega_k^+ \setminus \partial\Omega, \end{array} \tag{56}$$

plus the fixed boundary conditions on $\partial\Omega_k^+ \cap \partial\Omega$ and the conditions at infinity. The equation is quasilinear, uniformly elliptic, $A(x, 0) \equiv 0$, while $G(x, \nu)$ has a certain structure. Let $u_k \in C^{1,\alpha}(\overline{\Omega_k^+})$ be the solution of (56). We estimate that $\|u_k\|_{C^{1,\alpha}(\Omega_k^+)}$ is small if the perturbation is small, where we actually use appropriate weighted Hölder norms in the unbounded domains. Then we extend the function $\varphi_k := \varphi_0^+ + u_k$ from Ω_k^+ to Ω so that the $C^{1,\alpha}$ norm of $\varphi_k - \varphi_0^+$ in Ω is controlled by $\|u_k\|_{C^{1,\alpha}}$. We define $\Omega_{k+1}^+ := \{x \in \Omega \mid \varphi^k(x) < \varphi^-(x)\}$ for the next step. Note that, since $\|\varphi_k - \varphi_0^+\|_{C^{1,\alpha}(\Omega)}$ and $\|\varphi^- - \varphi_0^-\|_{C^{1,\alpha}(\Omega)}$ are small, we have $|D\varphi^-| - |D\varphi_k| \geq \delta > 0$ in Ω, and this nondegeneracy implies that $S_{k+1} := \partial\Omega_{k+1}^+ \setminus \partial\Omega$ is $C^{1,\alpha}$ and its norm is estimated in terms of the data of the problem.

The fixed point Ω^+ of this process determines a solution of the free boundary problem since the corresponding solution $\varphi(x)$ satisfies $\Omega^+ = \{\varphi < \varphi^-\}$ and the Rankine-Hugoniot condition holds on $S := \partial\Omega^+ \cap \Omega$.

On the other hand, the elliptic estimates alone are not sufficient to get the existence of a fixed point, because the right-hand side of the boundary condition in the problem (56) depends on the unit normal ν of the free boundary. We use the following feature of the flat or spherical shocks:

$$\rho(|D\varphi_0^+|^2)D\varphi_0^+ = \rho(|D\varphi_0^-|^2)D\varphi_0^- \quad \text{in } \Omega \tag{57}$$

to obtain better estimates for the iteration and to prove the existence of a fixed point. Note that this is a vector identity, and the Rankine-Hugoniot condition (8) is the normal part of (57) on the non-perturbed free boundary S_0.

The uniqueness and stability of solutions for the transonic shock problems are obtained by using the regularity and nondegeneracy of solutions.

For more details, see Chen-Feldman [10].

Acknowledgments

Gui-Qiang Chen's research was supported in part by the National Science Foundation under Grants DMS-0204225, DMS-0204455, and INT-9987378. Mikhail Feldman's research was supported in part by the National Science Foundation under Grants DMS-0200644 and DMS-0074037.

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