STEADY TRANSONIC SHOCKS AND FREE BOUNDARY PROBLEMS IN INFINITE CYLINDERS FOR THE EULER EQUATIONS

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Abstract. We establish the existence and stability of multidimensional transonic shocks (hyperbolic-elliptic shocks) for the Euler equations for steady potential fluids in infinite cylinders. The Euler equations, consisting of the conservation law of mass and the Bernoulli law for velocity, can be written as a second order nonlinear equation of mixed elliptic-hyperbolic type for the velocity potential. The transonic shock problem in an infinite cylinder can be formulated into the following free boundary problem: The free boundary is the location of the multidimensional transonic shock which divides two regions of $C^{1,\alpha}$ flow in the infinite cylinder, and the equation is hyperbolic in the upstream region where the $C^{1,\alpha}$ perturbed flow is supersonic. We develop a nonlinear approach to deal with such a free boundary problem in order to solve the transonic shock problem in unbounded domains. Our results indicate that there exists a solution of the free boundary problem such that the equation is always elliptic in the unbounded downstream region, the uniform velocity state at infinity in the downstream direction is uniquely determined by the given hyperbolic phase, and the free boundary is $C^{1,\alpha}$, provided that the hyperbolic phase is close in $C^{1,\alpha}$ to a uniform flow. We further prove that the free boundary is stable under the steady $C^{1,\alpha}$ perturbation of the hyperbolic phase. We also prove that if the steady perturbation of the hyperbolic phase is $C^{2,\alpha}$, the free boundary is also $C^{2,\alpha}$, and the solution is unique

1. Introduction

We are concerned with the existence and stability of multidimensional transonic shocks for the Euler equations for compressible fluids in unbounded domains. In this paper, we focus on inviscid steady potential fluid flows, which are governed by the Euler equations consisting of the conservation law of mass and the Bernoulli law for velocity. Then the Euler equations for the velocity potential $\varphi : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ can be formulated into the following second order nonlinear equations of mixed elliptic-hyperbolic type:

$$\operatorname{div}\left(\rho(|D\varphi|^2)D\varphi\right) = 0, \tag{1}$$

where the density $\rho(q^2)$ is

$$\rho(q^2) = \left(1 - \theta q^2\right)^{\frac{1}{2\theta}}, \tag{2}$$

and $\theta = \frac{\gamma-1}{2} > 0$ with the adiabatic exponent $\gamma > 1$.

The second order nonlinear equation (1) is elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0, \tag{3}$$

and is hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \tag{4}$$

We establish the existence and stability of multidimensional transonic shocks (hyperbolic-elliptic shocks) for the Euler equations (1) in infinite cylinders under $C^{1,\alpha}, \alpha \in (0,1)$, steady

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perturbations of velocity potential of the given upstream supersonic flows for which the equation (1) is hyperbolic. Our results indicate that, for any upstream supersonic flow which is sufficiently close in $C^{1,\alpha}$ to a uniform flow, there exists a multidimensional transonic shock solution such that the equation is always elliptic in the unbounded downstream region, the shock surface dividing the two regions is $C^{1,\alpha}$ and is close to the non-perturbed shock, and the velocity state at infinity in the downstream direction is uniquely determined by the hyperbolic phase and is close to the nonperturbed velocity state. These results show that the solution is stable under the $C^{1,\alpha}$ steady perturbation of the hyperbolic phase. Moreover, these results imply that, given any supersonic solution $\varphi^-(x)$ in the upstream region, the necessary and sufficient condition for the existence of a transonic shock solution in the infinite cylinder is that the uniform velocity state at infinity in the downstream direction is a certain specific state, uniquely determined by $\varphi^-(x)$. In other words, given a supersonic solution $\varphi^-(x)$ in the upstream region, there is no transonic shock solution which contains a transonic shock dividing the cylinder into the subsonic and supersonic regions, provided that one expects a different uniform velocity state at infinity in the downstream direction. This means that, for such a shock problem, one can apriori prescribe the uniformity condition of the flow, but can not apriori prescribe a velocity state, at infinity in the downstream direction in general. We also prove the regularity of free boundaries, that is, if the steady perturbation of the hyperbolic phase is $C^{2,\alpha}$, the free boundary is also $C^{2,\alpha}$. Moreover, in this case the solution is unique, and satisfies some stronger stability properties.

To achieve the existence and stability results, we reduce the transonic shock problem to a corresponding free boundary problem for nonlinear elliptic equations in the infinite cylinder. The free boundary is the location of the multidimensional transonic shock which divides two regions of smooth flow in the infinite cylinder, and the free boundary condition is the Rankine-Hugoniot jump condition on the shock surface. The equation is hyperbolic in the upstream region where the given $C^{1,\alpha}$ smooth perturbed flow is supersonic. We seek the location of free boundary such that the free boundary condition on the surface holds, the equation (1) is elliptic in the downstream region, and the uniform velocity state at infinity in the downstream direction is uniquely determined by the given hyperbolic phase.

In order to solve this free boundary problem, we first consider a one-phase free boundary problem in the infinite cylinder for a nonlinear, uniformly elliptic equation obtained by a modification of (1), which is the same equation as (1) in a uniform elliptic region. We solve first the one-phase free boundary problem in a sequence of bounded cylinders, which approximates the infinite cylinder, and obtain uniform estimates of the corresponding approximate solutions in some weighted Hölder norms, independent of the sequence of bounded cylinders.

For solving the free boundary problem in a fixed bounded cylinder we employ the iteration scheme developed in Chen-Feldman [5], which is based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound. Although the elliptic estimates alone are not sufficient to get the convergence to a fixed point since the right-hand side of the free boundary condition depends on the unit normal to the free boundary (see (8) below), the structure of our problem allows to obtain better estimates for the iteration and to prove the existence of a fixed point.

To obtain the uniform estimates of the solutions in some appropriate weighted Hölder norms, we first obtain a uniform L^2 estimate of the gradient of solutions. From that, we obtain uniform L^∞ estimates first for the gradient of a solution, and finally for the solution itself. By a scaling argument, we obtain the uniform estimates of the solutions and their derivatives in the weighted Hölder norms. With these uniform bounds, we then let the bounded cylinders tend to the infinite cylinder and prove that the corresponding approximate solutions converge to a $C^{1,\alpha}$ solution of the one-phase free boundary problem in the infinite cylinder for the modified elliptic equation, which decays faster than any algebraic order in the downstream direction. Finally, our gradient estimate in L^∞ ensures that the solution is in fact a solution of the original problem. The uniqueness and stability of the solution are also established.

Some efforts have been made in solving the nonlinear equation (1) of mixed type. Shiffman [29], Bers [2], and Finn-Gilbarg [12] proved the existence and uniqueness of solutions for the problem of subsonic flows of (1) past an obstacle. Morawetz in [26] showed that the flows of (1) past the obstacle may contain transonic shocks in general. Non-transonic shock (hyperbolic-hyperbolic shock) where studied in [6, 16, 19, 23, 28, 31] and the references cited therein.

Transonic shocks were considered in Canic-Keyfitz-Lieberman [4] for the two-dimensional transonic small-disturbance (TSD) equation. Technically, the main difference between the TSD model and (1) is that the coefficients of (1) depend on the gradient of the unknown function, while the coefficients of the TSD equation are independent of the gradient of the unknown function that generates additional compactness of solutions on which the effective approach in [4] relies. For other related results, we refer the reader to Majda [24] and Métivier [25].

For some fixed boundary value problems for second order elliptic equations in infinite cylinders or unbounded domains, see Oleinik [27] where the cases of the linear equations or semilinear equations of a certain structure were studied, Berestycki-Caffarelli-Nirenberg [3] where the qualitative properties of semilinear equations of the form $\Delta u + f(u) = 0$ were considered, and the references cited therein.

In this paper, we first set up the multidimensional transonic shock problem (Problem A) and introduce the main theorems (Theorems 2.1 and 2.2) in Section 2. In order to achieve the main theorems, we reformulate Problem A into a nonlinear free boundary problem (Problem C) for the modified nonlinear, uniformly elliptic equation using the techniques of reflection and truncation in Section 3. In Sections 4–5, we introduce the iteration procedure to construct the approximate solutions of the reformulated free boundary problem (Problem C) in the finite approximate cylinders Ω_e^R and make the uniform estimates of the solutions independent of R. In Section 6, we prove that the approximate solutions strongly converge to a solution of the original free boundary problem (Problem A) in the infinite cylinder. The uniqueness and stability of $C^{1,\alpha}$ solutions of the free boundary problem are established in Section 7. In Section 8, we establish a regularity theorem which indicates that, if the hyperbolic perturbation is in $C^{2,\alpha}$, then the free boundary is also in $C^{2,\alpha}$.

2. Main Theorems and Transonic Shocks

In this section, we first set up the multidimensional transonic shock problem and present the main theorems of this paper.

A function $\varphi \in W^{1,\infty}(\Omega)$ is a weak solution of (1) in an unbounded domain Ω if

(i) $|D\varphi(x)| \le 1/\sqrt{\theta}$ a.e.

(ii) For any $\zeta \in C_0^\infty(\Omega)$,

$$\int_\Omega \rho(|D\varphi|^2) D\varphi \cdot D\zeta \, dx = 0. \tag{5}$$

We are interested in weak solutions with shocks. Let Ω^+ and Ω^- be open subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \emptyset, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega},$$

and $S = \partial\Omega^+ \cap \Omega$. Let $\varphi \in W^{1,\infty}(\Omega)$ be a weak solution of (1) and be in $C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ so that $D\varphi$ experiences a jump across S that is an $(n-1)$-dimensional smooth surface. Then φ satisfies the following Rankine-Hugoniot conditions on S:

$$\varphi^+ = \varphi^- \qquad \text{on } S. \tag{6}$$

$$\Big[\rho(|D\varphi|^2)D\varphi\cdot\nu\Big]_S = 0, \tag{7}$$

where ν is the unit normal to S from Ω^- to Ω^+, and the bracket denotes the difference between the values of the function along S on the $\Omega^\pm$ sides, respectively. We can also write (7) as

$$\rho(|D\varphi^+|^2)\varphi^+_\nu = \rho(|D\varphi^-|^2)D\varphi^-\cdot\nu \qquad \text{on } S, \tag{8}$$

where $\varphi^\pm_\nu = D\varphi^\pm\cdot\nu$ are the normal derivatives on the $\Omega^\pm$ sides, respectively.

Note that the function

$$\Phi_1(p) := \left(1-\theta p^2\right)^{\frac{1}{2\theta}} p \tag{9}$$

is continuous on $\left[0, \sqrt{1/\theta}\right]$, and satisfies

$$\Phi_1(p) > 0 \quad \text{for } p \in \left(0, \sqrt{1/\theta}\right), \quad \Phi_1(0) = \Phi_1\left(\sqrt{1/\theta}\right) = 0, \tag{10}$$

$$0 < \Phi_1'(p) < 1 \text{ on } (0, p^1_{sonic}), \tag{11}$$

$$\text{and } \Phi_1'(p) < 0 \quad \text{on } \left(p^1_{sonic}, \sqrt{1/\theta}\right),$$

$$\Phi_1''(p) < 0 \quad \text{on } \quad (0, p^1_{sonic}], \tag{12}$$

where

$$p^1_{sonic} = \sqrt{1/(\theta+1)}. \tag{13}$$

Suppose that $\varphi(x)$ is a solution satisfying

$$|D\varphi(x)| < p^1_{sonic} \quad \text{in } \Omega^+, \qquad |D\varphi(x)| > p^1_{sonic} \quad \text{in } \Omega^-, \tag{14}$$

besides (6) and (7). Then $\varphi(x)$ is a *transonic shock solution* with *transonic shock* S dividing Ω into the *subsonic region* Ω^+ and the *supersonic region* Ω^- and satisfying the physical entropy condition (see Courant-Friedrichs [7]; also see Lax [18]):

$$\rho(|D\varphi_-|^2) < \rho(|D\varphi_+|^2) \qquad \text{along } S.$$

Note that the equation (1) is elliptic in the subsonic region and is hyperbolic in the supersonic region.

Let (x', x_n) be the coordinates in $\mathbf{R}^n$, where $x_n \in \mathbf{R}$ and $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $q_0^- \in \left(p^1_{sonic}, 1/\sqrt{\theta}\right)$ and $\varphi_0^-(x) := q_0^- x_n$. Then $\varphi_0^-(x)$ is a supersonic solution in Ω. According to (10)–(11), there exists a unique $q_0^+ \in (0, p^1_{sonic})$ such that

$$\rho\left((q_0^+)^2\right) q_0^+ = \rho\left((q_0^-)^2\right) q_0^-. \tag{15}$$

Thus, the function

$$\varphi_0(x) = \begin{cases} q_0^- x_n, & x \in \Omega_0^- := \Omega \cap \{x \,:\, x_n < 0\}, \\ q_0^+ x_n, & x \in \Omega_0^+ := \Omega \cap \{x \,:\, x_n > 0\} \end{cases} \tag{16}$$

is a plane transonic shock solution in Ω, Ω_0^+ and Ω_0^- are its subsonic and supersonic regions, respectively, and $S = \{x_n = 0\}$ is a transonic shock.

Defining $\varphi_0^+(x) := q_0^+ x_n$ in Ω, we have

$$\varphi_0(x) = \min(\varphi_0^+(x), \varphi_0^-(x)), \qquad \text{in } \Omega. \tag{17}$$

In order to deal with multidimensional transonic shocks in the unbounded domain Ω, we define the following weighted Hölder semi-norms and norms in the domain with form

$\mathcal{D} = \Omega \cap \{x_n > f(x')\}$, where $f(x')$ is a Lipschitz function. Denote $\delta_x = |x_n| + 1$ for $x = (x', x_n) \in \mathcal{D}$, and $\delta_{x,y} = \min(\delta_x, \delta_y)$ for $x, y \in \mathcal{D}$. For $k \in \mathbf{R}$, $\alpha \in (0,1)$, and $m \in \mathbf{N}$ (the set of nonnegative integers), we define

$$
\begin{aligned}
&[u]^{(k)}_{m;0;\mathcal{D}} = \sum_{|\beta|=m} \sup_{x\in\mathcal{D}} \left(\delta_x^{m+k} |D^\beta u(x)|\right), \\
&[u]^{(k)}_{m;\alpha;\mathcal{D}} = \sum_{|\beta|=m} \sup_{x,y\in\mathcal{D}, x\neq y} \left(\delta_{x,y}^{m+\alpha+k} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}\right), \\
&\|u\|^{(k)}_{m;0;\mathcal{D}} = \sum_{j=0}^{m} [u]^{(k)}_{j;0;\mathcal{D}}, \\
&\|u\|^{(k)}_{m;\alpha;\mathcal{D}} = \|u\|^{(k)}_{m;0;\mathcal{D}} + [u]^{(k)}_{m;\alpha;\mathcal{D}},
\end{aligned}
\tag{18}
$$

where $D^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$, $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index with $\beta_j \geq 0, \beta_j \in \mathbf{N}$, and $|\beta| = \beta_1 + \cdots + \beta_n$.

We denote by $\|u\|_{m,\alpha,\mathcal{D}}$ the (non-weighted) Hölder norms in any domain $\mathcal{D}$, i.e., the norms defined as above with $\delta_x = \delta_{x,y} = 1$.

In this paper, we focus on an infinite cylinder $(0,a)^{n-1} \times (-\infty, \infty)$. Since it is not necessary to require that the supersonic perturbation $\varphi^-(x)$ be defined in the whole infinite cylinder, we introduce a finite subcylinder $\Omega_1 := (0,a)^{n-1} \times (-1,1)$ and focus on the cylinder domain $\Omega := (0,a)^{n-1} \times (-1,\infty)$ without loss of generality. Then our multidimensional transonic shock problem can be formulated into the following form:

Problem A. Given a supersonic weak solution $\varphi^-(x)$ of (1) in Ω_1, which is a $C^{1,\alpha}$ perturbation of $\varphi_0^-(x)$ for some $\alpha \in (0,1)$:

$$\|\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_1} \leq \sigma, \tag{19}$$

with $\sigma > 0$ small, and satisfies

$$\partial_\nu \varphi^- = 0, \qquad \text{on} \quad \partial(0,a)^{n-1} \times [-1,1], \tag{20}$$

find a transonic shock solution $\varphi(x)$ in Ω such that, denoting by $\Omega^+ := \{x \in \Omega \ : \ |D\varphi(x)| < p^1_{sonic}\}$ and $\Omega^- := \Omega \setminus \Omega^+$ the subsonic and supersonic regions of $\varphi(x)$, we have

$$\Omega^- \subset \Omega_1, \qquad \varphi = \varphi^- \text{ in } \Omega^-,$$

and

$$\varphi = \varphi^-, \ \varphi_{x_n} = \varphi^-_{x_n} \qquad \text{on } (0,a)^{n-1} \times \{-1\}, \tag{21}$$

$$\partial_\nu \varphi = 0 \qquad \text{on } \partial(0,a)^{n-1} \times [-1,\infty), \tag{22}$$

$$\|\varphi(\cdot, x_n) - \omega x_n\|_{L^\infty((0,a)^{n-1})} \to 0 \qquad \text{as } x_n \to \infty, \tag{23}$$

for some $\omega \in (0, p^1_{sonic})$.

Then we have

Theorem 2.1. *Let $q_0^+ \in (0, p^1_{sonic})$ and $q_0^- \in \left(p^1_{sonic}, 1/\sqrt{\theta}\right)$ satisfy (15), and let $\varphi_0(x)$ be the transonic shock solution (16). Then there exist $\sigma_0 > 0$, $\hat{C}$, and C depending only on n, a, α, γ, and q_0^+ such that, for every $\sigma \in (0, \sigma_0)$ and any supersonic solution $\varphi^-(x)$ of (1) satisfying the conditions stated in Problem A, there exists a solution $\varphi(x)$ of Problem A satisfying*

$$\|\varphi - q_0^+ x_n\|^{(-1)}_{1;\alpha;\Omega^+(\varphi)} \leq \hat{C}\sigma. \tag{24}$$

Moreover, such solution $\varphi(x)$ satisfies the following properties:

(i) *The constant* ω *in* (23) *must be* q^+*:*

$$\omega = q^+, \tag{25}$$

where q^+ *is the unique solution in the interval* $(0, p^1_{sonic})$ *of the equation*

$$\rho((q^+)^2)q^+ = Q^+ \tag{26}$$

with

$$Q^+ = \frac{1}{a^{n-1}} \int_{(0,a)^{n-1}} \rho(|D\varphi^-(x', -1)|^2)\varphi^-_{x_n}(x', -1)dx'. \tag{27}$$

Thus $\varphi(x)$ *satisfies*

$$\|\varphi(\cdot, x_n) - q^+x_n\|_{L^\infty((0,a)^{n-1})} \to 0 \quad \text{as } x_n \to \infty, \tag{28}$$

and q^+ *satisfies*

$$|q^+ - q_0^+| \leq C\sigma. \tag{29}$$

(ii) *The subsonic region* $\Omega^+(\varphi) := \{x \in \Omega \;:\; |D\varphi(x)| < p^1_{sonic}\}$ *is of the form:*

$$\Omega^+(\varphi) = \{x_n > f(x')\} \cap \Omega \quad \text{with} \qquad \begin{array}{l} f \in C^{1,\alpha}([0,a]^{n-1}), \\ |Df|_{\partial(0,a)^{n-1}} = 0, \end{array} \tag{30}$$

where f *satisfies*

$$\|f\|_{1,\alpha,[0,a]^{n-1}} \leq C\sigma. \tag{31}$$

(iii) *For every* $k = 1, 2, \ldots.$

$$\|\varphi - q^+x_n\|^{(k)}_{1;\alpha;\Omega^+(\varphi)} \leq C_k\sigma, \tag{32}$$

where C_k *depends only on* k*,* n*,* a*,* γ*, and* q_0^+*. That is, when* $x_n \to \infty$*,* $\varphi(x)$ *uniformly converges to the linear function* q^+x_n *with respect to* $x' \in (0,a)^{n-1}$ *at a rate faster than any algebraic order.*

Furthermore, if $\varphi^- \in C^{2,\alpha}(\overline{\Omega_1})$ *in addition to the previous assumptions, then* $\varphi \in C^{2,\alpha}(\overline{\Omega^+(\varphi)})$*,* $f \in C^{2,\alpha}([0,a]^{n-1})$*, and*

$$\|D^2 f\|_{0,\alpha,[0,a]^{n-1}} \leq C(n, a, \alpha, \gamma, q_0^+, \sigma, \|D\varphi^-\|_{2,\alpha,\Omega_1}) < \infty, \tag{33}$$

$$\|D^2\varphi\|^{(k)}_{0;\alpha;\Omega^+(\varphi)} \leq C(n, a, \alpha, \gamma, q_0^+, \sigma, k, \|D\varphi^-\|_{2,\alpha,\Omega_1}) < \infty \tag{34}$$

for $k = 1, 2, 3, \ldots.$

Remark 2.1. *Theorem* 2.1 *indicates that, given any supersonic solution* $\varphi^-(x)$ *in the upstream region, the necessary and sufficient condition for the existence of a transonic shock solution in the infinite cylinder is that the velocity state* $\omega\nu_0, \nu_0 = (0, \ldots, 0, 1)$*, at infinity in the downstream direction must be* $q^+\nu_0$ *determined uniquely by* $\varphi^-(x)$*. In other words, given a supersonic solution* $\varphi^-(x)$ *in the upstream region, there is no transonic shock solution that contains a transonic shock dividing the subsonic phase from the supersonic phase, provided that one expects a different velocity state at infinity from* $q^+\nu_0$ *in the downstream direction. This means that, for this problem, one can apriori prescribe the uniformity condition of the flow, but can not prescribe a velocity state, at infinity in the downstream direction in general; otherwise the problem is overdetermined.*

Furthermore, as a consequence of the uniqueness, non-degeneracy, and regularity of solutions of the free boundary problem, we have the following stability theorem.

Theorem 2.2. *There exist a constant* $\sigma_0 > 0$ *and a nonnegative nondecreasing function* $\Psi \in C([0,\infty))$ *with* $\Psi(0) = 0$*, depending only on* n, a, α, γ*, and* q_0^+*, such that, if* $\sigma < \sigma_0$*, then:*

If the supersonic solution φ^-*, in addition to the conditions of Problem A, satisfies*

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega_1} \leq \sigma, \tag{35}$$

then the solution φ of Problem A, satisfying (24), is unique.

Moreover, if $\varphi^-(x)$ satisfies (35), *and $\hat{\varphi}^-(x)$ satisfies*

$$\|\varphi^- - \hat{\varphi}^-\|_{2,\alpha,\Omega_1} \leq \kappa, \tag{36}$$

with $\kappa < \sigma$, then the unique solutions $\varphi(x)$ and $\hat{\varphi}(x)$ of Problem A for $\varphi^-(x)$ and $\hat{\varphi}^-(x)$, respectively, satisfy

$$\|f_\varphi - f_{\hat{\varphi}}\|_{2,\alpha,(0,a)^{n-1}} \leq \Psi(\kappa), \tag{37}$$

where $f_\varphi(x')$ and $f_{\hat{\varphi}}(x')$ are the free boundary functions of $\varphi(x)$ and $\hat{\varphi}(x)$ in (30), *respectively.*

3. Reformulation of the Problem

In order to achieve the main theorems, we reformulate Problem A into a free boundary problem in an extended domain in this section.

3.1. **Extension to the Domain $\Omega_e = \mathbf{T}^{n-1} \times (-1, \infty)$.** We first extend the domain Ω of the transonic shock problem to the domain Ω_e to overcome the difficulty when the transonic shock intersects the fixed boundaries of the cylinder.

Observe first that any function $\phi \in C^{1,\alpha}(\overline{\Omega})$ (resp. $\phi \in C^{2,\alpha}(\overline{\Omega})$) with $\Omega = (0,a)^{n-1} \times (-1, \infty)$ and

$$\phi_\nu = 0 \qquad \text{on } \partial(0,a)^{n-1} \times [-1, \infty) \tag{38}$$

can be extended to $\mathbf{R}^{n-1} \times [-1, \infty)$ so that the extension (still denoted) $\phi(x)$ satisfies

$$\phi \in C^{1,\alpha}(\mathbf{R}^{n-1} \times [-1,\infty)) \ \ (\text{resp. } \phi \in C^{2,\alpha}(\mathbf{R}^{n-1} \times [-1,\infty))),$$

and, for every $j = 1, \dots, n-1$ and $k = 0, \pm 1, \pm 2, \dots,$

$$\phi(x_1, \dots, x_{j-1}, ka - z, x_{j+1}, \dots, x_n) = \phi(x_1, \dots, x_{j-1}, ka + z, x_{j+1}, \dots, x_n), \tag{39}$$

that is, $\phi(x)$ is symmetric with respect to every hyperplane $\{x_j = ka\}$.

Indeed, for $\mathbf{k} = (k_1, \dots, k_{n-1}, 0)$ with $k_1, \dots, k_{n-1}$ integers, we define

$$\phi(x + a\mathbf{k}) = \phi(\eta(x_1, k_1), \dots, \eta(x_{n-1}, k_{n-1}), x_n) \quad \text{for} \quad x \in (0,a)^{n-1} \times [-1, \infty),$$

with

$$\eta(t,k) = \begin{cases} t, & \text{if } k \text{ is even}, \\ a - t, & \text{if } k \text{ is odd}. \end{cases}$$

It follows from (39) that $\phi(x', x_n)$ is $2a$-periodic in each variable $x_1, \dots, x_{n-1}$:

$$\phi(x + 2ae_j) = \phi(x) \qquad \text{for } x \in \mathbf{R}^{n-1} \times [-1, \infty), \ j = 1, \dots, n-1,$$

where e_j is the unit vector in the direction of x_j.

Thus, with respect to this $2a$-periodicity, we can consider $\phi(x)$ as a function on $\Omega_e := \mathbf{T}^{n-1} \times [-1, \infty)$, where $\mathbf{T}^{n-1}$ is an $(n-1)$-dimensional flat torus with its coordinates given by the cube $(0, 2a)^{n-1}$. Note that (39) represents an extra symmetry condition, in addition to $\phi \in C^{1,\alpha}(\mathbf{T}^{n-1} \times [-1, \infty))$, and (39) implies (38).

Therefore, we can extend $\varphi^-(x)$ in this way, that is, $\varphi^-(x)$ is defined in $\Omega_{1,e} := \mathbf{T}^{n-1} \times (-1, 1)$, $\varphi^- \in C^{1,\alpha}(\Omega_{1,e})$, and satisfies (39). Furthermore, we can modify $\varphi^-(x)$ on the set $\mathbf{T}^{n-1} \times [3/4, 1]$ so that the modified function $\tilde{\varphi}^-(x)$ satisfies (19) with constant 2σ and $\tilde{\varphi}^- = q_0^- x_n$ on $\mathbf{T}^{n-1} \times [7/8, 1]$, and extend further $\tilde{\varphi}^-(x)$ to $\mathbf{T}^{n-1} \times [-1, \infty)$ by defining $\tilde{\varphi}^- = q_0^- x_n$ on $\mathbf{T}^{n-1} \times [1, \infty)$. The function $\tilde{\varphi}^-(x)$ then satisfies $\tilde{\varphi}^- \in C^{1,\alpha}(\mathbf{T}^{n-1} \times [-1, \infty))$ and

$$\|\tilde{\varphi}^- - \varphi_0^-\|_{1,\alpha,\Omega_e} \leq 2\sigma. \tag{40}$$

Also, $\varphi_0^\pm(x)$ can be considered as the functions in Ω_e satisfying (39), since $\varphi_0^\pm(x) = q_0^\pm x_n$ in $\mathbf{R}^{n-1} \times [-1, \infty)$, which are independent of x'.

Furthermore, given a domain $\mathcal{D} = \Omega \cap \{x_n > f(x')\}$, where $f(x')$ is a Lipschitz function, we can extend it (by reflection as above) to a domain $\mathcal{D}_e \subset \Omega_e$ with $\mathcal{D}_e = \Omega_e \cap \{x_n > f(x')\}$, where $f(x')$ is an extended function, i.e., $f : \mathbf{T}^{n-1} \to \mathbf{R}$ is Lipschitz and satisfies (39).

3.2. **Free Boundary Problems.** In order to construct a solution of Problem A, we reformulate it into a free boundary problem for the subsonic part of the solution. The main point is to replace the pointwise gradient condition $\{|D\varphi(x)| < p^1_{sonic}\}$ defining $\Omega^+(\varphi)$ by a condition stated in terms of $\varphi(x)$ so that our problem is formulated into the framework of free boundary problems, as in [1]. The following heuristic observation motivates our formulation: By (17) and $q_0^- > q_0^+$, we have $\Omega^+(\varphi_0) = \{x \in \Omega \ : \ \varphi_0(x) < \varphi_0^-(x)\}$. Since $\varphi^-(x)$ is a small $C^{1,\alpha}$ perturbation of $\varphi_0^-(x)$ and $q_0^- > q_0^+$, then we expect that $\varphi^+(x)$ is close to $\varphi_0^+(x)$ in $C^{1,\alpha}(\overline{\Omega^+(\varphi)})$, so that we can expect that $\Omega^+(\varphi) = \{x \in \Omega \ : \ \varphi(x) < \varphi^-(x)\}$.

Then it suffices for Problem A to solve the following free boundary problem in the extended domain Ω_e:

Problem B. Find $\varphi \in C(\overline{\Omega_e})$ such that

(i) $\varphi(x)$ satisfies

$$\varphi \le \varphi^- \qquad \text{in } \ \Omega_e, \tag{41}$$

and the following conditions on the boundary and at infinity:

$$\varphi = \varphi^-, \quad \varphi_{x_n} = \varphi^-_{x_n} \qquad \text{on } \mathbf{T}^{n-1} \times \{-1\}, \tag{42}$$

$$\|\varphi(\cdot, x_n) - \omega x_n\|_{L^\infty(\mathbf{T}^{n-1})} \to 0 \quad \text{as } x_n \to \infty; \tag{43}$$

(ii) $\varphi \in C^{2,\alpha}(\Omega^+) \cap C^{1,\alpha}(\overline{\Omega^+})$ is a solution of (1) in $\Omega^+ := \{\varphi < \varphi^-\} \cap \Omega_e$, the non-coincidence set, and satisfies the symmetry condition (39);

(iii) the free boundary $S = \partial\Omega^+ \cap \Omega$ is given by the equation $x_n = f(x')$ for $x' \in \mathbf{T}^{n-1}$ so that $\Omega^+ = \{x_n > f(x')\}$, where $f \in C^{1,\alpha}(\mathbf{T}^{n-1})$ and satisfies the symmetry condition (39);

(iv) The free boundary condition (7) holds on S.

We further modify the equation and the free boundary condition by a truncation technique to convert the problem into a free boundary problem for a nonlinear, uniformly elliptic equation.

3.3. **Truncation of Equation (1).** We modify the equation (1) to make it uniformly elliptic, and so that it coincides with the non-modified equation in the range $D\varphi$ in Ω^+ for φ satisfying (24) with sufficiently small σ. in the subsonic region. The details of the truncation procedure are in [5, Section 4.2]. The resulting equation is the following:

Let

$$\varepsilon = \frac{p^1_{sonic} - q_0^+}{2}. \tag{44}$$

There exists $\tilde\rho \in C^{1,1}([0,\infty))$ and $c_0, c_1, c_2 > 0$ depending only on q_0^+ and γ, such that

$$\tilde\rho(q^2) = \rho(q^2), \qquad \text{if} \quad 0 \le q < p^1_{sonic} - \varepsilon, \tag{45}$$

$$\tilde\rho(q^2) = c_0 + \frac{c_1}{q}, \qquad \text{if} \quad q > p^1_{sonic} - \varepsilon, \tag{46}$$

$$0 < c_0 \le \big(\tilde\rho(q^2)q\big)' \le c_2 \qquad \text{for} \quad q \in (0,\infty). \tag{47}$$

Then the equation

$$\tilde{\mathcal{L}}\varphi := \operatorname{div}\left(\tilde\rho(|D\varphi|^2)D\varphi\right) = 0 \tag{48}$$

is uniformly elliptic, whose ellipticity constants depend only on q_0^+ and γ.

We also perform the corresponding truncation of the free boundary condition (8):

$$\tilde\rho(|D\varphi|^2)\varphi_\nu = \rho(|D\varphi^-|^2)D\varphi^- \cdot \nu \qquad \text{on} \quad S. \tag{49}$$

On the right-hand side of (49) we use the non-truncated function ρ since $\rho \neq \tilde{\rho}$ on the range of $|D\varphi^-|^2$. Note that (49), with the right-hand side considered as a known function, is the conormal boundary condition for the uniformly elliptic equation (48).

Thus, we first solve the following free boundary problem, which is a truncated version of Problem B:

> **Problem C**. Find $\varphi \in C(\overline{\Omega_e})$ such that
> (i) $\varphi(x)$ satisfies (41) in Ω_e, (42) on $\partial\Omega_e$, and (43) at infinity;
> (ii) $\varphi \in C^{2,\alpha}(\Omega^+) \cap C^{1,\alpha}(\overline{\Omega^+})$ is a solution of (48) in $\Omega^+ := \{\varphi < \varphi^-\} \cap \Omega_e$, the non-coincidence set, and satisfies the symmetry condition (39);
> (iii) the free boundary $S = \partial\Omega^+ \cap \Omega_e$ is given by the equation $x_n = f(x')$ for $x' \in \mathbf{T}^{n-1}$ so that $\Omega^+ = \{x_n > f(x')\} \cap \Omega_e$, where $f \in C^{1,\alpha}(\mathbf{T}^{n-1})$ satisfies the symmetry condition (39);
> (iv) the free boundary condition (49) holds on S.

In this paper we combine the iteration techniques of [5] with uniform estimate techniques in the weighted Hölder norms to construct a unique solution of Problem C. Denote $\Omega_e^R := \Omega_e \cap \{x_n \leq R\}$ for large $R > 1$. We first construct a solution of the free boundary problem in the domain Ω_e^R which satisfies the estimates independent of R, and then let R tend to infinity. Finally, we use an estimate for $|D\varphi|$ to conclude that the solution of the truncated problem (Problem C) is actually a solution of Problem B, hence, the original problem (Problem A).

4. Iteration Procedure and Uniform Estimates in Ω_e^R

First we prove existence of the unique $q^+ \in (0, p^1_{sonic})$ satisfying

Lemma 4.1. *There exist σ_0 and C depending only on n, a, α, γ, and q_0^+ such that the equation (26) with Q^+ defined by (27) has a unique solution $q^+ \in (0, p^1_{sonic})$. Moreover, q^+ satisfies* (29).

Proof. We first estimate Q^+ defined by (27).

$$\begin{aligned} |Q^+ \;\; - & \;\; \rho((q_0^+)^2)q_0^+| \\ = & \;\; \frac{1}{a^{n-1}}\int_{(0,a)^{n-1}} \left|\rho(|D\varphi^-(x',-1)|^2)\varphi^-_{x_n}(x',-1) - \rho((q_0^+)^2)q_0^+\right| dx' \\ = & \;\; \frac{1}{a^{n-1}}\int_{(0,a)^{n-1}} \left|\rho(|D\varphi^-(x',-1)|^2)\varphi^-_{x_n}(x',-1) - \rho((q_0^-)^2)q_0^-\right| dx' \\ \leq & \;\; C\sigma, \end{aligned}$$

where we used (19), (15).

Thus, by (10) and (11) we obtain that, if σ is small depending only on the data, then there exists a unique solution $q^+ \in (0, p^1_{sonic})$ of the equation (93). It also follows that (29) holds. □

We now introduce an iteration procedure to construct approximate solutions of Problem C in the finite cylinders Ω_e^R and make uniform estimates of the solutions independent of R.

Let $M \geq 1$. We set

$$\mathcal{K}_M(R) := \left\{ \psi \in C^{1,\alpha}(\overline{\Omega_e^R}) \;\middle|\; \begin{array}{l} \psi \text{ satisfies (39)}, \\ \|\psi - q^+x_n\|^{(0)}_{1,\alpha,\Omega_e^R} \leq M\sigma \end{array} \right\}, \tag{50}$$

where q^+ is defined in Lemma 4.1. From the definition, $\mathcal{K}_M(R)$ is convex and compact in $C^{1,\beta}(\Omega_e^R)$ for $0 < \beta < \alpha$.

4.1. **Construction of the Iteration Scheme.** Let $\psi \in \mathcal{K}_M(R)$. Since $q_0^- > q_0^+$, it follows that, if

$$\sigma \le \frac{q_0^- - q_0^+}{C(M+1)}, \tag{51}$$

with large C depending only on n, then (50) and (19) imply

$$(\varphi^- - \psi)_{x_n}(x) \ge \frac{q_0^- - q_0^+}{2} > 0. \tag{52}$$

Then the set $\Omega_R^+(\psi) := \{\psi < \varphi^-\} \cap \Omega_e^R$ has the form:

$$\Omega^+(\psi) = \{x_n > f(x')\} \cap \Omega_e, \qquad \|f\|_{1,\alpha,\mathbf{T}^{n-1}} \le CM\sigma, \tag{53}$$

with C depending only on $q_0^- - q_0^+$. The corresponding unit normal $\nu_\psi(x')$ is

$$\nu_\psi(x') = \frac{(-Df(x'),1)}{\sqrt{1+|Df(x')|^2}} \in C^\alpha(\mathbf{T}^{n-1};\mathbf{S}^{n-1}),$$

and

$$\|\nu_\psi - \nu_0\|_{0,\alpha,\mathbf{T}^{n-1}} \le CM\sigma, \tag{54}$$

where ν_0 is defined by

$$\nu_0 := \frac{D\varphi_0^+}{|D\varphi_0^+|} = (0,\dots,0,1)^\top. \tag{55}$$

From the definition of $f(x')$, $\nu_\psi(\cdot)$ can be considered as a function on $S_f := \{x_n = f(x')\}$:

$$\nu_\psi(x) = \frac{D\varphi^-(x) - D\psi(x)}{|D\varphi^-(x) - D\psi(x)|} \qquad \text{for } x \in S_f. \tag{56}$$

By the definition of $\mathcal{K}_M(R)$ and (51), the formula (56) also defines $\nu_\psi(x)$ on Ω_e^1 and

$$\|\nu_\psi - \nu_0\|_{0,\alpha,\Omega_e^1} \le CM\sigma, \qquad \text{with } C = C(q_0^+, q_0^-). \tag{57}$$

Motivated by (49), we define the function

$$G_\psi(x) := \rho(|D\varphi^-(x)|^2) D\varphi^-(x)\cdot\nu_\psi(x) \qquad \text{on } \Omega_e^1. \tag{58}$$

Then we consider the following elliptic problem in the domain $\Omega_R^+(\psi)$:

$$\operatorname{div}(\tilde\rho(|D\varphi|^2)D\varphi) = 0 \qquad \text{in} \quad \Omega_R^+(\psi), \tag{59}$$

$$\tilde\rho(|D\varphi|^2)\varphi_\nu = G_\psi(x) \qquad \text{on} \quad S_f := \{x_n = f(x')\}, \tag{60}$$

$$\varphi = Rq^+ \qquad \text{on} \quad \partial\Omega_R^+(\psi)\setminus S_f \equiv \{x_n = R\}, \tag{61}$$

and show that the solution $\varphi(x)$ can be extended to the domain Ω_e^R so that $\varphi \in \mathcal{K}_M(R)$.

4.2. **Uniform Estimates of Solutions of the Elliptic Problem** (59)–(61) **in** Ω^R**.** Now our main objective is to obtain uniform apriori estimates independent of R for a weak solution of the conormal derivative problem (59)–(61) in the domain:

$$\Omega^R = \{(x', x_n) \in \mathbf{T}^{n-1}\times\mathbf{R} \;:\; f(x') < x_n < R\}, \;\text{ with }\; f \in C^{1,\alpha}(\mathbf{T}^{n-1}),$$

where $R > 3(\|f\|_{L^\infty} + 4)$.

We denote $\nu(x') = \nu_\psi(x')$ the inner unit normal to S_f with respect to domain Ω^R and $r_0 = 2(\|f\|_{L^\infty} + 2)$.

Lemma 4.2. *Let $u \in C^1(\overline{\Omega^R}) \cap C^2(\Omega^R)$ be a solution of*

$$\begin{array}{ll} \mathcal{N}(u) := \operatorname{div}(A(Du)) = 0 & \textit{in } \Omega^R, \\ A(Du)\cdot\nu = g & \textit{on } S_f, \\ u = 0 & \textit{on } \{x_n = R\}. \end{array} \tag{62}$$

Let $A = (A^1, \ldots, A^n) \in C^{1,\alpha}(\mathbf{R}^n; \mathbf{R}^n)$ *satisfy*

$$A(0) = 0, \tag{63}$$

and the ellipticity condition:

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} A^i_{P_j}(P)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for any} \quad P, \xi \in \mathbf{R}^n, \tag{64}$$

where $\Lambda > \lambda > 0$ *are constants, and*

$$A^i_{P_j}(P) = A^j_{P_i}(P), \tag{65}$$

$$(1 + |P|)|D_P A^i_{P_j}(P)| \leq \Lambda, \tag{66}$$

for any $P \in \mathbf{R}^n$ *and* $i, j = 1, \ldots, n$. *Let the function* $g \in C^\alpha(\overline{\Omega^{r_0}})$ *satisfy*

$$\int_{S_f} g \, d\mathcal{H}^{n-1} = 0. \tag{67}$$

Then

$$\|Du\|_{L^2(\Omega^R)} \leq C\|g\|_{0,\alpha,\Omega^{r_0}}, \tag{68}$$

where C *depends only on the data, i.e.* λ, Λ, n, *and* $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$, *but is independent of* R.

Proof. Since $u(x)$ is a solution of (62), then, for any $\phi \in C^1(\overline{\Omega^R})$,

$$\begin{aligned}
\int_{\Omega^R} A(Du) \cdot D\phi \, dx &= \int_{\mathbf{T}^{n-1}} A^n(Du(x', R))\phi(x', R) \, dx' - \int_{S_f} A(Du) \cdot \nu \, \phi \, d\mathcal{H}^{n-1} \\
&= \int_{\mathbf{T}^{n-1}} A^n(Du(x', R))\phi(x', R) \, dx' - \int_{S_f} g \, \phi \, d\mathcal{H}^{n-1}.
\end{aligned}$$

In particular, choosing $\phi = u$ and using $u(\cdot, R) \equiv 0$, we get

$$\int_{\Omega^R} A(Du) \cdot Du \, dx = -\int_{S_f} gu \, d\mathcal{H}^{n-1} = -\int_{S_f} g(u - Q) \, d\mathcal{H}^{n-1},$$

for any $Q \in \mathbf{R}$, where we used (67) in the last equality. Choosing $Q = (u)_{\Omega^{r_0}} := \dfrac{1}{|\Omega^{r_0}|}\displaystyle\int_{\Omega^{r_0}} u(x)dx$ and using the L^2 estimates of the boundary traces of functions in the Sobolev space $H^1(\Omega^{r_0})$, we get

$$\begin{aligned}
\int_{\Omega^R} A(Du) \cdot Du \, dx &\leq \left(\int_{S_f} g^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \left(\int_{S_f} (u - (u)_{\Omega^{r_0}})^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{S_f} g^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \left(\int_{\Omega^{r_0}} ((u - (u)_{\Omega^{r_0}})^2 + |Du|^2) dx\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{S_f} g^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \left(\int_{\Omega^{r_0}} |Du|^2 dx\right)^{\frac{1}{2}},
\end{aligned} \tag{69}$$

where we used the Poincaré inequality in the domain Ω^{r_0} in the last estimate, and the constants C depend only on n and $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$.

Now, using (63), we get

$$A(P) \cdot P = (A(P) - A(0)) \cdot P = \int_0^1 \left(\sum_{i,j=1}^{n} A^i_{P_j}(tP) \, P_i P_j\right) dt \geq \lambda|P|^2,$$

for any $P \in \mathbf{R}^n$. Combining this with estimate (69), we obtain

$$\int_{\Omega^R} |Du|^2 dx \leq \frac{1}{\lambda} \int_{\Omega^R} A(Du) \cdot Du\, dx \leq \frac{C}{\lambda} \left(\int_{S_f} g^2 d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \left(\int_{\Omega^{r_0}} |Du|^2 dx \right)^{\frac{1}{2}}.$$

Thus, we have

$$\|Du\|_{L^2(\Omega^R)} \leq C\|g\|_{L^2(S_f)} \leq C\|g\|_{0,\alpha,\Omega^{r_0}}.$$

□

Lemma 4.3. *Let $u(x)$, $A(P)$, and $g(x)$ be as in Lemma 4.2. Then there exists C depending only on λ, Λ, n, and $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$ such that*

$$\max_{\Omega^R} |u| \leq C\|g\|_{0,\alpha,\Omega^{r_0}}. \tag{70}$$

Proof. 1. Let $v := \partial_{x_k} u(x)$ for $k \in \{1, \dots, n\}$. Then $v(x)$ is a weak solution of the linear equation of divergence form:

$$\operatorname{div}(B(x)Dv) = 0 \qquad \text{in } \Omega^R,$$

where $B(x) = (A^i_{P_j}(Du(x)))$ is an $n \times n$ matrix. Clearly, the equation is elliptic with constants λ and Λ. Note that

$$\|v\|_{L^2(\Omega^R)} \leq \|Du\|_{L^2(\Omega^R)} \leq C\|g\|_{0,\alpha,\Omega^{r_0}},$$

by Lemma 4.2.

Since $r_0 = 2(\|f\|_{L^\infty} + 2) < R$, then, using the DeGiorgi-Nash-Moser estimate [14, Theorem 8.17] in the ball $B_1(x_*)$ for $x_* = (x', r_0)$ with $x' \in \mathbf{T}^{n-1}$ yields

$$|v(x', r_0)| \leq C\|v\|_{L^2(B_1(x_*))},$$

where C depends only on n and Λ/λ, which implies

$$\sup_{x' \in \mathbf{T}^{n-1}} |Du(x', r_0)| \leq C\|v\|_{L^2(\Omega^R)} \leq C\|g\|_{0,\alpha,\Omega^{r_0}}. \tag{71}$$

2. Now we show that there exists $x' \in \mathbf{T}^{n-1}$ such that $u(x', r_0) = 0$. On the contrary, we can assume

$$u(x', r_0) \geq \kappa > 0 \qquad \text{for any } \ x' \in \mathbf{T}^{n-1},$$

since the other case can be handled in a similar fashion. Let

$$\hat{w}(x) = -\frac{\kappa}{R - r_0} x_n + \frac{\kappa R}{R - r_0}. \tag{72}$$

Then

$$\begin{array}{ll} \operatorname{div}(A(D\hat{w})) = 0 & \text{in } \mathbf{T}^{n-1} \times (r_0, R), \\ \hat{w} = \kappa \leq u & \text{on } \{x_n = r_0\}, \\ \hat{w} = 0 = u & \text{on } \{x_n = R\}. \end{array}$$

By the maximum principle, we must have $u \geq \hat{w}$ in $\mathbf{T}^{n-1} \times (r_0, R)$. Hence

$$u_{x_n} \leq \hat{w}_{x_n} = -\frac{\kappa}{R - r_0} < 0 \qquad \text{on } \{x_n = R\}. \tag{73}$$

For any $a < 0$, using (63) and the ellipticity of $A(P)$ yields

$$\begin{aligned} A(ae_n) \cdot e_n &= (A(ae_n) - A(0)) \cdot e_n \\ &= a \int_0^1 \left(\sum_{i,j=1}^n A^i_{P_j}(tae_n)(e_n)_i(e_n)_j \right) dt \leq \lambda a < 0. \end{aligned}$$

Therefore, by (73),

$$\begin{aligned} A^n(u_{x_n}(x',R)e_n) &= A(u_{x_n}(x',R)e_n)\cdot e_n \\ &\le \lambda u_{x_n}(x',R) \le -\frac{\kappa}{R-r_0}\lambda < 0. \end{aligned} \tag{74}$$

On the other hand, since $u \in C^{1,\alpha}(\overline{\Omega^R})$ is a weak solution of (62), we use (67) to get

$$\begin{aligned} 0 &= \int_{\mathbf{T}^{n-1}} A^n(Du(x',R))dx' - \int_{S_f} A(Du)\cdot\nu\, d\mathcal{H}^{n-1} \\ &= \int_{\mathbf{T}^{n-1}} A^n(u_{x_n}(x',R)e_n)dx', \end{aligned}$$

which leads to a contradiction with (74).

Thus, $u(x',r_0)=0$ for some $x'\in\mathbf{T}^{n-1}$. Then, by (71), we obtain $|u|\le C\|g\|_{0,\alpha,\Omega^{r_0}}$ on $\{x_n=r_0\}$. Thus, by the maximum principle,

$$|u| \le C\|g\|_{0,\alpha,\Omega^{r_0}} \qquad \text{on } \mathbf{T}^{n-1}\times[r_0,R]. \tag{75}$$

3. Now it remains to bound u on Ω^{r_0}. Note that $u(x)$ satisfies

$$\begin{array}{ll} \operatorname{div}(A(Du))=0 & \text{in } \Omega^{r_0}, \\ A(Du)\cdot\nu = g & \text{on } S_f, \\ |u|\le K := C\|g\|_{0,\alpha,\Omega^{r_0}} & \text{on } \{x_n=r_0\}, \end{array}$$

where the inequality in the last line follows from (75). Since $S_f\in C^{1,\alpha}$, we can transform the above problem, by a map $\Phi:\Omega^{r_0}\to\mathbf{T}^{n-1}\times(0,1)$ such that the $C^{1,\alpha}$ norms of Φ and Φ^{-1} depend only on n and $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$, into the following problem for $w:=u\circ\Phi^{-1}\in C^2(\mathbf{T}^{n-1}\times(0,1])\cap C^1(\mathbf{T}^{n-1}\times[0,1])$:

$$\begin{array}{ll} \operatorname{div}(B(x,Dw))=0 & \text{in } \mathbf{T}^{n-1}\times(0,1), \\ B^n(x,Dw)=\tilde g & \text{on } \{x_n=0\}, \\ |w|\le K & \text{on } \{x_n=1\}. \end{array}$$

Here $B(x,P)$ is defined by

$$B(x,P) = \det(D\Phi(x))D\Phi^{-1}(\Phi(x))A\left(D\Phi^{-1}(\Phi(x))P\right),$$

and thus satisfies

$$\begin{aligned} &B(x,0)=0 \quad \text{for any} \quad x\in\mathbf{T}^{n-1}\times(0,1), \\ &\tilde\lambda|\xi|^2 \le \sum_{i,j=1}^n B^i_{P_j}(x,P)\xi_i\xi_j \le \tilde\Lambda|\xi|^2 \quad \text{for any } x\in\mathbf{T}^{n-1}\times(0,1),\ \xi\in\mathbf{R}^n, \end{aligned}$$

where $\tilde\Lambda\ge\tilde\lambda>0$ depend only on λ,Λ,n, and $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$, and the function $x\to B(\cdot,P)$ has a $C^\alpha(\mathbf{T}^{n-1}\times[0,1])$ norm depending only on $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$ and the constant in (66), but independent of $P\in\mathbf{R}^n$. Moreover, $B(x,P)$ satisfies (66) with constant Λ independent of $x\in\mathbf{T}^{n-1}\times(0,1)$. Also,

$$\|\tilde g\|_{0,\alpha,\mathbf{T}^{n-1}} \le C(\|f\|_{1,\alpha,\mathbf{T}^{n-1}},n)\|g\|_{0,\alpha,\Omega^{r_0}}.$$

4. The properties of $B(x,P)$ imply that $w(x)$ is a solution of the linear problem:

$$\begin{array}{ll} \displaystyle\sum_{i,j=1}^n (b_{ij}(x)w_{x_j})_{x_i}=0 & \text{in } \mathbf{T}^{n-1}\times(0,1), \\ \displaystyle\sum_{j=1}^n b_{nj}(x)w_{x_j}=\tilde g & \text{on } \{x_n=0\}, \\ |w|\le K & \text{on } \{x_n=1\}, \end{array}$$

where

$$b_{ij}(x) = b_{ji}(x) := \int_0^1 B^i_{P_j}(x, \tau Dw(x))d\tau, \qquad i, j = 1, \dots, n,$$

satisfy the uniform ellipticity condition with constants $\tilde{\lambda}$ and $\tilde{\Lambda}$.

Finally, there exists a function $\tilde{\psi} \in C^{1,\alpha}(\mathbf{T}^{n-1} \times [0,1]) \cap C^2(\mathbf{T}^{n-1} \times (0,1])$ such that

$$\begin{aligned}
&\|\tilde{\psi}\|_{0,\alpha,\mathbf{T}^{n-1}\times(0,1)} \le C(n)\|\tilde{g}\|_{0,\alpha,\mathbf{T}^{n-1}} \le C(\|f\|_{1,\alpha,\mathbf{T}^{n-1}}, n)\|g\|_{0,\alpha,\Omega^{r_0}},\\
&\sum_{j=1}^n b_{nj}(x)\psi_{x_j} = \tilde{g} \quad \text{on } \{x_n = 0\}.
\end{aligned}$$

Such a function $\tilde{\psi}$ exists (see [14, (6.68)]) with b_{nn} in the place of b_n, in virtue of $b_{nn} \ge \tilde{\lambda} > 0$). Then $\tilde{w} := w - \tilde{\psi}$ satisfies

$$\begin{aligned}
&\sum_{i,j=1}^n (b_{ij}(x)\tilde{w}_{x_j})_{x_i} = \sum_{i,j=1}^n (b_{ij}(x)\tilde{\psi}_{x_j})_{x_i} && \text{in } \mathbf{T}^{n-1} \times (0,1),\\
&\sum_{j=1}^n b_{nj}(x)\tilde{w}_{x_j} = 0 && \text{on } \{x_n = 0\},\\
&|\tilde{w}| \le C(\|f\|_{1,\alpha,\mathbf{T}^{n-1}}, n)\|g\|_{0,\alpha,\Omega^{r_0}} && \text{on } \{x_n = 1\}.
\end{aligned}$$

Extend b_{ij}, $\tilde{w}$, and $\tilde{\psi}$ to $\mathbf{T}^{n-1} \times (-1,1)$ by defining, for $(x', x_n) \in \mathbf{T}^{n-1} \times (0,1)$,

$$\begin{aligned}
&\tilde{w}(x', -x_n) := \tilde{w}(x', x_n),\\
&\tilde{\psi}(x', -x_n) := \tilde{\psi}(x', x_n),\\
&b_{ij}(x', -x_n) := b_{ij}(x', x_n) \quad \text{for } i, j = 1, \dots, n-1,\\
&b_{in}(x', -x_n) = b_{ni}(x', -x_n) := -b_{in}(x', x_n) \quad \text{for } i = 1, \dots, n-1,\\
&b_{nn}(x', -x_n) := b_{nn}(x', x_n).
\end{aligned}$$

Then $\tilde{w}, \tilde{\psi} \in H^1(\mathbf{T}^{n-1} \times (-1,1)) \cap C(\mathbf{T}^{n-1} \times [-1,1])$, where $H^1(\cdot)$ is the Sobolev space, and $b_{ij} \in L^\infty(\mathbf{T}^{n-1} \times (-1,1))$, $i, j = 1, \dots, n$, satisfy the $(\tilde{\lambda}, \tilde{\Lambda})$-ellipticity condition. Also,

$$\begin{aligned}
&\sum_{i,j=1}^n (b_{ij}(x)\tilde{w}_{x_j})_{x_i} = \sum_{i,j=1}^n (b_{ij}(x)\tilde{\psi}_{x_j})_{x_i} && \text{in } \mathbf{T}^{n-1} \times (-1,1),\\
&|\tilde{w}| \le C(\|f\|_{1,\alpha,\mathbf{T}^{n-1}}, n)\|g\|_{0,\alpha,\Omega^{r_0}} && \text{on } \{x_n = \pm 1\},
\end{aligned}$$

where the equation holds in the weak sense. Now, [14, Theorem 8.15] implies

$$\|\tilde{w}\|_{L^\infty(\mathbf{T}^{n-1}\times(-1,1))} \le C\|g\|_{0,\alpha,\Omega^{r_0}},$$

where C depends only on λ, Λ, n, and $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$. Therefore, a similar bound holds for $w = \tilde{w} + \tilde{\psi}$ in $\mathbf{T}^{n-1} \times (0,1)$ and hence for $u = w \circ \Phi$ in Ω^{r_0}. This completes the proof. □

In the proofs of Lemmas 4.2 and 4.3, we have employed some ideas in the proof of [10, Lemma 2.1] by Evans-Gangbo.

Remark 4.1. *The property proved in Step 2 in the proof of Lemma 4.3 is also true for the solutions of equations with variable coefficients. In particular, later we will need the following. Let $u \in C^1(\overline{\Omega_R}) \cap C^2(\Omega_R)$ satisfy*

$$\begin{aligned}
&\sum_{i,j=1}^n (a_{ij}(x)u_{x_j})_{x_i} = 0 \qquad \textit{in } \Omega^R,\\
&\sum_{i,j=1}^n a_{ij}u_{x_i}\nu_j = g \qquad \textit{on } S_f,\\
&u = 0 \qquad \textit{on } \{x_n = R\},
\end{aligned}$$

where $a_{ij} \in C^{1,\alpha}(\overline{\Omega_R})$ *satisfy* $a_{ij} = a_{ji}$ *and the ellipticity condition:*

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \textit{for any } x \in \Omega^R,\ \xi \in \mathbf{R}^n,$$

with $0 < \lambda < \Lambda$, *and* Ω^R *and* g *are as in Lemma* 4.2. *Then, for any* $r \in (1, R)$, *there exists* $x' \in \mathbf{T}^{n-1}$, *depending on* r, *such that*

$$u(x', r) = 0.$$

The proof basically repeats the arguments of Step 2 in the proof of Lemma 4.3. *The only difference is that, instead of* $\hat{w}(x)$ *defined by* (72), *we use the following function*

$$\hat{w}(x) = \frac{\kappa}{e^{Mr}}\left(e^{-Mx_n} - e^{-MR}\right),$$

where $M = \frac{1}{\lambda}\sum_{i=1}^{n}\|\partial_{x_n}a_{in}\|_{L^\infty(\Omega^R)}$. *Then, using* $a_{nn} \ge \lambda$, *we get*

$$\sum_{i,j=1}^{n}(a_{ij}\hat{w}_{x_j})_{x_i} = \kappa M e^{-M(x_n - r)}\left(M a_{nn} - \sum_i \partial_{x_n}a_{in}\right) \ge 0$$

in $\mathbf{T}^{n-1} \times (r, R)$, *i.e.,* $\hat{w}$ *is a subsolution in* $\mathbf{T}^{n-1} \times (r, R)$. *Then, assuming that* $u \ge \kappa \ge 0$ *on* $\{x_n = r\}$, *we obtain that, for any* $x' \in \mathbf{T}^{n-1}$,

$$\begin{aligned} \hat{w}(x', r) &= \kappa\left(1 - e^{-M(R-r)}\right) \le \kappa \le u(x', r), \\ \hat{w}(x', R) &= 0 = u(x', R). \end{aligned}$$

Thus $u \ge \bar{w}$ *on* $\mathbf{T}^{n-1} \times (r, R)$, *which implies*

$$\partial_{x_n}u(x', R) \le \partial_{x_n}\bar{w}(x', R) = -\kappa M e^{-M(R-r)} < 0 \quad \textit{for any } x' \in \mathbf{T}^{n-1}.$$

Then, following the arguments of Step 2, *we arrive at the result.*

In the next lemmas, we will use the norms defined in (18).

Lemma 4.4. *Let* $u(x)$, $A(P)$, $f(x)$, *and* $g(x)$ *be as in Lemma* 4.2. *Then, for any* $N > 0$, *there exists a constant* C *depending only on* n, N, λ, Λ, *and* $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$ *(but is independent of* R*) such that, if*

$$\|g\|_{0,\alpha,\Omega^{r_0}} \le N, \tag{76}$$

then

$$\|u\|^{(0)}_{1;\alpha;\Omega^R} \le C\|g\|_{0,\alpha,\Omega^{r_0}}. \tag{77}$$

Furthermore, for $m \in \{2, 3, \dots\}$, *if*

$$g \in C^{m-1,\alpha}(\overline{\Omega^{r_0}}), \quad f \in C^{m,\alpha}(\mathbf{T}^{n-1}), \quad \|g\|_{m-1,\alpha,\Omega^{r_0}} \le N, \tag{78}$$

then

$$\|u\|^{(0)}_{m;\alpha;\Omega^R} \le C\|g\|_{m-1,\alpha,\Omega^{r_0}}, \tag{79}$$

where C *depends only on* m, n, N, λ, Λ, *and* $\|f\|_{m,\alpha,\mathbf{T}^{n-1}}$.

Proof. 1. First we estimate the solution near S_f, say, in Ω^{r_0}. If $f \in C^{1,\alpha}(\mathbf{T}^{n-1})$ and $g \in C^\alpha(\overline{\Omega^{r_0}})$, then the estimates of [21] imply

$$\|u\|_{1,\alpha,\Omega^{r_0}} \le C,$$

where C depends only on n, λ, Λ, $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$, $\|g\|_{0,\alpha,\Omega^{r_0}}$, and $\|u\|_{L^\infty(\Omega^{r_0})}$. Since Lemma 4.3 provides the estimate of $\|u\|_{L^\infty(\Omega^R)}$, then C depends only on n, λ, Λ, $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$, and $\|g\|_{0,\alpha,\Omega^{r_0}}$.

Now we note that $u(x)$ satisfies

$$\begin{aligned}
&\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_j})_{x_i} = 0 \qquad \text{in } \Omega^{r_0}, \\
&\sum_{i,j=1}^{n} a_{ij}u_{x_i}\nu_j = g \qquad \text{on } S_f,
\end{aligned} \tag{80}$$

with C^α coefficients

$$a_{ij}(x) = \int_0^1 A^i_{P_j}(tDu(x))dt, \qquad i,j = 1,2,\dots,n.$$

Notice that the equation is elliptic with ellipticity constants λ and Λ, and the coefficients are Hölder since the estimate of $\|u\|_{1,\alpha,\Omega^{r_0}}$ above and (76) imply

$$\|a_{ij}\|_{0,\alpha,\Omega^{r_0}} \le C(n,N,\lambda,\Lambda,\|f\|_{1,\alpha,\mathbf{T}^{n-1}}).$$

Let $h := g/\|g\|_{0,\alpha,\Omega^{r_0}}$, and let $v(x)$ be a solution of the linear problem:

$$\begin{aligned}
&\sum_{i,j=1}^{n} (a_{ij}(x)v_{x_j})_{x_i} = 0 \qquad \text{in } \Omega^{r_0}, \\
&\sum_{i,j=1}^{n} a_{ij}v_{x_i}\nu_j = g/\|g\|_{0,\alpha,\Omega^{r_0}} \qquad \text{on } S_f, \\
&v = u/\|g\|_{0,\alpha,\Omega^{r_0}} \qquad \text{on } \{x_n = r_0\}.
\end{aligned}$$

Then $u = \|g\|_{0,\alpha,\Omega^{r_0}}v$. In particular, by (70), $\|v\|_{L^\infty(\Omega^{r_0})} \le C$, where C depends only on λ, Λ, n, and $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$. Thus, by the estimates of [21], we estimate $\|v\|_{1,\alpha,\Omega^{r_0}}$ in terms of the above constants and N to have

$$\|u\|_{1,\alpha,\Omega^{r_0}} \le C\|g\|_{0,\alpha,\Omega^{r_0}}.$$

If $f \in C^{2,\alpha}(\mathbf{T}^{n-1})$ and $g \in C^{1,\alpha}(\overline{\Omega^{r_0}})$, we use the local estimates for the oblique derivative problems from [22] to get

$$\|u\|_{2,\alpha,\Omega^{r_0}} \le C,$$

where C depends only on n, λ, Λ, $\|f\|_{2,\alpha,\mathbf{T}^{n-1}}$, and $\|u\|_{L^\infty}$, and hence, by Lemma 4.3, only on n, N, λ, Λ, and $\|f\|_{2,\alpha,\mathbf{T}^{n-1}}$.

Now the coefficients a_{ij} in (80) satisfy

$$\|a_{ij}\|_{1,\alpha,\Omega^{r_0}} \le C(n,N,\lambda,\Lambda,\|f\|_{2,\alpha,\mathbf{T}^{n-1}}).$$

Thus, we can rewrite the equation in (80) in non-divergence form with C^α coefficients and use Lemma 6.29 and Theorem 6.2 of [14] to obtain

$$\|u\|_{2,\alpha,\Omega^{r_0}} \le C(\|g\|_{1,\alpha,\Omega^{r_0}} + \|u\|_{L^\infty(\Omega^R)}) \le C\|g\|_{1,\alpha,\Omega^{r_0}},$$

where we used Lemma 4.3 in the last estimate. The higher regularity estimates for the data with higher regularity are now obtained from the linear theory [14, Chapter 6].

2. Now we obtain the interior estimates in the weighted norms. Let $x = (x', x_n)$ with $x_n \in [r_0, \frac{2R}{3}]$. For such x, we have $B_{\frac{\delta_x}{3}}(x) \subset \Omega^R$ with $\delta_x > 1$. We denote

$$d_x := \frac{\delta_x}{3},$$

and rescale to the unit ball $B_1 = B_1(0)$, i.e., define $v \in C^2(B_1(0))$ by

$$v(y) = \frac{1}{d_x}u(x + d_x y). \tag{81}$$

Then, in $B_1(0)$,

$$\mathrm{div}\,(A(Dv(y))) = d_x \sum_{i,j=1}^{n} A^i_{P_j}(Du(x+d_xy))u_{x_ix_j}(x+d_xy) = 0. \tag{82}$$

Using again the local estimates of [22] and Theorem 6.2 of [14], and using Lemma 4.3, we get, in the ball $B_{\frac{1}{2}} = B_{\frac{1}{2}}(0)$,

$$\|v\|_{2,\alpha,B_{\frac{1}{2}}} \leq C\|v\|_{L^\infty(B_1)} \leq \frac{C}{d_x}\|u\|_{L^\infty(B_{d_x}(x))} \leq \frac{C}{d_x}\|g\|_{0,\alpha,\Omega^{r_0}}.$$

The higher regularity estimates for the data with higher regularity are now obtained from the linear theory [14, Theorem 6.17]. Re-scaling back and using that, for any $y \in B_{\frac{d_x}{2}}(x)$, there holds $1/4 \leq d_x/\delta_y \leq 1/2$, we obtain

$$\begin{aligned}&\sum_{|\beta|\leq m} \sup_{y\in B_{\frac{\delta_x}{6}}(x)} \left(\delta_y^{|\beta|}|D^\beta u(y)|\right)\\ &+\sum_{|\beta|=m} \sup_{y,z\in B_{\frac{\delta_x}{6}}(x), y\neq z} \left(\delta_{y,z}^{m+\alpha}\frac{|D^\beta u(y)-D^\beta u(z)|}{|y-z|^\alpha}\right) \leq C_m\|g\|_{0,\alpha,\Omega^{r_0}},\end{aligned} \tag{83}$$

for any $x \in \Omega^R$ with $x_n \in [r_0, \frac{2R}{3}]$ and for any $m = 1, 2, \ldots$.

3. In order to obtain the estimates of u in $\Omega^R \cap \{x_n \geq \frac{2R}{3}\}$, we estimate $u(y)$ in every half-ball $B^-_{\frac{R}{3}}(x) := B_{\frac{R}{3}}(x) \cap \{(y', y_n) \;:\; y_n < R\}$, with $x = (x', R)$, $x' \in \mathbf{T}^{n-1}$. We have $B^-_{\frac{2R}{3}}(x) \subset \Omega^R$. We rescale to the unit half-ball $B^-_1 = B^-_1(0) = B_1(0) \cap \{(y', y_n) \;:\; y_n < 0\}$, i.e. define $v \in C^2(\overline{B^-_1(0)})$ by (81) with $d_x := \frac{2R}{3}$. Then (82) holds in B^-_1, and $v \equiv 0$ on $\{y_n = 0\}$. Now we use the local estimates for the Dirichlet problem in [14] to show that, for $m = 1, 2, \ldots$,

$$\|v\|_{m,\alpha,B^-_{\frac{1}{2}}} \leq C\|v\|_{L^\infty(B^-_1)} \leq \frac{C}{R}\|u\|_{L^\infty(\Omega^R)} \leq \frac{C}{R}\|g\|_{0,\alpha,\Omega^{r_0}}.$$

Re-scaling back and taking into account that, for any $y \in B_{\frac{d_x}{2}}(x)$, there holds $1/2 \leq R/\delta_y \leq 3/2$, we obtain the estimate similar to (83) in the half-ball $B^-_{\frac{R}{3}}(x)$ for any $x = (x', R)$, $x' \in \mathbf{T}^{n-1}$.

4. The estimates in Steps 1–3 imply (77) and (79). Indeed, for $m = 1, 2, \ldots$, it only remains to estimate

$$\delta_{x,y}^{m+\alpha}\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \qquad \text{for } |\beta| = m,$$

in the case $\delta_x > r_0$, $\delta_x \geq \delta_y$, $|x-y| > \frac{\delta_x}{6}$. We have proved that $|D^\beta u(z)| \leq C\|g\|_{0,\alpha,\Omega^{r_0}}/\delta_z^m$ for every $z \in \Omega^R$. Thus,

$$\delta_{x,y}^{m+\alpha}\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \leq C\|g\|_{0,\alpha,\Omega^{r_0}}\delta_y^{m+\alpha}\frac{\frac{1}{\delta_x^m}+\frac{1}{\delta_y^m}}{\delta_x^\alpha} \leq C\|g\|_{0,\alpha,\Omega^{r_0}}.$$

□

Lemma 4.5. *Let $u(x)$, $A(P)$, $f(x)$, and $g(x)$ be as in Lemma 4.2. Fix $N > 0$ and assume that* (76) *holds. Then, for every $k = 1, 2, \ldots$, there exists $C > 0$ such that*

$$\|u\|^{(k)}_{1;\alpha;\Omega^R} \leq C\|g\|_{0,\alpha,\Omega^{r_0}}, \tag{84}$$

where C depends only on n, N, λ, Λ, $\|f\|_{1,\alpha,\mathbf{T}^{n-1}}$, and k. If, in addition, (78) *holds, then, for every $k = 1, 2, \dots$,*

$$\|u\|^{(k)}_{m;\alpha;\Omega^R} \le C_m \|g\|_{m-1,\alpha,\Omega^{r_0}}, \tag{85}$$

where C_m depends only on m, n, N, λ, Λ, and k.

Proof. It suffices to improve the estimates away from S_f in (77) and (79), i.e. the estimates obtained in Steps 2 and 3 in the proof of Lemma 4.4. We prove (84) by induction. The case $k = 0$ has been proved in Lemma 4.4.

Assume that (84) has been proved for $k = k_0 \in \{0, 1, \dots\}$. Then, in particular, for any $x \in \Omega^R$,

$$|Du(x)| \le \frac{C}{(\delta_x)^{k_0+1}} \|g\|_{0,\alpha,\Omega^{r_0}}, \tag{86}$$

where C depends only on the data and k_0.

Similar to the proof of Lemma 4.3, we show that, for every $x_n \in (r_0, R)$, there exists $x' \in \mathbf{T}^{n-1}$ such that

$$u(x', x_n) = 0.$$

Then (86) implies that

$$|u(x', x_n)| \le \frac{C}{\delta_x^{k_0+1}} \|g\|_{0,\alpha,\Omega^{r_0}} \quad \text{for any } (x', x_n) \in (2, R) \times \mathbf{T}^{n-1}. \tag{87}$$

Now, similar to Step 2 in the proof of Lemma 4.4, we estimate u in every ball $\tilde{B} := B_{\frac{\delta_x}{3}}(x) \subset \Omega^R$, for $x \in \Omega^R$ with $x_n \in (r_0, 2R/3)$. Re-scaling as in the proof of Lemma 4.4, we obtain the estimate for the function $v(y)$ defined by (81) with $d_x = \delta_x/3$:

$$\|v\|_{m,\alpha,B_{\frac{1}{2}}} \le C\|v\|_{L^\infty(B_1)} \le \frac{C}{d_x}\|u\|_{L^\infty(\tilde{B})} \le \frac{C}{d_x^{k_0+2}} \|g\|_{0,\alpha,\Omega^{r_0}},$$

for any $m = 1, 2, \dots$, where we used (87). Re-scaling back and using that, for any $y \in B_{\frac{d_x}{2}}(x)$, there holds $1/4 \le d_x/\delta_y \le 1/2$, we obtain

$$\sum_{|\beta| \le m} \sup_{y \in B_{\frac{\delta_x}{6}}(x)} \left(\delta_y^{k_0+1+|\beta|} |D^\beta u(y)| \right)$$

$$+ \sum_{|\beta|=m} \sup_{y,z \in B_{\frac{\delta_x}{6}}(x), y \ne z} \left(\delta_{y,z}^{k_0+1+m+\alpha} \frac{|D^\beta u(y) - D^\beta u(z)|}{|y-z|^\alpha} \right) \le C_m \|g\|_{0,\alpha,\Omega^{r_0}},$$

for any $x \in \Omega^R$ with $x_n \in [r_0, \frac{2R}{3}]$ and for any $m = 1, 2, \dots$.

A similar estimate is obtained in each half-ball $B^-_{\frac{R}{6}}(x)$ with $x = (x', R)$, $x' \in \mathbf{T}^{n-1}$, by combining (87) with the argument of Step 3 in the proof of Lemma 4.4.

Repeating the argument of Step 4 in the proof of Lemma 4.4, we obtain (84) and (85) for $k = k_0 + 1$. □

Now we show Lemmas 4.5–4.7 about the behavior of the solutions in $\Omega := \Omega^\infty$ in the limit $R \to \infty$, which will be used for proving the uniqueness and stability of solutions of Problem A.

Lemma 4.6. *Let $A(P)$ and $f(x')$ be as in Lemma* 4.2*. Then, for any $g \in C^\alpha(\overline{\Omega^{r_0}})$ satisfying* (76) *and* (67)*, there exists a solution $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ of*

$$\begin{aligned} &\operatorname{div}(A(Du)) = 0 && \text{in } \Omega, \\ &A(Du) \cdot \nu = g && \text{on } S_f, \end{aligned} \tag{88}$$

satisfying

$$u(\cdot, x_n) \to 0 \qquad \text{uniformly on } \mathbf{T}^{n-1} \text{ when } x_n \to \infty. \tag{89}$$

This solution is unique in the class of weak solutions in $C^{1,\alpha}(\overline{\Omega})$. Moreover, we have

(i) *For any $R > 2$, there exists a unique solution $u_R(x) \in C^{1,\alpha}(\overline{\Omega_R}) \cap C^{2,\alpha}(\Omega_R)$ of problem* (62)*,*
(ii) $u = \lim_{R\to\infty} u_R$*, where the convergence is in $C^{1,\beta}$ on compact subsets of $\overline{\Omega}$ for any $0 \le \beta < \alpha$.*
(iii) *For every $R_0 \ge 1$, there exists $x' \in \mathbf{T}^{n-1}$ such that $u(x', R_0) = 0$.*
(iv) *For every $k = 0, 1, 2, \ldots$, there exists C_k such that*

$$\|u\|^{(k)}_{1;\alpha;\Omega} \le C_k \|g\|_{0,\alpha,\Omega^{r_0}} . \tag{90}$$

Proof. The proof of the existence of a solution $u_R \in C^{1,\alpha}(\overline{\Omega^R}) \cap C^{2,\alpha}(\Omega^R)$ of the problem described in Lemma 4.2 for any $R > 2$ can be obtained as following: if $g \in C^{1,\alpha}(\overline{\Omega^{r_0}})$ and $f \in C^{2,\alpha}(\mathbf{T}^{n-1})$, then the existence and uniqueness of $u_R \in C^{2,\alpha}(\overline{\Omega^R})$ follows from [14, Theorem 17.30]; and if $g \in C^{\alpha}(\overline{\Omega^{r_0}})$ and $f \in C^{1,\alpha}(\mathbf{T}^{n-1})$, then we approximate g and f by sequences of smooth functions $g_\ell \to g$ in $C^{\alpha}(\overline{\Omega^{r_0}})$ and $f_\ell \to f$ in $C^{1,\alpha}(\mathbf{T}^{n-1})$, respectively, solve the problem in the smooth domain $\Omega^R(f_\ell)$ with the right-hand side g_ℓ to obtain solutions $u^\ell_R \in C^{2,\alpha}(\overline{\Omega^R(f_\ell)})$, and take the limit in a subsequence of u^ℓ_R as $\ell \to \infty$, with the aid of estimate (77). More precisely, we can map $\overline{\Omega^R} = \overline{\Omega^R(f)}$ to $\overline{\Omega^R(f_\ell)}$ by a mapping Φ_ℓ such that $\Phi_\ell \to Id$ in $C^{1,\alpha}(\overline{\Omega^R}; \mathbf{R}^n)$. Then, by (77) applied to u^ℓ_R, the functions $v^\ell := u^\ell_R \circ \Phi_\ell^{-1}$ are uniformly bounded in $C^{1,\alpha}(\overline{\Omega^R})$ and satisfy

$$\begin{array}{ll} \operatorname{div}(B_\ell(x, Dv^\ell)) = 0 & \text{in } \Omega^R, \\ B_\ell(x, Dv^\ell)\cdot\nu = g_\ell & \text{on } S_f, \\ v^\ell = 0 & \text{on } \{x_n = R\}, \end{array}$$

where B_ℓ and g_ℓ are obtained by the procedure similar to the one in Step 3 in the proof of Lemma 4.3, and thus $B_\ell \to A$ in $C^1(\overline{\Omega^R} \times \mathbf{R}^n)$ (where we used (66)) and $g_\ell \to g$ in $C^{\alpha}(\overline{\Omega^R})$. Thus, a subsequence $v^{\ell_j}(x)$ converges in $C^1(\overline{\Omega^R})$ and its limit $u_R \in C^{1,\alpha}(\overline{\Omega^R})$ is a weak solution of (62) in Ω^R. The standard interior regularity estimates imply $u_{R_j} \in C^{1,\alpha}(\overline{\Omega^R}) \cap C^{2,\alpha}(\Omega^R)$.

Now, let $R_j \to \infty$. Using the estimates of Lemmas 4.4 and 4.5, we can extract from $\{u_{R_j}\}$ a subsequence converging in $C^{1,\frac{\alpha}{2}}$ on compact subsets of $\overline{\Omega}$ to a function $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ that satisfies (90) and especially (89). Indeed, this can be achieved by extracting a subsequence converging in $C^{1,\alpha}(\overline{\Omega^1})$, a further subsequence converging in $C^{1,\alpha}(\overline{\Omega^2})$, etc., and using the diagonal procedure. Obviously, $u(x)$ is a weak solution of (88). The standard interior regularity estimates imply $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.

Moreover, this solution $u(x)$ satisfies property (iii): Indeed, for fixed $R_0 \ge 1$, $R_j \ge R_0$ for sufficiently large j. For such j, by Step 2 in the proof of Lemma 4.3, there exists $x'_j \in \mathbf{T}^{n-1}$ such that $u_{R_j}(x'_j, R_0) = 0$. Then there exists a subsequence of $\{x'_j\}$ converging to some point $x' \in \mathbf{T}^{n-1}$. Since $u_{R_j}(\cdot, R_0) \to u(\cdot, R_0)$ uniformly on $\mathbf{T}^{n-1}$, we have $u(x', R_0) = 0$. Thus, the solution $u(x)$ satisfies all the properties asserted.

In order to complete the proof, it remains to show the uniqueness of solution $u(x)$ of (88)–(89) and of solution $u_R(x)$ of (62) for all $R > 2$. Both follow from the comparison principle of Lemma 4.7 below. □

Lemma 4.7. (i) *Let $\Omega \subset \mathbf{R}^n$ be an unbounded domain with $\partial\Omega \in C^{1,\alpha}$, and let $\nu(x)$ be the interior unit normal to $\partial\Omega$. Let $A(x, P)$ satisfy* (63)*–*(66) *as in Lemma* 4.2*.*

Let $u, v \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ *satisfy*

$$\begin{aligned}
&\operatorname{div}(A(x, Du)) \geq \operatorname{div}(A(x, Dv)) \qquad in \ \ \Omega,\\
&A(x, Du)\cdot\nu \geq A(x, Dv)\cdot\nu \qquad on \ \ \partial\Omega,\\
&\lim_{R\to\infty} \sup_{\Omega\setminus B_R(0)} (u - v) \leq 0.
\end{aligned}$$

Then $u \leq v$ *in* Ω.

(ii) *Let* Ω^R *be a domain as in Lemma* 4.2. *Let* $A(x, P)$ *satisfy* (63)–(66). *Let* $u, v \in C^{1,\alpha}(\overline{\Omega^R}) \cap C^{2,\alpha}(\Omega^R)$ *satisfy*

$$\begin{aligned}
&\operatorname{div}(A(x, Du)) \geq \operatorname{div}(A(x, Dv)) \qquad in \ \ \Omega,\\
&A(x, Du)\cdot\nu \geq A(x, Dv)\cdot\nu \qquad on \ \ S_f,\\
&u \leq v \qquad on \ \ \{x_n = R\}.
\end{aligned}$$

Then $u \leq v$ *in* Ω^R.

Proof. We prove (i). If $\sup_\Omega(u - v)$ is attained at $x_0 \in \Omega$, then $u - v \equiv const.$ in Ω by the strong maximum principle, which implies $u \leq v$ by the condition at infinity. If $\sup_\Omega(u - v)$ is attained at $x_0 \in \partial\Omega$, then the tangential derivatives $D_\tau(u - v)(x_0) = 0$ to $\partial\Omega$ implies $D(u - v)(x_0) = (u - v)_\nu(x_0)\nu(x_0)$. Then it follows that

$$\begin{aligned}
0 &\leq (A(x_0, Du(x_0)) - A(x_0, Dv(x_0)))\cdot\nu(x_0)\\
&= (v - u)_\nu(x_0)\int_0^1 \left(\sum_{i,j=1}^n A^j_{P_i}(x_0, tDu(x_0) + (1 - t)Dv(x_0))\nu_i(x_0)\nu_j(x_0)\right) dt.
\end{aligned}$$

Now the ellipticity of $A(x_0, P)$ implies that $(u - v)_\nu(x_0) \geq 0$. This contradicts Lemma A.1(ii), since $w(x) = u(x) - v(x)$ is a subsolution of the linear elliptic equation:

$$\sum_{i,j=1}^n (a_{ij}(x)w_{x_j})_{x_i} \geq 0 \qquad \text{in } \ \Omega,$$

with coefficients $a_{ij}(x) = \int_0^1 A^j_{P_i}(x, tDu(x) + (1 - t)Dv(x))dt \in C^\alpha(\overline{\Omega})$. The remaining possibility is that there exist $x_j \to \infty$ such that $(u - v)(x_j) \to \sup_\Omega(u - v)$. From our conditions at infinity, it follows that $\lim_{j\to\infty}(u - v)(x_j) \leq 0$ and hence $\sup_\Omega(u - v) \leq 0$. Thus $u \leq v$ in Ω.

The proof of (ii) is similar. □

Lemma 4.8. *Let* $\Omega = \Omega^\infty$, *and let* $A(P)$ *and* $f(x')$ *be as in Lemma* 4.2. *Let* $u \in C^{1,\alpha}(\overline{\Omega})$ *be bounded and satisfy*

$$\operatorname{div}(A(Du)) = 0$$

in the weak sense in Ω. *Then,*

(i) *for* $R_0 = \|f\|_{L^\infty} + 1$,

$$\|u\|^{(0)}_{2;\alpha;\mathbf{T}^{n-1}\times(R_0,\infty)} \leq C\|u\|_{L^\infty(\Omega)}; \tag{91}$$

(ii) *for* $S_f := \partial\Omega = \{(x', f(x')) \ : \ x' \in \mathbf{T}^{n-1}\}$ *and the inner unit normal* $\nu(x)$ *on* $\partial\Omega$, $g := A(Du)\cdot\nu \in C^\alpha(S_f)$ *must satisfy* (67).

Proof. By the interior estimates, we get $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$. Similar to the proof of Lemma 4.4, we use the re-scaling (81) to show (91). From (91), we see that $Du(\cdot, x_n) \to 0$ uniformly on $\mathbf{T}^{n-1}$ as $x_n \to \infty$. Now, for any $R > R'$,

$$0 = \int_{\Omega\cap\{x_n<R'\}} \operatorname{div}(A(Du))dx = \int_{\mathbf{T}^{n-1}\times\{R\}} A^n(Du)dx' - \int_{S_f} g\, d\mathcal{H}^{n-1}.$$

Letting $R \to \infty$, we get (67). □

4.3. **Existence of the Iteration Map.** Now we show the existence of the iteration map.

Proposition 4.9. *There exist constants C_k depending only on n, γ, q_0^+, α, a and $k = 1, 2, \dots$, such that for any $M > 1$, there exists $\sigma_0 > 0$ depending only on the data (i.e. on n, γ, q_0^+, α and a) and M so that, for any $\sigma < \sigma_0$, $R > 1$, any $\psi \in \mathcal{K}_M(R)$, and any $\varphi^-(x)$ satisfying* (19)*, the problem* (59)–(61) *has a unique solution $\varphi \in C^{1,\alpha}(\overline{\Omega_R^+(\psi)}) \cap C^{2,\alpha}(\Omega_R^+(\psi))$ that satisfies* (39) *and*

$$\|\varphi - q^+ x_n\|^{(k)}_{1;\alpha;\Omega_R^+(\psi)} \le C_k \sigma \tag{92}$$

for $k = 0, 1, \dots$.

Proof. 1. By (53), we can choose σ so small that, for $\psi \in \mathcal{K}_M(R)$, the corresponding boundary $f(x')$ satisfies

$$\|f\|_{1,\alpha,\mathbf{T}^{n-1}} \le \frac{1}{4}.$$

Then $\Omega_R^-(\psi)$ is a bounded domain with Lipschitz boundary.

For Q^+, defined by (27), we have:

$$Q^+ = \frac{1}{a^{n-1}} \int_{S_\psi} G_\psi \, d\mathcal{H}^{n-1}. \tag{93}$$

Indeed, since, $\varphi^- \in C^{1,\alpha}(\overline{\Omega^-})$ is a weak solution of (1), we have

$$\int_{\Omega_R^-(\psi)} \rho(|D\varphi^-|^2) D\varphi^- \cdot D\psi \, dx = \int_{\partial\Omega_R^-(\psi)} \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu \, \psi \, d\mathcal{H}^{n-1}$$

for any $\psi \in C^\infty(\mathbf{R}^n)$. Choosing $\psi \equiv 1$ and using the boundary condition (20), we get

$$\begin{aligned} Q^+ &= \frac{1}{a^{n-1}} \int_{(0,a)^{n-1} \times \{-1\}} \rho(|D\varphi^-|^2) \varphi^-_{x_n} dx' \\ &= \frac{1}{a^{n-1}} \int_{S_\psi} \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu \, d\mathcal{H}^{n-1}, \end{aligned}$$

and (93) is proved.

2. Now we can follow the proof of [5, Proposition 4.1]. We first rewrite the problem (59)–(61) in terms of the function $v := \varphi - q^+ x_n$. The problem then takes the form:

$$\operatorname{div} A(Dv) = 0 \qquad \text{in} \quad \Omega_R^+(\psi), \tag{94}$$

$$A(Dv) \cdot \nu = g_\psi(x) \qquad \text{on} \quad S_\psi, \tag{95}$$

$$v = 0 \qquad \text{on} \quad \partial\Omega_R^+(\psi) \setminus S_\psi = \mathbf{T}^{n-1} \times \{R\}, \tag{96}$$

where

$$\begin{aligned} &A(P) = \tilde\rho(|P + q^+\nu_0|^2)(P + q^+\nu_0) - \tilde\rho((q^+)^2) q^+ \nu_0 \quad \text{for} \ P \in \mathbf{R}^n, \\ &g_\psi(x) = G_\psi(x) - \rho((q^+)^2) q^+ \nu \cdot \nu_0. \end{aligned}$$

Thus, $v(x)$ satisfies a uniformly elliptic equation with the same ellipticity constants as in (48). Moreover, by (45), (46), the $A(P)$ defined above satisfies (66). Also, from the definition, $A(P)$ satisfies (63), (65). The function $g_\psi(x)$ satisfies

$$\|g_\psi\|_{0,\alpha,\Omega_e} \le C\sigma, \tag{97}$$

where C depends only on the data. To see this, we first note that

$$\rho(|D\varphi_0^+|^2) D\varphi_0^+ = \rho(|D\varphi_0^-|^2) D\varphi_0^- \qquad \text{in } \Omega_e, \tag{98}$$

since both sides of (98) are equal to $\rho((q_0^+)^2)q_0^+\nu_0$. Using this and (58) yields

$$\begin{aligned}\|g_\psi\|_{0,\alpha,\Omega_e} &= \|(\rho(|D\varphi^-|^2)D\varphi^- - \rho((q^+)^2)q^+\nu_0)\cdot\nu\|_{0,\alpha,\Omega_e}\\ &\le \|(\rho(|D\varphi^-|^2)D\varphi^- - \rho((q_0^-)^2)q_0^-\nu_0)\cdot\nu\|_{0,\alpha,\Omega_e}\\ &\quad +\|\rho((q_0^+)^2)q_0^+ - \rho((q^+)^2)q^+\|_{0,\alpha,\Omega_e}\\ &\le C\sigma(1+M\sigma),\end{aligned}$$

by (19), (57), and (29). Choosing $\sigma \le \dfrac{1}{M}$ we get (97).

Moreover, by the definition of $g_\psi(x)$ and q^+ and by (93), and since $S_\psi = \{x_n = f(x') \mid x' \in (0,a)^{n-1}\}$ and $\nu\cdot\nu_0 = 1\Big/\sqrt{1+|Df|^2}$ on S_ψ, we have

$$\begin{aligned}\int_{S_\psi} g_\psi d\mathcal{H}^{n-1} &= \int_{S_\psi} G_\psi d\mathcal{H}^{n-1} - \rho((q^+)^2)q^+\int_{S_\psi}\nu\cdot\nu_0\, d\mathcal{H}^{n-1}\\ &= \int_{S_\psi} G_\psi d\mathcal{H}^{n-1} - a^{n-1}\rho((q^+)^2)q^+ = 0.\end{aligned}$$

3. Now, by Lemma 4.6(i), we obtain the existence of a solution $v \in C^{1,\alpha}(\overline{\Omega_R^+(\psi)}) \cap C^{2,\alpha}(\Omega_R^+(\psi))$ of the problem (94)–(96), and Lemmas 4.4 and 4.5 imply that, for $k = 0, 1, \dots,$

$$\|v\|^{(k)}_{2;\alpha;\Omega^R} \le C_k\|g\|_{1,\alpha,\Omega^{r_0}} \le C_k\sigma,$$

if σ is small depending only on the data and M, where the constants C_k depend only on the data and k. Thus, the function $\varphi(x) := v(x) + q^+x_n$ is a solution of (59)–(61) and satisfies (92).

The uniqueness of solutions of (59)–(61) follows from Lemma 4.7. The property (39) of $\varphi(x)$ follows from the uniqueness as in the proof of [5, Proposition 4.1].

Then Proposition 4.9 is proved. □

5. Existence of Solutions of the Free Boundary Problem in Ω^R

Now we define an extension operator $\mathcal{P}_\psi$, similar to [5, Proposition 4.5], to obtain

Proposition 5.1. *Let $\sigma > 0$ and $\varphi^-(x)$ be as in Proposition 4.9. Let $\psi \in \mathcal{K}_M(R)$, and let $\varphi(x)$ be a solution of the problem (59)–(61) in the domain $\Omega_R^+(\psi)$. Then $\varphi(x)$ can be extended to the domain Ω_e^R, and this extension $\mathcal{P}_\psi\varphi$ satisfies the following two properties:*

(i) *There exists $\hat{C}$, depending only on $n, \gamma, q_0^+, a, \alpha$ and independent of M, R, σ, and ψ, such that*
$$\|\mathcal{P}_\psi\varphi - q^+x_n\|^{(0)}_{0;0;\Omega_e^R} \le \hat{C}\sigma; \tag{99}$$

(ii) *Let $\beta \in (0,\alpha)$. If a sequence $\psi_j \in \mathcal{K}_M(R)$ converges in $C^{1,\beta}(\overline{\Omega_e^R})$ to $\psi \in \mathcal{K}_M(R)$, and $\varphi_j \in C^{1,\alpha}(\overline{\Omega_R^+(\psi_j)})$ and $\varphi \in C^{1,\alpha}(\overline{\Omega_R^+(\psi)})$ are the weak solutions of the problems (59)–(61) for ψ_j and ψ, respectively, then $\mathcal{P}_{\psi_j}\varphi_j \to \mathcal{P}_\psi\varphi$ in $C^{1,\beta}(\overline{\Omega_e^R})$.*

Proof. The operator $\mathcal{P}_\psi$ is a $C^{1,\alpha}$ - version of the operator defined in [5, Proposition 4.5]. We define it as following. Let

$$\Omega_0 := \mathbf{T}^{n-1}\times(-1,1).$$

We first employ the extension map in [14, pp. 136–137] to define our extension operator $\mathcal{E}_1 : C^{1,\beta}(\overline{\Omega_0^+}) \to C^{1,\beta}(\overline{\Omega_0})$ for any $\beta \in (0,1)$. Let $v \in C^{1,\beta}(\overline{\Omega_0^+})$. Define $\mathcal{E}_1 v = v$ in Ω_0^+. For $x_n \in [-1,0)$, define

$$\mathcal{E}_1 v(x', x_n) = \sum_{i=1}^{2} c_i v(x', -\frac{x_n}{i}),$$

where $c_1 = -3$ and $c_2 = 4$, which are determined by

$$\sum_{i=1}^{2} c_i \left(-\frac{1}{i}\right)^m = 1, \quad m = 0, 1.$$

Starting from this definition, we follow the proof of [5, Proposition 4.5] to define first the extension operator $\mathcal{E}_\psi : C^{1,\beta}(\overline{\Omega_R^+(\psi)}) \to C^{1,\beta}(\overline{\Omega_e})$ for $\psi \in \mathcal{K}_M(R)$. Noting that, since for any $R > 2$ and $\psi \in \mathcal{K}_M(R)$ we have $\Omega_e \setminus \Omega_R^+(\psi) \subset \mathbf{T}^{n-1} \times (-1, 1)$, one actually needs only to prove the non-weighted Holder estimates for the extended function $\mathcal{E}_\psi v$ restricted to $\mathbf{T}^{n-1} \times (-1, 1)$, we obtain

$$\|\mathcal{E}_\psi v\|^{(0)}_{1,\beta,\Omega_e} \le C \|v\|^{(0)}_{1,\beta,\Omega_R^+(\psi)} \tag{100}$$

and other properties of $\mathcal{E}_\psi$ as in the proof of [5, Proposition 4.5]. We define $\mathcal{P}_\psi : C^{1,\beta}(\overline{\Omega_R^+(\psi)}) \to C^{1,\beta}(\overline{\Omega_e})$ for $\psi \in \mathcal{K}_M(R)$ and $\beta \in (0, \alpha]$ by $\mathcal{P}_\psi v = \mathcal{E}_\psi(v - q^+ x_n) + q^+ x_n$, and then (99) follows from (100) and (92) with $k = 0$.

In order to prove assertion (ii) of the proposition, we also follow the proof of [5, Proposition 4.5], noting that the uniqueness of solutions of the problem (59)–(61) follows from Lemma 4.7(ii). □

Proposition 5.2. *There exists $M > 0$ depending only on n, γ, q_0^+, α and a such that the following holds. Let $\sigma > 0$ and $\varphi^-(x)$ be as in Proposition 4.9. Let $R > 4$. There exists a solution $\tilde{\varphi} \equiv \tilde{\varphi}_R$ of Problem C in the domain Ω^R satisfying*

$$\tilde{\varphi} \equiv \tilde{\varphi}_R = \min(\varphi^-, \varphi), \tag{101}$$

where $\varphi \equiv \varphi_R \in \mathcal{K}_M(R)$. Furthermore, for any $k = 0, 1, \dots,$

$$\|\tilde{\varphi} - q^+ x_n\|^{(k)}_{1;\alpha;\Omega_R^+(\tilde{\varphi})} \le C_k \sigma, \tag{102}$$

where C_k depend only on n, a, α, γ, and q_0^+ and k, but not on R.

Proof. Choose M to be the constant $\hat{C}$ in (99), and fix this choice of M from now on.

Let $\psi \in \mathcal{K}_M(R)$, and let $\varphi(x)$ be the corresponding solution defined in Proposition 4.9, and consider its extension $\mathcal{P}_\psi \varphi(x)$, which we also denote $\varphi(x)$. From Proposition 5.1, it follows that $\varphi \in \mathcal{K}_M(R)$ if $\sigma > 0$ is sufficiently small depending only on the data (since M is now fixed).

Now we can define a map $J_R : \mathcal{K}_M(R) \to \mathcal{K}_M(R)$ by

$$J_R \psi = \varphi, \tag{103}$$

where $\psi(x)$ and $\varphi(x)$ are as above. From Proposition 5.1(ii), the map J_R is continuous on $\mathcal{K}_M(R)$ in the $C^{1,\beta}$ norm for any positive $\beta < \alpha$.

In order to find a suitable solution of Problem C in Ω^R, we seek a fixed point of the map J_R. If $\varphi(x)$ is such a fixed point, then (101) defines a solution of Problem C in Ω^R. We use the Schauder Fixed Point Theorem [14, Theorem 11.1] in the following setting:

Let $\sigma > 0$ satisfy the conditions of Proposition 5.1. Let $\beta \in (0, \alpha)$. Since $\overline{\Omega_e^R}$ is a compact manifold with boundary, the set $\mathcal{K}_M(R)$ is a compact convex subset of $C^{1,\beta}(\overline{\Omega_e})$. We have shown that $J_R(\mathcal{K}_M(R)) \subset \mathcal{K}_M(R)$ and J_R is continuous in the $C^{1,\beta}(\overline{\Omega_e})$ norm. Then, by the Schauder Fixed Point Theorem, J_R has a fixed point $\varphi \in \mathcal{K}_M(R)$.

Now, (102) follows from Proposition 4.9 since $J_R \varphi = \varphi$. □

6. Existence of Solutions of Problem A in the Infinite Cylinder Ω_e

Let $\sigma > 0$ and $\varphi^-(x)$ be as in Proposition 4.9. Fix M as in Proposition 5.2. Then, by Proposition 5.2, for each $R > 4$, there exists a solution $\tilde\varphi_R(x)$ of Problem C in Ω^R, such that

$$\tilde\varphi_R = \min(\varphi^-, \varphi_R),$$

where $\varphi_R \in \mathcal{K}_M(R)$, and (102) holds. Clearly, if $4 < R_1 < R_2$, then the restrictions of functions from $\mathcal{K}_M(R_2)$ to $\Omega_e^{R_1}$ belong to $\mathcal{K}_M(R_1)$. Let $\beta = \frac{\alpha}{2}$. Since $\mathcal{K}_M(10)$ is a compact subset of $C^{1,\beta}(\overline{\Omega_e^{10}})$, there exists a sequence $\tilde R_j \to \infty$ such that $\varphi_{\tilde R_j}(x)$ converge in $C^{1,\beta}(\overline{\Omega_e^{10}})$. A further subsequence can be extracted to converge in $C^{1,\beta}(\overline{\Omega_e^{20}})$, etc. By a diagonal procedure, we obtain a sequence $R_j \to \infty$ such that $\varphi_{R_j}(x)$ converge in $C^{1,\beta}$ to $\varphi(x)$ on any compact subset of Ω_e^∞. Then $\varphi \in C^{1,\alpha}(\overline{\Omega_e^\infty})$ and $\varphi(x)$ satisfies (99) in Ω_e^∞. Now, from (52) and (53), it follows that the free boundaries $S_{\varphi_{R_j}}$ of $\varphi_{R_j}(x)$ converge in $C^{1,\beta}$ to the free boundary S_φ of $\varphi(x)$, i.e., the corresponding functions $f_{\varphi_{R_j}}(x')$ converge in $C^{1,\beta}(\mathbf{T}^{n-1})$ to $f_\varphi(x')$. Then it follows that $\varphi(x)$ satisfies the weak form of the equation (1) in $\Omega^+ \equiv \Omega_\varphi^+$ and the free boundary condition (7) on S_φ.

Thus

$$\tilde\varphi(x) := \min(\varphi^-(x), \varphi(x))$$

is a classical solution of Problem C, and S_φ is its free boundary.

It follows that $\tilde\varphi(x)$ is a solution of Problem B, provided that σ is small enough so that (32) with $k=1$ implies that $|D\varphi(x)| < p^1_{sonic} - \varepsilon$, where ε is defined by (44). Indeed, then (45) implies that $\varphi(x)$ lies in the non-truncated region for the equation (48) and the free boundary condition (49).

For such σ, the function $\tilde\varphi(x)$ is a solution of Problem A. Indeed, $|D\tilde\varphi(x)| < p^1_{sonic} - \varepsilon$ on $\Omega^+(\tilde\varphi) := \{\tilde\varphi < \varphi^-\}$ since $\tilde\varphi = \varphi$ on $\Omega^+(\tilde\varphi)$.

Note that φ satisfies (32), since each $\tilde\varphi_R$ satisfies (102) in Ω_e^R with constants C_k independent of R and $f_{\varphi R_j} \to f_\varphi$ as $j \to \infty$ in $C^{1,\beta}(\mathbf{T}^{n-1})$. Finally, (32) and (29) imply (24.

The existence part of Theorem 2.1 and the estimates (24), (32) are now proved.

7. Uniqueness and Stability

We first assume that φ^- satisfies (19), and prove assertions (i) and (iii) of Theorem 2.1. Then we assume that φ^- satisfies a stronger condition (35), and prove uniqueness of solution of Problem A, satisfying (24).

Lemma 7.1. *Let φ^- satisfy (19). Let $\varphi \in W^{1,\infty}(\Omega)$ be a solution of Problem A with $\omega \in (0, p^1_{sonic})$. Assume that $\varphi(x)$ satisfies* (24), *where $\hat C$ is the constant obtained in Section* 6. *There exists $\sigma_0 > 0$, C depending only on n, a, α, γ, and q_0^+ such that, if $\sigma \in (0, \sigma_0)$, then $\varphi(x)$ satisfies* (30), (31), *and*

$$\|\varphi - \omega x_n\|^{(k)}_{1;\alpha;\Omega^+(\varphi)} \le C_k(\sigma + |\omega - q_0^+|), \tag{104}$$

where C_k depend only on n, a, α, γ, q_0^+, and k for $k = 1, 2, \dots$.

Proof. Extend $\varphi(x)$ to the domain Ω_e as in §3.1. Then the extension is a solution of Problem B in Ω_e and satisfies (23) in the extended domains (i.e. $[0,a]^{n-1}$ replaced by $\mathbf{T}^{n-1}$). From now on, we consider the extensions of $\varphi(x)$ and $\varphi^-(x)$.

From (24) and (19), $\Omega^-(\varphi) \subset \Omega_1 \equiv \Omega \cap \{x_n < 1\}$ if σ is small. Let $\varphi^+(x)$ denote $\varphi(x)$ restricted to $\Omega_1^+(\varphi) := \Omega^+(\varphi) \cap \{x_n < 1\}$. Then, using (24) we extend φ^+ from $\Omega_1^+(\varphi)$, which we consider now as a subset of $\mathbf{R}^n$, into $\mathbf{R}^n$ so that the extension $\mathcal{E}\varphi$ satisfies

$$\|\mathcal{E}\varphi^+ - \varphi_0^+\|_{1;\alpha;\mathbf{R}^n} \le C\sigma,$$

where C depends only on n (see e.g. [30, Chap. 6, Theorem 4]). Now, if σ is sufficiently small, it follows from (19) that $\Omega^+(\varphi)\cap\Omega_1 = \{\mathcal{E}\varphi^+ < \varphi^-\}\cap\Omega_1$, and the Implicit Function Theorem implies (30) and (31).

Define

$$g := \rho(|D\varphi^-|^2)D\varphi^-\cdot\nu - \rho(\omega^2)\omega\nu_0\cdot\nu \qquad \text{on } \ S_\varphi := \partial\Omega^+(\varphi), \tag{105}$$

where ν is the unit normal to S_φ in the direction of $\Omega^+(\varphi)$. Then we obtain from (19), (31), and (15) that

$$\|g\|_{0,\alpha,S_\varphi} \le C(\sigma + |\omega - q_0^+|). \tag{106}$$

If σ is sufficiently small, then, by (24), the equation (1) is uniformly elliptic in $\Omega^+(\varphi)$. Then the function $v := \varphi - \omega x_n \in C^{1,\alpha}(\overline{\Omega^+(\varphi)})$ is a weak solution in $\Omega^+(\varphi)$ of the conormal derivative problem for a uniformly elliptic equation:

$$\begin{aligned}&\operatorname{div}(A(Dv)) = 0 \qquad \text{in } \ \Omega^+(\varphi),\\ &A(Dv)\cdot\nu = g \qquad \text{on } S_\varphi,\\ &v(\cdot,x_n)\to 0 \qquad \text{uniformly on } \mathbf{T}^{n-1} \text{ when } x_n\to\infty,\end{aligned}$$

with

$$A(P) := \tilde\rho(|P+\omega\nu_0|^2)(P+\omega\nu_0) - \tilde\rho(\omega^2)\omega\nu_0, \qquad P\in\mathbf{R}^n,$$

and with g is defined by (105). By (23), $v(x)$ is bounded in $\Omega^+(\varphi)$. Thus, by Lemma 4.8, g satisfies (67). Hence, by (23) and (106), we can apply Lemma 4.6 to $v(x)$ in $\Omega^+(\varphi)$ to obtain (104). □

Now we show that ω obtained in Section 6 is uniquely determined by $\varphi^-(x)$.

Proposition 7.2. *There exist σ_0 depending only on the data as in Lemma* 7.1 *such that if φ^- satisfies (19), and $\varphi(x)\in W^{1,\infty}(\Omega)$ is a solution of Problem A with $\omega\in(0,p^1_{sonic})$, then $\omega = q^+$ in* (23)*, where q^+ is defined by (26). In particular, $\varphi(x)$ satisfies* (32).

Proof. We have $((0,a)^{n-1}\times\{-1\})\cap\overline{\Omega^+(\varphi)} = \emptyset$ by (30) and (31) for small σ. Note that $\varphi=\varphi^-$ in Ω^-, $\varphi^-\in C^{1,\alpha}(\overline{\Omega^-})$ is a weak solution of (1), and Ω^- is a bounded domain. Thus,

$$\int_{\Omega^-}\rho(|D\varphi^-|^2)D\varphi^-\cdot D\psi\,dx = \int_{\partial\Omega^-}\rho(|D\varphi^-|^2)D\varphi^-\cdot\nu\,\psi\,d\mathcal{H}^{n-1}$$

for any $\psi\in C^\infty(\mathbf{R}^n)$. Choosing $\psi\equiv1$ and using the boundary condition (20), we get

$$\begin{aligned}0 &= \int_{\partial\Omega^-}\rho(|D\varphi^-|^2)D\varphi^-\cdot\nu\,d\mathcal{H}^{n-1}\\ &= \int_{S_\varphi}\rho(|D\varphi^-|^2)D\varphi^-\cdot\nu\,d\mathcal{H}^{n-1} - \int_{(0,a)^{n-1}\times\{-1\}}\rho(|D\varphi^-|^2)\varphi^-_{x_n}dx',\end{aligned}$$

where $S_\varphi = \partial\Omega^+\setminus\partial\Omega$ is the shock surface. The Rankine-Hugoniot condition (7) on S_φ implies

$$\begin{aligned}\int_{S_\varphi}\rho(|D\varphi^+|^2)D\varphi^+\cdot\nu\,d\mathcal{H}^{n-1} &= -\int_{S_\varphi}\rho(|D\varphi^-|^2)D\varphi^-\cdot\nu\,d\mathcal{H}^{n-1}\\ &= -\int_{(0,a)^{n-1}\times\{-1\}}\rho(|D\varphi^-|^2)\varphi^-_{x_n}dx'.\end{aligned}$$

Since $(0,a)^{n-1}\times[1,R]\subset\Omega^+$ if σ is small and $R>1$, and since $\varphi^+(x)\in C^{1,\alpha}(\overline{\Omega^+})$ is a weak solution of (1) in $\Omega^+ = \{x_n > f(x')\}\cap\Omega$, then using (22) and performing the calculation as above yield

$$\int_{(0,a)^{n-1}\times\{R\}}\rho(|D\varphi|^2)\varphi_{x_n}dx' = -\int_{S_\varphi}\rho(|D\varphi|^2)D\varphi\cdot\nu\,d\mathcal{H}^{n-1} \qquad \text{for } \ R>1.$$

Thus, for any $R > 1$,

$$\int_{(0,a)^{n-1}\times\{R\}} \rho(|D\varphi|^2)\varphi_{x_n}\,dx' = \int_{(0,a)^{n-1}\times\{-1\}} \rho(|D\varphi^-|^2)\varphi^-_{x_n}\,dx'. \tag{107}$$

By (104), $D\varphi(\cdot, R) \to \omega\nu_0$ uniformly on $(0,a)^{n-1}$ as $R \to \infty$. Now we can pass to the limit as $R \to \infty$ in (107) and get $\omega = q^+$ where q^+ satisfies (26) and (27).

Now, (32) follows from (104) and (29). □

Now we prove uniqueness of solution of Problem A, if (35) holds.

Proposition 7.3. *Let q_0^+ and q_0^- be as in Theorem* 2.1. *There exists $\sigma_0 > 0$ such that the solution $\varphi(x)$ of Problem A, satisfying* (24), *is unique for any $\sigma \in (0, \sigma_0)$ and any supersonic solution $\varphi^-(x)$ of* (1) *satisfying (35) and the boundary conditions stated in Problem A.*

Proof. 1. Let $\varphi \in W^{1,\infty}(\Omega)$ be a solution of Problem A satisfying (24) and (28). By Lemma 7.1 and Proposition 7.2, if $\sigma > 0$ is sufficiently small, then $\varphi(x)$ satisfies (28)–(32).

Now we follow the scheme of the proof of [5, Theorem 5.1]; However, we need some new estimates because the domain is now an infinite cylinder.

2. Let $\varphi \neq \hat\varphi$ be two solutions of Problem A satisfying (30)–(32). Extend them to the domain Ω_e as in §3.1. Then, by Step 1, the extensions satisfy (32) and are solutions of Problem C in Ω_e.

By (31), $\Omega_e \cap \{x_n > 1/2\} \subset \Omega^+(\varphi)$ and $\Omega_e \cap \{x_n > 1/2\} \subset \Omega^+(\hat\varphi)$. In particular, by (19) and (32), for sufficiently small σ, we have $\varphi, \hat\varphi < q_0^- x_n$ on $\mathbf{T}^{n-1} \times [3/4, 1]$. Then, as in §3.1, we can modify φ^- on the set $\mathbf{T}^{n-1} \times [3/4, 1]$ and extend to $\mathbf{T}^{n-1} \times [-1, \infty)$ to obtain $\tilde\varphi^- \in C^{1,\alpha}(\mathbf{T}^{n-1} \times [-1, \infty))$ satisfying $\tilde\varphi^- \equiv \varphi^-$ on $\mathbf{T}^{n-1} \times [-1,\ 3/4]$,

$$\|\tilde\varphi^- - \varphi_0^-\|_{1,\alpha,\Omega_e} \le 2\sigma, \tag{108}$$

and

$$\tilde\varphi^- = q_0^- x_n \qquad \text{on} \quad \mathbf{T}^{n-1} \times [7/8, \infty). \tag{109}$$

Then

$$\begin{aligned}
&\varphi, \hat\varphi \le \tilde\varphi^- \quad \text{on} \ \ \Omega_e,\\
&\varphi = \tilde\varphi^- \ \ \text{on} \ \ \Omega^-(\varphi), \qquad \varphi < \tilde\varphi^+ \ \ \text{on} \ \ \Omega^-(\varphi),\\
&\hat\varphi = \tilde\varphi^- \ \ \text{on} \ \ \Omega^-(\hat\varphi), \qquad \hat\varphi < \tilde\varphi^+ \ \ \text{on} \ \ \Omega^-(\hat\varphi).
\end{aligned}$$

We write $\varphi^-(x)$ for both the original function and the modified and extended function $\tilde\varphi^-(x)$.

Define

$$u := \varphi^- - \varphi, \qquad \hat u := \varphi^- - \hat\varphi \qquad \text{in } \Omega_e.$$

Then

$$u \ge 0, \qquad \hat u \ge 0 \qquad \text{in } \ \Omega_e,$$

and

$$\begin{aligned}
\Omega^+(u) := \{u > 0\} \cap \Omega_e = \Omega^+(\varphi), \qquad \Omega^+(\hat u) := \{\hat u > 0\} \cap \Omega_e = \Omega^+(\hat\varphi),\\
S(u) := \partial\{u > 0\} \cap \Omega_e = S(\varphi), \qquad S(\hat u) := \partial\{\hat u > 0\} \cap \Omega_e = S(\hat\varphi).
\end{aligned}$$

The definitions of $u(x)$ and $\hat u(x)$ with (32), (19), and (109) imply

$$\begin{aligned}
\|u - (q_0^- - q^+)x_n\|^{(1)}_{1;\alpha;\Omega^+(u)} \le C\sigma,\\
\|\hat u - (q_0^- - q^+)x_n\|^{(1)}_{1;\alpha;\Omega^+(\hat u)} \le C\sigma.
\end{aligned} \tag{110}$$

3. By (32) and (45), if σ is sufficiently small, then the function $\varphi \in C^{1,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ satisfies

$$\begin{aligned} \operatorname{div}(\tilde{\rho}(|D\varphi|^2)D\varphi) = 0 \qquad & \text{in} \quad \Omega^+(\varphi), \\ \tilde{\rho}(|D\varphi|^2)\varphi_\nu = G_\varphi \quad & \text{on} \quad S_\varphi := \{x_n = f(x')\}, \end{aligned} \tag{111}$$

where G_φ is defined by (58) for $\psi = \varphi$. Rewrite this problem in terms of $u = \varphi^- - \varphi$. Then, if σ is sufficiently small, $u(x)$ in $\Omega^+(u)$ is the solution of the following problem:

$$\operatorname{div}(A(x, Du)) = 0 \qquad \text{in} \quad \Omega^+(u), \tag{112}$$

$$u_\nu = H_f(x') \qquad \text{on} \ \ S(u), \tag{113}$$

where $S(u) = \{(x', f(x')) \ : \ x' \in \mathbf{T}^{n-1}\}$ by (30),

$$A(y, P) = \tilde{\rho}(|D\varphi^-(y) - P|^2)(D\varphi^-(y) - P) \qquad \text{for} \ \ y \in \Omega_e, \ \ P \in \mathbf{R}^n, \tag{114}$$

and

$$H_f(x') := D\varphi^-(x) \cdot \nu_f(x) - G(|D\varphi^-(x)|^2, D\varphi^-(x) \cdot \nu_f(x)), \tag{115}$$

for $x' \in \mathbf{T}^{n-1}, x = (x', f(x')) \in S(u)$, and $\nu_f(x)$ is the inward unit normal to $S(u)$ at x. Here $G : \mathbf{R}^2 \to \mathbf{R}$ is a smooth function, constructed in [5, Theorem 5.1, Step 2], which depends only on $\tilde{\rho}$, q^+, and q^-.

Note that (112) is uniformly elliptic with the same ellipticity constants as those for (48).

The function $\hat{u}(x)$ is a solution of the similar problem in $\Omega^+(\hat{u})$, i.e., $\hat{u}(x)$ satisfies the equation (112) in $\Omega^+(\hat{u})$, and

$$\hat{u}_\nu = H_{\hat{f}}(x') \qquad \text{on} \quad S(\hat{u}),$$

where

$$H_{\hat{f}}(x') = D\varphi^-(x) \cdot \nu_{\hat{f}}(x) - G(|D\varphi^-(x)|^2, D\varphi^-(x) \cdot \nu_{\hat{f}}(x)),$$

for $x' \in \mathbf{T}^{n-1}, x = (x', \hat{f}(x')) \in S(\hat{u})$, and the inward unit normal $\nu_{\hat{f}}(x)$ to $S(\hat{u})$ at x.

4. We may assume $f \neq \hat{f}$; otherwise $\varphi = \hat{\varphi}$ by the uniqueness of solutions of the problem (112) and (113) satisfying (110) in the domain $\Omega^+(\varphi) = \Omega^+(u)$, which follows from Lemma 4.7, applied to $u - (q_0^- - q^+)x_n$ and $\hat{u} - (q_0^- - q^+)x_n$ that are also solutions of the conormal problems for an elliptic equation with similar form to (112) and (113). Thus, we may assume that

$$\delta := \|\hat{f} - f\|_{L^\infty(\mathbf{T}^{n-1})} > 0, \tag{116}$$

and, moreover,

$$\delta = \hat{f}(x'_*) - f(x'_*) \qquad \text{for some } x'_* \in \mathbf{T}^{n-1},$$

since the opposite case can be handled similarly. Applying (31) to both $f(x')$ and $\hat{f}(x')$, we have $\delta \le C\sigma$. We assume that σ is small so that $C\sigma < 1$.

We shift the domain $\Omega^+(\hat{u})$ in the direction $-\nu_0$ by the distance δ. Then the resulting domain $\mathcal{B}$ contains $\Omega^+(u)$ and $\partial\mathcal{B} \cap S(u) \neq \emptyset$. Precisely, define $v : \Omega_e \to \mathbf{R}$ by

$$v(x', x_n) = \hat{u}(x', x_n + \delta).$$

Denote $h(x') := \hat{f}(x') - \delta$. Clearly,

$$\Omega^+(v) := \{(x', x_n) \ : \ v(x', x_n) > 0\} = \{(x', x_n) \ : \ x_n > h(x')\} \cap \Omega_e.$$

It follows that $\Omega^+(u) \subset \Omega^+(v)$. By construction, $f(x') \ge h(x')$, and there exists $x'_* \in \mathbf{T}^{n-1}$ such that $f(x'_*) = h(x'_*)$. Denote $x_* := (x'_*, f(x'_*)) \in \mathbf{T}^{n-1} \times \mathbf{R}$. Then the smooth surface $S(u)$ touches the smooth surface $S(v)$ at x_*. Denote the common unit normal to $S(u)$ and $S(v)$ at x_* in the direction of $\Omega^+(v)$ by $\nu(x_*)$. Since $S(\hat{u}) = S(v) + \delta\nu_0$, it follows that the inward unit normal $\nu_{\hat{f}}(x_* + \delta\nu_0)$ to $S(\hat{u})$ at $x_* + \delta\nu_0 = (x', f(x') + \delta)$ is equal to $\nu(x_*)$. Then, from the definition of $H_f(x')$ and $H_{\hat{f}}(x')$,

$$|H_f(x'_*) - H_{\hat{f}}(x'_*)| \le C|D\varphi^-(x'_*, f(x'_*)) - D\varphi^-(x'_*, f(x'_*) + \delta)| \le C\delta\sigma,$$

where we used (35) in the last inequality.

Also, since $\hat{u}(x)$ satisfies the free boundary condition $\hat{u}_\nu(x') = H_{\hat{f}}(x')$ and $v(x) = \hat{u}(x + \delta\nu_0)$ for any x, we have

$$v_\nu(x_*) := Dv(x_*) \cdot \nu(x_*) = D\hat{u}(x_* + \delta\nu_0) \cdot \nu_{\hat{f}}(x_* + \delta\nu_0) = H_{\hat{f}}(x'_*).$$

Since $u(x)$ satisfies $u_\nu(x_*) := Du(x_*) \cdot \nu(x_*) = H_f(x'_*)$, we have

$$|v_\nu(x_*) - u_\nu(x_*)| = |H_f(x'_*) - H_{\hat{f}}(x'_*)| \le C\delta\sigma. \tag{117}$$

We will come to a contradiction for small σ by showing that $v_\nu(x_*) - u_\nu(x_*) \ge c\delta$ with $c > 0$.

5. Note that $x_* \in S(u)$ and $\nu(x_*)$ is the inward normal to $S(u)$ at x_*. Also

$$v|_{S(u)} \ge u|_{S(u)}. \tag{118}$$

Indeed, $v \ge 0 = u$ on $S(u)$ from the definition of $\Omega^+(u)$ and $v(x)$.

Since $\hat{u}(x)$ satisfies (112) in $\Omega^+(\hat{u})$, then $v(x)$ satisfies

$$\operatorname{div}\left(A(x + \delta\nu_0, Dv)\right) = 0 \qquad \text{in } \ \Omega^+(u).$$

We write this equation in the form:

$$\operatorname{div}\left(A(x, Dv)\right) = \operatorname{div}\psi(x) \qquad \text{in } \ \Omega^+(u), \tag{119}$$

where $A(y, P)$ is the function (114) and

$$\psi(x) := A(x, Dv(x)) - A(x + \delta\nu_0, Dv(x)) = -\delta \int_0^1 D_y A(x + \delta t\nu_0, Dv(x)) \cdot \nu_0 \, dt.$$

Note that (109) implies

$$\psi(x', x_n) = 0, \qquad \text{if} \quad x_n > 2, \tag{120}$$

since $\delta \le C\sigma \le 1$.

By (114) and (35), for any $L_0 > 0$,

$$\sup_{|P| \le L_0} \|D_y A(\cdot, P)\|_{0,\alpha,\Omega_e} \le C(L_0)\sigma,$$
$$\sup_{|P| \le L_0} \|D^2_{yP} A(\cdot, P)\|_{0,0,\Omega_e} \le C(L_0)\sigma.$$

From this, we use $|Dv(x)| \le q^- - q^+ + CM\sigma$ with $M\sigma \le 1$ to conclude

$$|\psi| \le C\delta\sigma \qquad \text{in } \ \Omega_e,$$

and

$$\begin{aligned} |\psi(x) \quad - \quad & \psi(\hat{x})| \\ \le \quad & \delta \left| \int_0^1 \left[D_y A(x + \delta s\nu_0, Dv(x)) - D_y A(\hat{x} + \delta s\nu_0, Dv(x))\right] \cdot \nu_0 \, ds \right| \\ & + \delta \left| \int_0^1 \left(D_y A(\hat{x} + \delta s\nu_0, Dv(x)) - D_y A(\hat{x} + \delta s\nu_0, Dv(\hat{x}))\right) \cdot \nu_0 \, ds \right| \\ \le \quad & C\delta\sigma|x - \hat{x}|^\alpha + C\delta|Dv(x) - Dv(\hat{x})| \\ \le \quad & CM\delta\sigma|x - \hat{x}|^\alpha, \end{aligned}$$

where we used (110) and the fact that $v(x) = \hat{u}(x + \delta\nu_0)$ in the last inequality. Thus, we have

$$\|\psi\|_{0,\alpha,\Omega_e} \le C\delta\sigma. \tag{121}$$

Denoting $w := v - u$, we obtain

$$\sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j} w) = \operatorname{div}\psi(x) \quad \text{ in } \quad \Omega^+(u), \tag{122}$$

where

$$a_{ij}(x) = \int_0^1 A^i_{P_j}(x, sDv(x) + (1-s)Du(x))ds. \tag{123}$$

Thus, the equation (122) is uniformly elliptic with the $C^\alpha(\overline{\Omega^+(u)})$ norms of $a_{ij}(x)$ depending only on n, q_0^+, γ, and Ω, where we used (110) to estimate the $C^\alpha(\overline{\Omega^+(u)})$ norms of a_{ij}.

By (110) and the definition of $v(x)$

$$\|w - (q_0^- - q^+)\delta\|^{(1)}_{1;\alpha;\Omega^+(u)} \le C\sigma. \tag{124}$$

6. By (124), there exists $L > 0$ such that $w(x', x_n) > (q_0^- - q^+)\delta/2 > 0$ for any $x' \in \mathbf{T}^{n-1}$ and $x_n \ge L$.

Denote $\Omega_L := \Omega^+(u) \cap \{x_n < L\}$. Then $w = w_1 + w_2$ in Ω_L, where $w_1, w_2 \in C^{1,\alpha}(\overline{\Omega_L})$ are respectively unique weak solutions of the following problems:

$$\begin{aligned} \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j} w_1) &= 0 \qquad \text{in} \quad \Omega_L, \\ w_1 &= w \qquad \text{on} \quad \partial\Omega_L; \end{aligned} \tag{125}$$

and

$$\begin{aligned} \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j} w_2) &= \operatorname{div}\psi(x) \qquad \text{in} \quad \Omega_L, \\ w_2 &= 0 \qquad \text{on} \quad \partial\Omega_L. \end{aligned} \tag{126}$$

The existence of $w_1(x)$ and $w_2(x)$ in the periodic setting $\Omega_L \subset \mathbf{T}^{n-1}\times\mathbf{R}$ is proved as in [5, Proof of Theorem 5.1, Step 5].

By (120) and Appendix A, we can estimate $w_2(x)$ as

$$\|w_2\|_{1,\alpha,\Omega_L} \le C\|\psi\|_{0,\alpha,\Omega_e},$$

where C depends only on the dimension, ellipticity constants, and $\|f\|_{1,0,\mathbf{T}^{n-1}}$, and is independent of L. Furthermore, using $\|f\|_{1,\alpha,\mathbf{T}^{n-1}} \le C\sigma$ and choosing σ small yield that C depends only on n, q_0^+, q_0^-, and σ_0. Thus, we use (121) to obtain

$$\|w_2\|_{1,\alpha,\Omega^+(u)} \le C\sigma\delta. \tag{127}$$

7. Now we estimate $(w_1)_\nu(x_*)$ from below. By (118), $w_1 \ge 0$ on $\partial\Omega^+(u)$. On $\{x_n = L\}$,

$$w_1 = w - w_2 \ge \frac{q_0^- - q^+}{2}\delta - C\sigma\delta > 0,$$

if σ is small. Thus, $w_1 \ge 0$ on $\partial\Omega_L$. Since $w_1(x)$ is a solution of (125) in Ω_L, then $w_1(x) \ge 0$ in Ω_L by the maximum principle.

By (120), the functions $u(x)$ and $v(x)$ satisfy

$$\operatorname{div}(A(x, Du)) = \operatorname{div}(A(x, Dv)) = 0 \qquad \text{in} \quad \mathbf{T}^{n-1}\times(2,\infty).$$

By (109) and the definition (114) of $A(x,P)$, it follows that $A(x,P)$ is independent of x on $\{x_n > 2\}$, thus we define

$$A(P) := A(3e_n, P).$$

Denoting

$$\tilde u := u - (q_0^- - q^+)x_n, \qquad \tilde v := v - (q_0^- - q^+)x_n$$

we find that $\tilde u(x)$ and $\tilde v(x)$ satisfy the elliptic equation:

$$\operatorname{div}(\tilde A(D\tilde u)) = \operatorname{div}(\tilde A(D\tilde v)) = 0 \quad \text{in} \quad \mathbf{T}^{n-1}\times(2,\infty),$$

where $\tilde{A}(P) = A(P + (q_0^- - q^+)e_n) - A((q_0^- - q^+)e_n)$ satisfies $\tilde{A}(0) = 0$ and

$$\tilde{u}(\cdot, x_n) \to 0 \text{ and } \tilde{v}(\cdot, x_n) \to (q_0^- - q^+)\delta$$

uniformly on $\mathbf{T}^{n-1}$ as $x_n \to \infty$. Denote $g_1 = \tilde{A}^n(D\tilde{u})$ and $g_2 = \tilde{A}^n(D\tilde{v})$. By Lemma 4.8(ii),

$$\int_{\mathbf{T}^{n-1}} g_k(x', 2)dx' = 0, \quad k = 1, 2. \tag{128}$$

Denote by $u_R(x)$ (resp. $v_R(x)$) the unique solution of problem (62) in the domain $\mathbf{T}^{n-1} \times (2, R)$, with the equation and conormal boundary condition defined by $\tilde{A}(P)$, and $g = g_1$ (resp. $g = g_2$). By (128) and Lemma 4.6(ii), we conclude that

$$u_R \to \tilde{u} \text{ and } v_R \to \tilde{v} - (q_0^- - q^+)\delta$$

in $C^{1,\frac{\alpha}{2}}$ on compact subsets of $\mathbf{T}^{n-1} \times [2, \infty)$ as $R \to \infty$. Thus, denoting $w_R := v_R - u_R$ we have

$$w_R \to w - (q_0^- - q^+)\delta \qquad \text{in } C^{1,\frac{\alpha}{2}} \text{ on compact subsets of } \mathbf{T}^{n-1} \times [2, \infty). \tag{129}$$

Denoting $a_{ij}^R(x) = \int_0^1 \tilde{A}^i_{P_j}(sD\tilde{v}(x) + (1-s)D\tilde{u}(x))ds$, we get

$$\begin{aligned}
&\sum_{i,j=1}^n \partial_{x_i}(a_{ij}^R(x)\partial_{x_j} w^R) = \operatorname{div} \tilde{A}(D\tilde{v}) - \operatorname{div} \tilde{A}(D\tilde{u}) = 0 \quad \text{in} \quad \mathbf{T}^{n-1} \times (2, R),\\
&\sum_{i=1}^n a_{nj}^R(x)\partial_{x_j} w^R = \tilde{A}^n(D\tilde{v}) - \tilde{A}^n(D\tilde{u}) = g_2 - g_1 \quad \text{on} \quad \mathbf{T}^{n-1} \times \{2\},\\
&w^R = 0 \quad \text{on} \quad \mathbf{T}^{n-1} \times \{R\}.
\end{aligned}$$

Clearly, $a_{ij}^R(x)$ are uniformly elliptic coefficients, and $a_{ij}^R(x) = a_{ji}^R(x)$. Also, $a_{ij}^R \in C^{1,\alpha}(\mathbf{T}^{n-1} \times [2, R])$ since $u, v \in C^{2,\alpha}(\mathbf{T}^{n-1} \times [2, R])$. By (128), we can apply Remark 4.1 and conclude that there exists $x'_R \in \mathbf{T}^{n-1}$ such that $w_R(x'_R, 3) = 0$. Then, for some sequence $R_j \to \infty$, we have $x'_{R_j} \to \hat{x}' \in \mathbf{T}^{n-1}$ and, by (129), we conclude that $w(\hat{x}', 3) = (q_0^- - q^+)\delta$. Thus,

$$w_1(\hat{x}', 3) = w(\hat{x}', 3) - w_2(\hat{x}', 3) \geq (q_0^- - q^+)\delta - C\sigma\delta \geq \frac{q_0^- - q^+}{2}\delta, \tag{130}$$

if $\sigma < (q_0^- - q^+)/(2C)$. Note that $w_1(x)$ is a nonnegative solution of (125). Without loss of generality, we assume $L > 5$. Then, by the Harnack inequality and (130), we have

$$w_1 \geq c\delta \qquad \text{in } \mathbf{T}^{n-1} \times [2, 4].$$

Since the geometry is fixed by the inclusion $\mathbf{T}^{n-1} \times [1, \infty) \subset \Omega^+(u)$, the constant $c > 0$ depends only on n, q_0^+, q_0^-, and σ_0. Now Lemma A.1(i) implies

$$(w_1)_\nu(x_*) \geq c\delta, \tag{131}$$

where $c > 0$ depends only on n, q_0^+, q_0^-, and σ_0.

Combining (131) with (127), we obtain

$$(v - u)_\nu(x_*) = \partial_\nu(w_1 + w_2)(x_*) \geq (c - C\sigma)\delta \geq \frac{c}{2}\delta,$$

if σ is small. If $\delta > 0$ and σ is small, this contradicts with (117). Thus $\delta = 0$. This completes the proof. $\square$

As a consequence of the uniqueness, non-degeneracy, and regularity of solutions of the free boundary problem, we can conclude Theorem 2.2. Its proof repeats the one of [5, Theorem 6.1] by using in particular that the perturbed supersonic solutions $\varphi^-(x)$ and $\hat{\varphi}^-(x)$ are considered on the compact set $\Omega_1 := \mathbf{T}^{n-1} \times [-1, 1]$ in the present case and define unique solutions of Problem A, or B, by Theorem 2.1, respectively, and also using the fact that, under the conditions of Theorem 2.2, the higher regularity estimates (33)–(34), proved in Section 8 below, hold.

8. Higher regularity

In this section, we prove the higher regularity assertion of Theorem 2.1 and the estimates (33) and (34).

We continue to consider the function $\varphi^- = \tilde{\varphi}^-$, which is extended to the domain $\mathbf{T}^{n-1} \times (-1, \infty)$ and satisfies (108) and (109). The assumption $\varphi^- \in C^{2,\alpha}(\overline{\Omega_1})$ implies that the extended function satisfies $\varphi^- \in C^{2,\alpha}(\mathbf{T}^{n-1} \times (-1, \infty))$. The function $u := \varphi^- - \varphi \geq 0$ is a solution of the following problem in the domain $\Omega^+(u) = \{u > 0\}$:

$$\begin{aligned} &\operatorname{div}(A(x, Du)) = 0 \qquad \text{in} \quad \Omega^+(u), \\ &B(x, Du) = 0 \qquad \text{on} \quad S(u), \end{aligned}$$

where $S(u) = \{(x', f(x')) \,:\, x' \in \mathbf{T}^{n-1}\}$, $A(x, P)$ is defined by (114), and

$$B(x, P) = \left[A(x, P) - \rho(|D\varphi^-(x)|^2) D\varphi^-(x)\right] \cdot \frac{P}{|P|}$$

for $P \in \mathbf{R}^n \setminus \{0\}$. Note that $\frac{Du}{|Du|} = \nu_f$ is the unit inner normal to $S(u)$. Since $\varphi^- \in C^{2,\alpha}(\mathbf{T}^{n-1} \times (-1, \infty))$, then $A \in C^{1,\alpha}(\overline{\Omega^+(u)} \times \mathbf{R}^n)$ and $B \in C^{1,\alpha}(\overline{\Omega^+(u)} \times (\mathbf{R}^n \setminus \{0\}))$.

Since $u(x)$ satisfies (110), then, for sufficiently small σ,

$$2(q_0^- - q^+) \geq \partial_{x_n} u \geq \frac{q_0^- - q^+}{2} > 0 \qquad \text{in} \quad \overline{\Omega^+(u)}. \tag{132}$$

In particular, we can modify $B(x, P)$ near $P = 0$ so that the modification does not affect the values of $B(x, Du(x))$ for $x \in \overline{\Omega^+(u)}$, and the modified $B(x, P)$ is in $C^{1,\alpha}(\overline{\Omega^+(u)} \times \mathbf{R}^n)$.

By (132), we can perform the transformation $\Phi : \Omega^+(u) \to \mathbf{T}^{n-1} \times (0, \infty)$ introduced in Kinderlehrer-Nirenberg [17, Section 3] in the whole domain $\Omega^+(u)$:

$$\Phi(x', x_n) = (y', y_n), \qquad \text{with} \qquad y' = x', \;\; y_n = u(x', x_n).$$

By (132), $\Phi(\Omega^+(u)) = \mathbf{T}^{n-1} \times (0, \infty)$ and Φ, Φ^{-1} are $C^{1,\alpha}$.

Then, as in [17, Section 3], we can express $x_n = w(y)$ and the derivatives of $u(x)$ through the derivatives of $w(x)$ and in particular obtain by (132), (110)

$$\begin{aligned} &\frac{1}{2(q_0^- - q^+)} \leq \partial_{y_n} w \leq \frac{2}{q_0^- - q^+} \qquad \text{in} \quad \mathbf{T}^{n-1} \times [0, \infty), \\ &\left\| w - \frac{1}{q_0^- - q^+} y_n \right\|^{(1)}_{1;\alpha;\mathbf{T}^{n-1} \times [0,\infty)} \leq C\sigma. \end{aligned} \tag{133}$$

Then we rewrite the problem (112) in terms of $w(x)$:

$$\begin{aligned} &\textstyle\sum_{i,j=1}^n a_{ij}(y', w, Dw) w_{y_i y_j} + b(y', w, Dw) = 0 \qquad \text{in} \quad \mathbf{T}^{n-1} \times (0, \infty), \\ &G(y', w, Dw) = 0 \qquad \text{in} \quad \mathbf{T}^{n-1} \times \{y_n = 0\}, \end{aligned} \tag{134}$$

where, by (133) and [17, Section 3], the coefficients $a_{ij}(y', z, P)$ are elliptic for $(y', z, P) \in \{(y', w(y), Dw(y)) \mid y \in \mathbf{T}^{n-1} \times (0, \infty)$ if σ is sufficiently small, and the ellipticity constants depend only on those of the equation (112) and the bounds in (133). The boundary function is

$$G(y', z, P) \;\; = \;\; \left[A\left(y', z, -\frac{P'}{P_n}, \frac{1}{P_n}\right) - \rho(|D\varphi^-|^2) D\varphi^-\right] \cdot \frac{(-P', 1)}{|(-P', 1)|},$$

with notation $P = (P', P_n) \in \mathbf{R}^n$.

We now check the strict obliqueness condition on $\mathbf{T}^{n-1} \times \{y_n = 0\}$: Then, denoting $\eta = (-P', 1)$ and using the ellipticity of $A(x, P)$, we obtain

$$G_P \cdot \nu = G_{P_n} = -\frac{1}{P_n^2 |\eta|} \sum_{i,j=1}^{n} A^i_{P_j} \eta_i \eta_j \geq \frac{\lambda |(-P', 1)|}{P_n^2}.$$

By (133), for P in the range of $Dw(y)$, we have

$$G_P \cdot \nu = G_{P_n} \geq \kappa(\lambda, q_0^{\pm}) > 0.$$

We may modify G away from this range of P so that the above inequality holds for all $(y', z, P) \in \mathbf{T}^{n-1} \times \mathbf{R} \times \mathbf{R}^n$, and $w(x)$ still satisfies the boundary condition in (134).

Also, $a_{ij}, G \in C^{1,\alpha}(\mathbf{T}^{n-1} \times \mathbf{R} \times \mathbf{R}^n)$ and $b \in C^{\alpha}(\mathbf{T}^{n-1} \times \mathbf{R} \times \mathbf{R}^n)$ with the norms depending only on the norms of $A(x, P)$ in $C^{1,\alpha}(\overline{\Omega(u)} \times \mathbf{R}^n)$ and $D\varphi^-(x)$ in $C^{1,\alpha}(\overline{\Omega(u)})$, respectively, as well as the bounds in (133).

Since $w(y', 0) = f(y')$, the estimates (31) and (133) imply that

$$|w(y', y_n)| \leq C\sigma + \frac{2}{q_0^- - q^+} y_n \qquad \text{for } y_n > 0.$$

Now it follows from the local boundary estimates of [20, Theorem 2] that

$$\|w\|_{2,\alpha,\mathbf{T}^{n-1}\times[0,1]} \leq C(n, \alpha, q_0^+, q_0^-, \gamma, \sigma, \|D\varphi^-\|_{1,\alpha,\Omega_1}).$$

Again, since $w(y', 0) = f(y')$, then the estimate (33) is proved.

Now, (34) follows from (85) applied to the problem (111) rewritten in the terms of the function $v := \varphi - q^+ x_n$, i.e.,

$$\begin{aligned} &\operatorname{div}(A(Dv)) = 0 \qquad &&\text{in} \quad \Omega^+(\varphi) \equiv \Omega^+(u), \\ &A(Dv) \cdot \nu = g(x) &&\text{on} \quad S(u), \end{aligned}$$

where

$$\begin{aligned} &A(P) = \tilde{\rho}(|P + q^+\nu_0|^2)(P + q^+\nu_0) - \tilde{\rho}((q^+)^2) q_\psi^+ \nu_0 \quad \text{for} \quad P \in \mathbf{R}^n, \\ &g(x) = G_\varphi(x) - \rho((q^+)^2) q^+ \nu \cdot \nu_0, \end{aligned}$$

and G_φ is defined by (58) for $\psi = \varphi$. Note that $g(x)$ satisfies (67) by the definition of q^+.

Appendix A. Hopf-type lemma for the equations of divergence form in domains with $C^{1,\alpha}$ boundaries

The following lemma is a refined version of [11, Lemma 7] with more precise estimates of the normal derivative of solutions at boundary points for our applications above.

Lemma A.1. *Let $\Omega \subset \mathbf{R}^n$ be an open set, and let $\Omega \cap B_R(0) = \{x_n > f(x')\} \cap B_R(0)$ with $f \in C^{1,\alpha}(\mathbf{R}^{n-1})$, $f(0) = 0$, and $Df(0) = 0$.*

(i) *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a weak solution of*

$$\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} = 0 \quad \text{in} \quad \Omega, \tag{135}$$

where $A = (a_{ij}) \in C^{\alpha}(\overline{\Omega})$ satisfies the ellipticity condition:

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \qquad \text{for any} \quad x \in \Omega, \ \xi \in \mathbf{R}^n, \tag{136}$$

with $\Lambda \geq \lambda > 0$ and $a_{ij} = a_{ji}, i, j = 1, \ldots, n$. Assume that $u \geq 0$ in $\Omega \cap B_R(0)$ and $u(0) = 0$. Then

$$\partial_{x_n} u(0) \geq c\, u(\frac{R}{2} e_n),$$

where $c > 0$ depends only on $n, \lambda, \Lambda, R, \alpha, \|a_{ij}\|_{0,\alpha,\Omega\cap B_R}$, and $\|f\|_{1,\alpha,B_R^{n-1}}$.

(ii) *If $u \in C^{1,\alpha}(\overline{\Omega})$ is a weak supersolution of*

$$\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} \leq 0 \quad in \quad \Omega,$$

where $A(x) = (a_{ij}(x))$ is as above, and if $u > 0$ in $\Omega \cap B_R(0)$ with $u(0) = 0$, then

$$\partial_{x_n} u(0) > 0.$$

Proof. We prove (i). In the following proof, all constants $C, c, c_1, etc.$ are positive and depend only on $n, \lambda, \Lambda, R, \alpha, \|a_{ij}\|_{0,\alpha,\Omega\cap B_R}$, and $\|f\|_{1,\alpha,B_R^{n-1}}$, unless other dependence is specified.

The change of variables $y = \Phi(x', x_n) = (x', x_n - f(x'))$, defined on $B_{2\rho(0)}$, satisfies $\|\Phi - Id\|_{1,\alpha,B_{2\rho(0)}} < 1/10$ and

$$\Phi(\Omega) \cap B_\rho(0) \subset \{x_n > 0\} \cap B_\rho(0),$$

where $\rho > 0$ is small, depending only on n, R, and $\|f\|_{1,\alpha,B_R^{n-1}}$. Then the equation for $v(y) := u \circ \Phi^{-1}$ in $\{y_n > 0\} \cap B_\rho(0)$ remains the same form as the original equation for $u(x)$, with new ellipticity constants and norms of coefficients in C^α depending only on n, R, $\|f\|_{1,\alpha,B_R^{n-1}}$, and the original λ, Λ, and $\|a_{ij}\|_{0,\alpha,\Omega\cap B_R}$. By the Harnack inequality in $B_{R-\frac{\rho}{10}}(0) \subset B_R(0)$,

$$u(\frac{R}{2}e_n) \leq C \inf_{B_{\frac{\rho}{8}}(\frac{\rho}{2}e_n)} u,$$

and thus

$$v(\frac{\rho}{2}e_n) = u(\frac{\rho}{2}e_n) \geq \frac{1}{C}u(\frac{R}{2}e_n).$$

Now, noting that e_n is the normal vector to both Ω and $\Phi(\Omega)$ at 0, and $v_{y_n}(0) = u_{y_n}(0)$, it suffices to prove the lemma for $v(y)$ in $\{y_n > 0\} \cap B_\rho(0)$, i.e., to show that

$$v_{y_n}(0) \geq cv(\frac{\rho}{2}e_n). \tag{137}$$

Furthermore, by performing a linear transform $T : \mathbf{R}^n \to \mathbf{R}^n$ of the form $Ty = OA^{-\frac{1}{2}}y$ where O is an appropriate orthogonal transform, we can achieve that

$$\{y_n > 0\} \cap B_{\rho_1}(0) \subset T(\{y_n > 0\} \cap B_\rho(0)),$$

with $\rho_1 = \frac{\rho}{C(n,\lambda,\Lambda)}$. Then $Te_n = Ke_n$ where $K_1(n, \lambda, \Lambda) \geq K \geq K_2(n, \lambda, \Lambda) > 0$, and thus $|T^{-1}(\rho_1 e_n) - \rho e_n| \leq \rho - \varepsilon$ with $\varepsilon = \varepsilon(n, \lambda, \Lambda) > 0$. Defining $w = v \circ T$, we have that the equation for $w(x)$ is of form (135) with $a_{ij}(0) = \delta_{ij}$. Applying the Harnack inequality for $v(y)$ in $B_{\rho-\varepsilon}(\rho e_n)$, we see that $w(\rho_1 e_n) = v(T^{-1}(\rho_1 e_n)) \geq cv(\rho e_n)$, where $c = c(n, \lambda, \Lambda) > 0$. Also, $\partial_{y_n} w(0) = Dw(0) \cdot e_n = Dv(0) \cdot (Te_n) = K\partial_{y_n} v(0)$. Thus, it suffices to prove (137) for $w(y)$ and ρ_1.

Thus, in the original notations, we reduced the proof to the case

$$\Omega \cap B_R(0) := \{x_n > 0\} \cap B_R(0), \qquad a_{ij}(0) = \delta_{ij}. \tag{138}$$

Let $r > 0$ be chosen small below depending only on the data. Let

$$\mathcal{D}_r(re_n) = B_r(re_n) \setminus B_{\frac{r}{2}}(re_n).$$

Let $w(y)$ satisfy

$$\begin{aligned}
&\sum_{i,j=1}^{n} (a_{ij}(y)w_{y_i})_{y_j} = 0 \quad \text{in} \quad \mathcal{D}_r(re_n),\\
&w = 1 \quad \text{on} \quad \partial B_{\frac{r}{2}}(re_n),\\
&w = 0 \quad \text{on} \quad \partial B_r(re_n).
\end{aligned}$$

Rescale to $\mathcal{D}_1 = B_1(0) \setminus B_{\frac{1}{2}}(0)$:

$$\tilde{w}(y) = w(\frac{y - re_n}{r}), \qquad \tilde{a}_{ij}(y) = a_{ij}(\frac{y - re_n}{r}).$$

Then we have

$$\begin{aligned} &\sum_{i,j=1}^n (\tilde{a}_{ij}(y)\tilde{w}_{y_i})_{y_j} = 0 \quad \text{in} \quad \mathcal{D}_1, \\ &\tilde{w} = 1 \quad \text{on} \quad \partial B_{\frac{1}{2}}(0), \\ &\tilde{w} = 0 \quad \text{on} \quad \partial B_1(0), \end{aligned}$$

and

$$\|\tilde{a}_{ij}\|_{C^0(\mathcal{D}_1)} = \|a_{ij}\|_{C^0(\mathcal{D}_r(re_n))}, \quad [\tilde{a}_{ij}]_{0,\alpha,\mathcal{D}_1} \le \|a_{ij}\|_{0,\alpha,\mathcal{D}_r(re_n)} r^\alpha, \tag{139}$$

$$\tilde{a}_{ij}(-e_n) = a_{ij}(0) = \delta_{ij}. \tag{140}$$

We assume $r < 1$.

Let $G(x)$ satisfy

$$\begin{aligned} &\Delta G = 0 \quad \text{in} \quad \mathcal{D}_1, \\ &G = 1 \quad \text{on} \quad \partial B_{\frac{1}{2}}(0), \\ &G = 0 \quad \text{on} \quad \partial B_1(0). \end{aligned}$$

Then

$$\|G\|_{1,\alpha,\mathcal{D}_1} \le C(n), \quad \partial_{y_n} G(-e_n) = c(n) > 0.$$

By (140), the function $W = \tilde{w} - G$ satisfies

$$\begin{aligned} &\sum_{i,j=1}^n (\tilde{a}_{ij}(y)W_{y_i})_{y_j} = -\sum_{i,j=1}^n ((\tilde{a}_{ij}(y) - \tilde{a}_{ij}(-e_n))G_{y_i})_{y_j} \quad \text{in} \quad \mathcal{D}_1, \\ &W = 0 \quad \text{on} \quad \partial B_{\frac{1}{2}}(0) \cup \partial B_1(0). \end{aligned}$$

Using (139), we get

$$\|\tilde{a}_{ij}(\cdot) - \tilde{a}_{ij}(-e_n)\|_{1,\alpha,\mathcal{D}_1} \le Cr^\alpha,$$

and thus

$$\|(\tilde{a}_{ij}(\cdot) - \tilde{a}_{ij}(-e_n))G_{x_i}\|_{1,\alpha,\mathcal{D}_1} \le Cr^\alpha.$$

Now, by [14, Theorem 8.33],

$$\|W\|_{1,\alpha,\mathcal{D}_1} \le Cr^\alpha.$$

Thus,

$$\partial_{y_n}\tilde{w}(-e_n) \ge \partial_{y_n}\tilde{G}(-e_n) - \|W\|_{1,0,\mathcal{D}_1} \ge c(n) - Cr^\alpha \ge \frac{c(n)}{2},$$

if $r > 0$ is sufficiently small depending only on the data.

Rescaling back to $\mathcal{D}_r(re_n)$, we get

$$\partial_{x_n} w(0) \ge \frac{c(n)}{2r} \ge \frac{c(n)}{2}.$$

Applying the Harnack inequality to u in $B_{\frac{R}{2}-\frac{r}{4}}(\frac{R}{2}e_n)$, we have

$$\min_{\partial B_{\frac{r}{2}}(re_n)} u \ge c\,u(\frac{R}{2}e_n).$$

Thus, by the maximum principle, $u \ge c\,u(\frac{R}{2}e_n)w$ in $\mathcal{D}_r(re_n)$. Since $u(0) = w(0) = 0$ and $u, w \ge 0$ in $\mathcal{D}_r(re_n)$, we get

$$\partial_{x_n} u(0) \ge c\,u(\frac{R}{2}e_n)\partial_{x_n} w(0) \ge c_1 u(\frac{R}{2}e_n).$$

This proves (i).

To prove (ii), we repeat the transformations that lead to (138) and check that, after the transformations, $u(x)$ is still a positive weak supersolution of an elliptic equation of form (135) in $\Omega \cap B_R(0)$ and $u(0) = 0$. Thus, we assume (138) without loss of generality. Denote $\varepsilon := \min_{\partial B_{\frac{r}{2}}(re_n)} u > 0$. Then, using $w(x)$ defined above yields $u \geq \varepsilon w$ in $\mathcal{D}_r(re_n)$ and $u(0) = w(0) = 0$. Thus, $\partial_{x_n} u(0) \geq \varepsilon \partial_{x_n} w(0) > 0$. □

Appendix A. Bounds of weak solutions in cylindrical domains

Lemma A.1. *Let $\Omega_R := \mathbf{T}^{n-1} \times (-\infty, R) \cap \{x_n > f(x')\}$, where $f \in C^1(\mathbf{T}^{n-1})$ with $\|f\|_{L^\infty(\mathbf{T}^{n-1})} \leq r/2$, where $R > r > 0$. Let $u(x)$ be a weak subsolution of*

$$\begin{aligned} \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) &= \phi(x) + \operatorname{div}\psi(x) \qquad \text{in} \quad \Omega_R, \qquad (141)\\ u &= 0 \qquad \text{on} \quad \partial\Omega_R, \end{aligned}$$

where $A(x) = (a_{ij}(x))$ satisfies the uniform ellipticity condition (136). Assume $\phi \in L^{\frac{q}{2}}(\Omega_R; \mathbf{R})$, $\psi := (\psi_1, \ldots, \psi_n) \in L^q(\Omega_R; \mathbf{R}^n)$ with $q > n$, and

$$\phi = 0, \ \psi_j = 0, \quad \text{a.e. on } \mathbf{T}^{n-1} \times (r, R), j = 1, \ldots, n.$$

Then

$$\sup_{\Omega_R} u \leq C\kappa,$$

where

$$\kappa = \|\phi\|_{L^{\frac{q}{2}}(\Omega_r)} + \|\psi\|_{L^q(\Omega_r)}, \quad \Omega_r := \Omega_R \cap \{x_n < r\},$$

and C depends only on $n, \lambda, \Lambda, q, r, \|f\|_{C^1}$, and is independent of R.

Proof. The proof follows the Moser iteration technique [14, Theorem 8.15 and 8.16], with some adjustments, to show the independence of R in the estimate. We will only sketch the proof, pointing out the features which are special in the present case.

We first show that

$$\sup_{\Omega_R} u \leq C(\|u^+\|_{L^2(\Omega_r)} + \kappa), \tag{142}$$

where $C = C(n, q, r, \|f\|_{C^1})$ and $u^+ = \max(u, 0)$.

For $\beta \geq 1$ and $N > \kappa > 0$, we define $H \in C^1([\kappa, \infty))$ by $H(w) = w^\beta - \kappa^\beta$ for $w \in [\kappa, N]$ and taking H to be linear for $z \geq N$. Setting $w = u^+ + \kappa$ and $\hat{b} = \lambda^{-2}\kappa^{-2}|\psi|^2 + \lambda^{-1}\kappa^{-1}|\phi|$, and arguing as in the proof of [14, Theorem 8.15], we obtain the estimate

$$\int_{\Omega_R} |DH(w)|^2 dx \leq 6 \int_{\Omega_R} \hat{b}\, |H'(w)w|^2 dx.$$

Since $\hat{b} = 0$ a.e. on $\mathbf{T}^{n-1} \times (r, R)$, we get

$$\int_{\Omega_r} |DH(w)|^2 dx \leq 6 \int_{\Omega_R} \hat{b}\, |H'(w)w|^2 dx = 6 \int_{\Omega_r} \hat{b} |H'(w)w|^2 dx.$$

Note that $H(w) \in H^1(\Omega_r)$ and $H(w) = 0$ on $S_f = \{(x', f(x')) \mid x' \in \mathbf{T}^{n-1}\} \subset \partial\Omega_r$ with $\mathcal{H}^{n-1}(S_f) > 0$. Then, applying the Sobolev-Poincare inequality in Ω_r, we obtain

$$\|H(w)\|_{L^s(\Omega_r)} \leq C\|DH(w)\|_{L^2(\Omega_r)},$$

where $s = 2n/(n-2)$ for $n > 2$ and any $s > 2$ for $n = 2$, and $C = C(s, n, \Omega_r, S_f)$. In fact, the dependence C on S_f is actually the dependence on r and $\|f\|_{C^1}$, which can be verified by mapping Ω_r onto $\mathbf{T}^{n-1} \times (0, 1)$ by a map Φ such that $\Phi(S_f) = \mathbf{T}^{n-1} \times \{0\}$ and the C^1 norms of Φ and Φ^{-1} depend only on r and $\|f\|_{C^1}$. Thus $C = C(s, n, r, \|f\|_{C^1})$.

For $n = 2$, we choose $\hat{q}$ so that $2 < \hat{q} < q$. Now, denoting $\hat{n} = n$ for $n > 2$ and $\hat{n} = \hat{q}$ for $n = 2$, we get

$$\begin{aligned}\|H(w)\|_{L^{\frac{2\hat{n}}{\hat{n}-2}}(\Omega_r)} &\leq C\left(\int_{\Omega_r} \hat{b}\,|H'(w)w|^2 dx\right)^{\frac{1}{2}} \\ &\leq C\|\hat{b}\,\|^{1/2}_{L^{q/2}(\Omega_r)}\|H(w)w\|_{L^{2q/(q-2)}(\Omega_r)},\end{aligned}$$

where $C = C(n, r, \|f\|_{C^1})$. Starting from this estimate, we follow the proof of [14, Theorem 8.15] to perform the Moser iteration in domain Ω_r to obtain

$$\sup_{\Omega_r} u \leq C(\|u^+\|_{L^2(\Omega_r)} + \kappa),$$

with $C = C(n, q, r, \|f\|_{C^1})$. Since, on $\mathbf{T}^{n-1} \times [r, R]$, the equation is homogeneous and $u = 0$ on $\{x_n = R\}$, we obtain the same bound on $\mathbf{T}^{n-1} \times [r, R]$ from the maximum principle. Thus (142) is proved.

It remains to drop the term $\|u^+\|_{L^2(\Omega_r)}$ in (142). Denoting

$$M = \sup_{\Omega_R} u = \sup_{\Omega_r} u,$$

and proceeding as in the proof of [14, Theorem 8.16], we obtain

$$\begin{aligned}\frac{\lambda}{2}\int_{\Omega_R} \frac{|Du^+|^2}{(M+\kappa-u^+)^2}dx &\leq \frac{1}{M+\kappa}\int_{\Omega_R}\left(\frac{u^+|\phi|}{(M+\kappa-u^+)}\right. \\ &\quad \left. + \frac{(M+\kappa)|\psi|^2}{2\lambda(M+\kappa-u^+)^2}\right)dx.\end{aligned}$$

On the right-hand side, the integrand vanishes on $\{x_n > r\}$, and hence we can change the domain of integration to Ω_r. And, since the integrand on the left-hand side is nonnegative, we can integrate it over Ω_r so that the inequality is still kept. Then, by the definition of κ, we get

$$\int_{\Omega_r} \frac{|Du^+|^2}{(M+\kappa-u^+)^2} \leq C(|\Omega_r|).$$

Defining $w = \log\left(\dfrac{M+\kappa}{M+\kappa-u^+}\right)$, we have

$$\int_{\Omega_r} |Dw^2| \leq C(|\Omega_r|).$$

Also, $w = 0$ on S_f. Thus, by the Poincare inequality in Ω_r,

$$\int_{\Omega_r} |w^2| \leq C(n, \Omega_r, S_f) = C(n, r, \|f\|_{C^1}). \tag{143}$$

Now we follow the argument of [14, Page 193] and show that $w(x)$ in Ω_R is a subsolution of an equation of form (141) with the right-hand side $\hat{\phi} + \operatorname{div}\hat{\psi}$, where $\hat{\phi} = |\phi|/\kappa + |\psi|^2/2\lambda\kappa^2$ and $\hat{\psi} = \psi/(M+\kappa-u^+)$. Thus, $\hat{\phi}$ and $\hat{\psi}$ vanish on $\{x_n > r\}$, and $w(x)$ satisfies (142). Since, from the definition, $\|\hat{\phi}\|_{L^{\frac{q}{2}}(\Omega_r)} \leq 2\lambda$ and $\|\psi\|_{L^q(\Omega_r)} \leq \lambda$, we obtain

$$\sup_{\Omega_R} w \leq C(n, q, r, \|f\|_{C^1})(1 + \|w\|_{L^2(\Omega_r)}) \leq C,$$

where we used (143) to get the second inequality. Thus, $(M+\kappa)/\kappa \leq C$, and Lemma is proved. $\square$

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