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MULTIDIMENSIONAL TRANSONIC SHOCKS AND FREE BOUNDARY PROBLEMS FOR NONLINEAR EQUATIONS OF MIXED TYPE

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1. Introduction

We are concerned with the existence and stability of multidimensional transonic shocks for the Euler equations for steady potential compressible fluids. The Euler equations, consisting of the conservation law of mass and the Bernoulli law for the velocity, can be written as the following second-order nonlinear equation of mixed elliptic-hyperbolic type for the velocity potential $\varphi : \Omega \subset \mathbf{R}^n \to \mathbf{R}$:

$$\operatorname{div}\left(\rho(|D\varphi|^2)D\varphi\right) = 0, \tag{1.1}$$

where the density function $\rho(q^2)$ is

$$\rho(q^2) = \left(1 - \theta q^2\right)^{\frac{1}{2\theta}} \tag{1.2}$$

with $\theta = \frac{\gamma-1}{2} > 0$ for the adiabatic exponent $\gamma > 1$.

The second-order nonlinear equation (1.1) is strictly elliptic at $D\varphi$ with $|D\varphi| = q$ if

$$\rho(q^2) + 2q^2\rho'(q^2) > 0; \tag{1.3}$$

and is strictly hyperbolic if

$$\rho(q^2) + 2q^2\rho'(q^2) < 0. \tag{1.4}$$

The elliptic regions of the equation (1.1) correspond to the subsonic flow, and the hyperbolic regions of (1.1) to the supersonic flow.

Some efforts have been made in solving the nonlinear equation (1.1) of mixed type. Shiffman [30], Bers [5], and Finn-Gilbarg [16] proved the existence and uniqueness of solutions for the problem of subsonic flows of (1.1) passing an obstacle and showed that, if the uniform outflow speed at infinity is sufficiently subsonic, then there exists a unique subsonic solution of this problem, in which the nonlinear equation (1.1) is uniformly elliptic; also see Dong [15] for further results. When the uniform outflow speed at infinity is near sonic, Morawetz in [27] showed that the flows of (1.1) past the obstacle may contain transonic shocks in general. One exception is that, when the obstacle forms a wedge or conical body with a sharp

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head and an angle smaller than a certain degree, the uniform outflow with supersonic speed may produce a non-transonic shock (hyperbolic-hyperbolic shock) attached to the sharp head; the existence and stability of such shocks can be found in [11, 12, 19, 22, 25, 29, 32] and the references cited therein.

Transonic shocks also arise in many other situations of physical importance. For example, when a plane shock hits a wedge head on, a self-similar pattern of regularly reflected shock travels outward as the shock moves forward in time, provided either the wedge angle is large or the strength of the plane shock is large; then some part or all of the reflected shock may form a transonic shock dividing two regions of smooth flow, which is hyperbolic outside the shock and elliptic inside it (see [18, 28]). Steady transonic shocks are also very useful for constructing global solutions for some time-dependent problems (see [10]).

In this paper, we prove the existence and stability of steady multidimensional transonic shocks (hyperbolic-elliptic shocks) for (1.1) under a $C^{2,\alpha}, \alpha \in (0,1)$, steady perturbation of the upstream supersonic flow. We consider (1.1) in a bounded domain $\Omega := Q^{n-1} \times (-N_1, N_2)$ with $Q^{n-1} = (0,a)^{n-1}$, with Neumann boundary conditions $\partial_\nu u = 0$ on $\partial Q^{n-1} \times (-N_1, N_2)$, and Dirichlet conditions on $x_n = -N_1$ and $x_n = N_2$; that is, a flow in a channel (with quadratic cross-section if $n = 3$). Our results indicate that, for any given upstream supersonic flow φ^- which is sufficiently close in $C^{2,\alpha}$ to a uniform flow in the direction x_n, there exists a unique solution φ of (1.1) with boundary data described above, such that $\varphi = \varphi^-$ in the supersonic region Ω^- of φ, and the equation (1.1) is elliptic in $\Omega^+ := \Omega \setminus \overline{\Omega^-}$. The shock surface S dividing Ω^+ and Ω^- is $C^{2,\alpha}$; that is, S is a graph $x_n = f(x_1, \cdots, x_{n-1})$ with $f \in C^{2,\alpha}$. The solution φ is stable under the $C^{2,\alpha}$ steady perturbation of the supersonic flow φ^-.

The transonic shock problem can be formulated into a free boundary problem: The free boundary is the location of the transonic shock, and the free boundary condition is the Rankine-Hugoniot jump condition on the shock. The equation is hyperbolic in the upstream region where the given $C^{2,\alpha}$ perturbed flow is supersonic. We are looking for the location of free boundary such that the free boundary condition holds and the equation (1.1) is elliptic in the downstream region.

In order to solve this free boundary problem, we first consider a one-phase problem for a uniformly elliptic equation obtained by a modification of (1.1) away from the elliptic region: a solution satisfies the modified equation in the downstream region and the modified free boundary condition, and coincides with the given hyperbolic phase in the upstream region. Then, by a gradient estimate, we show that the solution in fact solves the original problem. In order to avoid the difficulties related to the study of the free boundary up to the fixed boundary, we use a reflection technique to extend the domain so that the whole free boundary lies in the interior of the extended domain.

One of the main difficulties of the modified free boundary problem (Problem C below) is that it does not directly fit into the variational framework in Alt-Caffarelli [1] and Alt-Caffarelli-Friedman [2]. Indeed, according to Remark 3.1, Problem C can be reformulated into the following form: Find a non-negative solution $u \in C(\Omega)$

satisfying suitable boundary conditions on $\partial\Omega$ and

$$\operatorname{div} A(x, Du) = f(x) \qquad \text{in } \Omega^+ := \{u > 0\}, \tag{1.5}$$

$$A(x, Du) \cdot \nu = G(x, \nu) \qquad \text{on } S := \partial\Omega^+ \setminus \partial\Omega, \tag{1.6}$$

where ν is the unit normal vector to S towards the unknown phase. The equation is quasilinear, uniformly elliptic, while the dependence on ν in the function $G(x, \nu)$ has a certain structure. The methods of [1] and [2] are directly applicable if $A(x, Du) = a(x, |Du|)Du$, where $a(x, s)$ is a scalar function, and G is independent of ν. However, both conditions do not hold in our problem. On the other hand, the nonlinearity in our problem makes it difficult to apply the Harnack inequality approach of Caffarelli [6]. In particular, a boundary comparison principle for positive solutions of elliptic equations in Lipschitz domains is unavailable yet in the case that nonlinear equations are not homogeneous with respect to D^2u, Du and u, which is the our case.

The approach we develop here is an iteration scheme, which is based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of a solution across the free boundary has a strictly positive lower bound. The non-degeneracy is also essential in other approaches to free boundary problems, for example, see Alt-Caffarelli [1], Alt-Caffarelli-Friedman [2, 3], and Caffarelli [6]. In terms of the problem (1.5) and (1.6), the iteration process is the following: Let the domain Ω_i^+ be given so that $S_i := \partial\Omega_i^+ \setminus \partial\Omega$ is in $C^{2,\alpha}$. We solve the oblique derivative problem (1.5) and (1.6) in Ω_i^+ (where the fixed boundary conditions on $\partial\Omega_i^+ \cap \partial\Omega$ is also used) to obtain the solution $u_i \in C^{2,\alpha}(\overline{\Omega_i^+})$. However, u_i is not identically zero on S_i in general. Using the nondegeneracy and geometry of our problem, we extend u_i to the whole domain Ω so that $u_i \in C^{2,\alpha}(\overline{\Omega})$ and $\partial_{x_n} u_i \geq a > 0$ in Ω. Thus, the level set $S_{i+1} := \{u_i = 0\} \cap \Omega$ is a $C^{2,\alpha}$ surface. We define $\Omega_{i+1}^+ := \{u > 0\}$ for the next step. The fixed point Ω^+ of this process determines a solution of the free boundary problem, since the corresponding solution u of (1.5) and (1.6) satisfies $u > 0$ on Ω^+ and $u = 0$ on S. On the other hand, since the right-hand side of the free boundary condition (1.6) depends on ν, the elliptic estimates alone are not sufficient to get the existence of a fixed point. However, the structure of our problem allows to obtain better estimates for the iteration and to prove the existence of a fixed point.

The uniqueness and stability of solutions of the free boundary problem are obtained by using the regularity and nondegeneracy of solutions.

The nonlinear approach we develop here can be applied for solving other multidimensional transonic shock problems. As a direct example, in Section 7, we establish the existence and stability of multidimensional transonic shocks near spherical or circular transonic shocks. Another advantage of this approach is that it can be applied to multidimensional free boundary problems with more general fixed boundary conditions. Furthermore, our approach and results in this paper can extend to the problems with a steady $C^{1,\alpha}, \alpha \in (0, 1)$, perturbation of the upstream supersonic flow and/or the problem with unbounded domains (see Chen-Feldman [9]).

A similar problem was considered in Canic-Keyfitz-Lieberman [8] for the two-dimensional transonic small-disturbance (TSD) equation, which governs the behavior of the first non-trivial term in the geometric optics expansion to (1.1) near a certain physical point. The TSD model can be written as a second-order nonlinear equation of mixed type in two dimension with coefficients depending only upon the unknown function itself. The main difference between the TSD model and (1.1) is that the coefficients of (1.1) depend on the gradient of the unknown function, while the coefficients of the TSD equation are independent of the gradient of the unknown function that generates additional compactness on which the approach in [8] relies. For other related results, we refer the reader to Majda [26] on the existence and stability, locally in time, of multidimensional shock fronts for the Euler equations for compressible fluids.

In Section 2, we formulate the multidimensional transonic shock problem into a free boundary problem, and then we describe the main theorems of this paper. In Section 3, we introduce a subsonic truncation procedure and an extension procedure to reformulate the free boundary problem for the equation of mixed type into the free boundary problem for a second-order, nonlinear, uniformly elliptic equation with a non-degenerate free boundary condition and to resolve the difficulties for the study of the free boundary up to the fixed boundary. In Section 4, we introduce an iteration scheme and prove the existence of a fixed point, that is, the existence of a solution of the truncated free boundary problem. By choosing the $C^{2,\alpha}$ perturbation small in the hyperbolic region, we obtain an a priori gradient estimate to ensure that our solution is the solution of the original free boundary problem. We show the stability and uniqueness of the solution of the free boundary problem in Sections 5 and 6. In Section 7, we give another application of our approach for establishing the existence and stability of multidimensional transonic shocks near spherical or circular transonic shocks.

2. Transonic Shocks, Free Boundaries, and Main Theorems

In this section we formulate the multidimensional transonic shock problem into a free boundary problem for (1.1), and then we describe the main theorems of this paper.

A function $\varphi \in W^{1,\infty}(\Omega)$ is a weak solution of (1.1) if

(i) $|D\varphi(x)| \le 1/\sqrt{\theta}$ a.e. $x \in \Omega$ (physical region);

(ii) for any $w \in C_0^\infty(\Omega)$,

$$\int_\Omega \rho(|D\varphi|^2) D\varphi \cdot Dw \, dx = 0. \tag{2.1}$$

We are interested in the weak solutions with shocks. Let Ω^+ and Ω^- be open nonempty subsets of Ω such that

$$\Omega^+ \cap \Omega^- = \emptyset, \qquad \overline{\Omega^+} \cup \overline{\Omega^-} = \overline{\Omega},$$

and $S = \partial\Omega^+ \setminus \partial\Omega$. Let $\varphi \in W^{1,\infty}(\Omega)$ be a weak solution of (1.1) so that $\varphi \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm})$ and $D\varphi$ has a jump across S.

We first derive the conditions on S which is an $(n-1)$-dimensional smooth surface. First, the requirement $\varphi \in W^{1,\infty}(\Omega)$ yields $curl(D\varphi) = 0$ in the sense of distributions, which implies

$$\varphi_\tau^+ = \varphi_\tau^- \qquad \text{on} \quad S, \tag{2.2}$$

where

$$\varphi_\tau^\pm := D\varphi^\pm - (D\varphi^\pm \cdot \nu)\nu$$

are the tangential gradients of φ on S in the tangential space with $(n-1)$-dimension on the $\Omega^{\pm}$ sides, respectively, and ν is the unit normal vector to S from Ω^- to Ω^+. Then we simply write $\varphi_\tau := \varphi_\tau^{\pm}$ on S and assume

$$\varphi^+ = \varphi^- \qquad \text{on} \quad S. \tag{2.3}$$

Now, for $w \in C_0^\infty(\Omega)$, we use (1.1) and (2.1) to compute

$$\begin{aligned} 0 &= \int_\Omega \rho(|D\varphi|^2) D\varphi \cdot Dw\, dx \\ &= \left(\int_{\Omega^+} + \int_{\Omega^-}\right) \rho(|D\varphi|^2) D\varphi \cdot Dw\, dx \\ &= -\int_{\partial\Omega^+} \rho(|D\varphi|^2) D\varphi \cdot \nu\, w\, d\mathcal{H}^{n-1} + \int_{\partial\Omega^-} \rho(|D\varphi|^2) D\varphi \cdot \nu\, w\, d\mathcal{H}^{n-1} \\ &= \int_S \left(-\rho(|D\varphi^+|^2) D\varphi^+ \cdot \nu + \rho(|D\varphi^-|^2) D\varphi^- \cdot \nu\right) w\, d\mathcal{H}^{n-1}. \end{aligned}$$

Thus, another condition on S, which measures the jump of the normal derivative of φ across S, is

$$\Big[\rho(|D\varphi|^2) D\varphi \cdot \nu\Big]_S = 0, \tag{2.4}$$

where the bracket denotes the difference between the values of the function along S on the Ω^+ and Ω^- sides, respectively. That is,

$$\rho(|D\varphi^+|^2)\varphi_\nu^+ = \rho(|D\varphi^-|^2)\varphi_\nu^- \qquad \text{on} \quad S, \tag{2.5}$$

where $\varphi_\nu^{\pm} = D\varphi^{\pm} \cdot \nu$ are the values of the normal derivative of φ on the $\Omega^{\pm}$ sides, and

$$\rho(|D\varphi^{\pm}|^2) = \left(1 - \theta|\varphi_\tau^{\pm}|^2 - \theta|\varphi_\nu^{\pm}|^2\right)^{\frac{1}{2\theta}},$$

respectively.

Lemma 2.1. *Let $K > 0$. Then the function*

$$\Phi_K(p) := \left(K - \theta p^2\right)^{\frac{1}{2\theta}} p,$$

defined for $p \in \left[0, \sqrt{K/\theta}\right]$, satisfies

(i) $\lim_{p\to 0} \Phi_K(p) = \lim_{p\to\sqrt{K/\theta}} \Phi_K(p) = 0$;

(ii) $\Phi_K(p) > 0$ *for* $p \in \left(0, \sqrt{K/\theta}\right)$;

(iii) $0 < \Phi_K'(p) \le K^{\frac{1}{2\theta}}$ *for* $p \in (0, p_{sonic}^K)$, *and* $\Phi_K'(p) < 0$ *for* $p \in \left(p_{sonic}^K, \sqrt{K/\theta}\right)$;

(iv) $\Phi_K''(p) < 0$ *for* $p \in (0, p_{sonic}^K]$,

where

$$p_{sonic}^K := \sqrt{K/(\theta+1)}. \tag{2.6}$$

Proof. Properties (i) and (ii) are obvious. For $p \in \left(0, \sqrt{K/\theta}\right)$, we compute

$$\begin{aligned} \Phi_K'(p) &= \left(K - \theta p^2\right)^{\frac{1}{2\theta}-1} \left(K - (\theta+1)p^2\right), \\ \Phi_K''(p) &= p\left(K - \theta p^2\right)^{\frac{1}{2\theta}-2} \left((\theta+1)p^2 - 3K\right), \end{aligned}$$

and note that $p^K_{sonic} = \sqrt{K/(\theta+1)} \in \left(0, \sqrt{K/\theta}\right)$. Then (iii) and (iv) follow. □

Suppose that φ is a solution satisfying

$$|D\varphi| < p^1_{sonic} := 1/\sqrt{\theta+1} \quad \text{in } \Omega^+, \qquad |D\varphi| > p^1_{sonic} \quad \text{in } \Omega^-, \tag{2.7}$$

and

$$D\varphi^\pm \cdot \nu > 0 \quad \text{on } S, \tag{2.8}$$

besides (2.2) and (2.4). Then φ is a *transonic shock solution* with *transonic shock* S, which divides the *subsonic region* Ω^+ and the *supersonic region* Ω^-. In addition, φ satisfies the physical entropy condition (see Courant-Friedrichs [13]; also see Lax [21]):

$$\rho(|D\varphi^-|^2) < \rho(|D\varphi^+|^2) \tag{2.9}$$

which implies, by (2.8), that the density ρ increases in the flow direction.

Note that the equation (1.1) is elliptic in the subsonic region Ω^+ and hyperbolic in the supersonic region Ω^-.

Let (x', x_n) be the coordinates of $\mathbf{R}^n$ with $x_n \in \mathbf{R}$ and $x' = (x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}$. From now on, we denote $\Omega := (0,a)^{n-1} \times (-N_1, N_2)$.

Let $q^- \in \left(p^1_{sonic}, 1/\sqrt{\theta}\right)$ and $\varphi_0^-(x) := q^- x_n$. Then φ_0^- is a supersonic solution in Ω. According to Lemma 2.1, there exists a unique $q^+ \in (0, p^1_{sonic})$ such that

$$\left(1-\theta(q^+)^2\right)^{\frac{1}{2\theta}} q^+ = \left(1-\theta(q^-)^2\right)^{\frac{1}{2\theta}} q^-. \tag{2.10}$$

Define $\varphi_0^+(x) := q^+ x_n$ in Ω. Then the function

$$\varphi_0(x) = \min(\varphi_0^+(x), \varphi_0^-(x)) \;\text{ for } x \in \Omega \tag{2.11}$$

is a transonic shock solution in Ω, in which $\Omega_0^\pm = \{\pm x_n > 0\} \cap \Omega$ are the subsonic and supersonic regions of φ_0, respectively. Also note that, on $\partial(0,a)^{n-1} \times (-N_1, N_2)$, the boundary condition $(\varphi_0)_\nu = 0$ holds.

We study perturbations of the solution (2.11). We use the following Hölder norms: for $\alpha \in (0,1)$ and any non-negative integer k,

$$\begin{aligned}
&[u]_{k,0,\Omega} = \sum_{|\beta|=k} \sup_{x\in\Omega} |D^\beta u(x)|, \\
&[u]_{k,\alpha,\Omega} = \sum_{|\beta|=k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \\
&\|u\|_{k,0,\Omega} = \sum_{j=0}^{k} [u]_{j,0,\Omega}, \qquad \|u\|_{k,\alpha,\Omega} = \|u\|_{k,0,\Omega} + [u]_{k,\alpha,\Omega},
\end{aligned} \tag{2.12}$$

where $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_l \geq 0$ integers, $D^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$, and $|\beta| = \beta_1 + \cdots + \beta_n$.

Our transonic shock problem is the following.

Problem A. Given a supersonic solution φ^- of (1.1) in Ω, which is a $C^{2,\alpha}$ perturbation of φ_0^-, for some $\alpha > 0$:

$$\|\varphi^- - \varphi_0^-\|_{2,\alpha,\Omega} \leq \sigma, \tag{2.13}$$

with $\sigma > 0$ small, and satisfies

$$\varphi_\nu^- = 0 \qquad \text{on} \quad \partial(0,a)^{n-1} \times (-N_1, N_2), \tag{2.14}$$

find a transonic shock solution φ in Ω such that

$$\varphi = \varphi^- \qquad \text{in} \ \ \Omega^-,$$

where Ω^- is the supersonic region of φ in Ω, defined by $\Omega^- := \Omega\backslash\overline{\Omega^+}$ with $\Omega^+ := \{x \in \Omega \ : \ |D\varphi(x)| < p^1_{sonic}\}$ which is the subsonic region of φ, and

$$\begin{array}{lll} \varphi = \varphi^- & \text{on} & (0,a)^{n-1} \times \{-N_1\}, \\ \varphi = \varphi_0^+ & \text{on} & (0,a)^{n-1} \times \{N_2\}, \\ \varphi_\nu = 0 & \text{on} & \partial(0,a)^{n-1} \times (-N_1, N_2). \end{array} \tag{2.15}$$

In order to construct a solution of Problem A, we reformulate it into a more general free boundary problem for the subsonic part of the solution. The following heuristic observation motivates our formulation: Since $\varphi = \varphi^-$ in Ω^-, $|D\varphi| < p^1_{sonic} < |D\varphi^-|$ in Ω^+, $|D\varphi^-| \sim \partial_{x_n}\varphi^- > p^1_{sonic}$ in Ω, and we expect that $\Omega^+ = \{x_n > f(x')\} \cap \Omega$ and $|D\varphi| \sim \partial_{x_n}\varphi < p^1_{sonic}$ in Ω^+ with (2.3) across the free boundary, then φ should satisfy

$$\varphi(x) \le \varphi^-(x) \qquad \text{for} \ \ x \in \Omega. \tag{2.16}$$

Now we can formulate the following free boundary problem:

Problem B. Find $\varphi \in C(\overline{\Omega})$ such that
- (i) φ satisfies (2.16) in Ω and (2.15) on $\partial\Omega$;
- (ii) $\varphi \in C^{2,\alpha}(\overline{\Omega^+})$ is a solution of (1.1) in $\Omega^+ := \{x \in \Omega \ : \ \varphi(x) < \varphi^-(x)\}$, the non-coincidence set;
- (iii) the free boundary $S = \partial\Omega^+ \cap \Omega$ is given by the equation $x_n = f(x')$ for $x' \in (0,a)^{n-1}$ so that $\Omega^+ = \{x_n > f(x')\} \cap \Omega$, where $f \in C^{2,\alpha}([0,a]^{n-1})$;
- (iv) the free boundary condition (2.4) holds on S.

Note that the definitions of the regions $\Omega^\pm$ in Problems A and B are apriori different, since the formula of Ω^+ given in the formulation of Problem B is a new definition, rather than an expression of the region Ω^+ defined in Problem A above. In particular, in the free boundary problem (Problem B), we do not require that the phase φ^- should be a solution of (1.1) and that φ be subsonic in Ω^+, although we require it in Problem A so that the free boundary is a transonic shock.

We will show that, if the perturbation $\varphi^- - \varphi_0^-$ is small enough in $C^{2,\alpha}$, then the free boundary problem (Problem B) has a solution so that Problem A has a transonic shock solution. Furthermore, the transonic shock is stable under any small $C^{2,\alpha}$ perturbation of φ^-. Precisely, we have the following theorem.

Theorem 2.1. *Let $q^+ \in (0, p^1_{sonic})$ and $q^- \in \left(p^1_{sonic}, 1/\sqrt{\theta}\right)$ satisfy (2.10), and let φ_0 be the transonic shock solution (2.11). Then there exist positive constants σ_0, C_1, and C_2 depending only on n, γ, q^+, and Ω such that, for every $\sigma \le \sigma_0$ and any supersonic solution φ^- of (1.1) satisfying the conditions stated in Problem A, there exists a unique solution φ of Problem A satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+} \le C_1\sigma.$$

In addition, $\Omega^+ = \{x_n > f(x')\} \cap \Omega$ *where* $f : \mathbf{R}^{n-1} \to \mathbf{R}$ *satisfies*

$$\|f\|_{2,\alpha,\mathbf{R}^{n-1}} \le C_2\sigma,$$
$$D_{x'}f(x') = 0 \quad \text{on } \partial(0,a)^{n-1},$$

that is, the shock surface $S = \{(x', x_n) \ : \ x_n = f(x'), x' \in \mathbf{R}^{n-1}\} \cap \Omega$ *is in* $C^{2,\alpha}$ *and* S *is orthogonal to* $\partial\Omega$ *at their intersection points.*

Theorem 2.1 is a corollary of the following corresponding theorem for Problem B.

Theorem 2.2. *Let* q^+ *and* q^- *be as those in Theorem* 2.1*. Then there exist positive constants* σ_0*,* C_1*, and* C_2 *depending only on* n*,* γ*,* q^+*, and* Ω *such that, for every* $\sigma \le \sigma_0$ *and any function* φ^- *satisfying* (2.13) *and* (2.14)*, there exists a unique solution* φ *of Problem B satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+} \le C_1\sigma.$$

In addition, $\Omega^+ = \{x_n > f(x')\} \cap \Omega$ *where* $f : \mathbf{R}^{n-1} \to \mathbf{R}$ *satisfies*

$$\|f\|_{2,\alpha,\mathbf{R}^{n-1}} \le C_2\sigma,$$
$$D_{x'}f(x') = 0 \quad \text{on } \partial(0,a)^{n-1}.$$

Remark 2.1. *If the hyperbolic phase is* C^∞*, then the solution and the corresponding free boundary in Theorem* 2.2 *are also* C^∞*. Furthermore, our results can extend to the problem with a steady* $C^{1,\alpha}, \alpha \in (0,1)$*, perturbation of the upstream supersonic flow and/or general Dirichlet data* $h(x'), x' \in \mathbf{R}^{n-1}$*, at* $x_n = N_2$ *satisfying*

$$\|h - \varphi_0^+\|_{1,\alpha,\mathbf{R}^{n-1}} \le C\sigma.$$

Also, the Dirichlet data in Problem B may be replaced by the corresponding Neumann data satisfying the global solvability condition. Furthermore, the bounded domain in the problem can be replaced by unbounded domains, especially the unbounded cylinder up to $x_n = \infty$*. See Chen-Feldman* [9] *for the details.*

The following features of the equation (1.1) and the free boundary condition (2.5) will be employed in the proof of Theorems 2.1 and 2.2. The nonlinear equation (1.1) is uniformly elliptic only if $|D\varphi| < p^1_{sonic} - \varepsilon$ in Ω^+ for some $\varepsilon > 0$; the quantity $|D\varphi^+| = \left((\varphi_\nu^+)^2 + |\varphi_\tau|^2\right)^{1/2}$ on S is subsonic only if φ_τ is sufficiently small; and the free boundary condition (2.5) is uniformly non-degenerate (i.e., $\varphi_\nu^- - \varphi_\nu^+$ is bounded from below by a positive constant on S) only if $\varphi_\nu^- > p^K_{sonic} - \varepsilon$ on S for some $\epsilon > 0$ with $K = 1 - \theta|\varphi_\tau|^2$. By (2.13), these conditions hold if, for any $x \in S$, the unit normal vector $\nu(x)$ to S is sufficiently close to being orthogonal to $\{x_n = 0\}$.

To establish Theorem 2.2 (hence Theorem 2.1), we first introduce and solve a truncated problem, by modifying both the nonlinear equation (1.1) and the free boundary condition (2.5), in order to make the equation uniformly elliptic and the free boundary condition non-degenerate. Then, for small σ, a gradient bound for the solution implies that it indeed solves the original free boundary problem, Problem B, hence Problem A.

3. Subsonic Truncations

In this section, we introduce a truncated free boundary problem, by modifying both the nonlinear equation (1.1) and the free boundary condition (2.5), to make the equation uniformly elliptic and the free boundary condition non-degenerate; we also extend the domain Ω to the domain Ω_e to overcome the difficulties for the study of the free boundary up to the fixed boundary for Problem B.

3.1. **Truncation of the Equation (1.1).** The truncation procedure below is motivated by the argument in [3, pp. 87–90].

First, we note that the ellipticity condition for (1.1) at $|D\varphi| = q$ is (1.3), which is equivalent to

$$\Phi_1'(q) > 0, \tag{3.1}$$

where $\Phi_K(p)$ is the function introduced in Lemma 2.1.

By Lemma 2.1(iii), the inequality (3.1) holds for $q \in (0, p^1_{sonic})$. We modify the function $\Phi_1(q)$ so that the new function $\tilde{\Phi}_1(q)$ satisfies (3.1) uniformly for all $q > 0$ and, around q^+, the function $\tilde{\Phi}_1(q) = \Phi_1(q)$.

Let

$$\varepsilon = \frac{p^1_{sonic} - q^+}{2}. \tag{3.2}$$

Let $y = c_0 q + c_1$ be the tangent line of the graph of $y = \Phi_1(q)$ at $q = p^1_{sonic} - \varepsilon$. Then, using Lemma 2.1(iii), we obtain

$$c_0 = \Phi_1'(p^1_{sonic} - \varepsilon) > 0.$$

Now the function $\tilde{\Phi}_1 : [0, \infty) \to \mathbf{R}$, defined by

$$\tilde{\Phi}_1(q) = \begin{cases} \Phi_1(q) & \text{if } \ 0 \le q < p^1_{sonic} - \varepsilon, \\ c_0 q + c_1 & \text{if } \ q > p^1_{sonic} - \varepsilon, \end{cases} \tag{3.3}$$

satisfies $\tilde{\Phi}_1 \in C^{1,1}([0, \infty))$.

Define

$$\tilde{\rho}(q^2) = \frac{\tilde{\Phi}_1(q)}{q} \qquad \text{for } \ q \in [0, \infty), \tag{3.4}$$

that is,

$$\tilde{\rho}(s) = \begin{cases} \rho(s) & \text{if } \ 0 \le s < (p^1_{sonic} - \varepsilon)^2, \\ c_0 + \dfrac{c_1}{\sqrt{s}} & \text{if } \ s > (p^1_{sonic} - \varepsilon)^2. \end{cases}$$

Then $\tilde{\rho} \in C^{1,1}([0, \infty))$, and

$$\tilde{\rho}(q^2) = \rho(q^2) \qquad \text{if } 0 \le q < p^1_{sonic} - \varepsilon. \tag{3.5}$$

By Lemma 2.1(iii)–(iv) and the definition (3.3) of $\tilde{\Phi}_1$,

$$0 < c_0 \le \tilde{\Phi}_1'(q) = \tilde{\rho}(q^2) + 2q^2 \tilde{\rho}'(q^2) \le C, \quad \text{for } q \in (0, \infty),$$

for some constant $C > 0$. Thus, the equation

$$\tilde{\mathcal{L}}\varphi := \operatorname{div}(\tilde{\rho}(|D\varphi|^2) D\varphi) = 0 \tag{3.6}$$

is uniformly elliptic, with ellipticity constants depending only on q^+ and γ.

We also perform the corresponding truncation of the free boundary condition (2.5):

$$\tilde{\rho}(|D\varphi|^2)\varphi_\nu = \rho(|D\varphi^-|^2)D\varphi^- \cdot \nu \qquad \text{on} \quad S. \tag{3.7}$$

On the right-hand side of (3.7), we use the non-truncated function ρ since $\rho \neq \tilde{\rho}$ on the range of $|D\varphi^-|^2$. Note that (3.7), with the right-hand side considered as a known function, is the conormal boundary condition for the uniformly elliptic equation (3.6).

Thus, we solve the following free boundary problem, which is a truncated version of Problem B.

Problem C. Find $\varphi \in C(\overline{\Omega})$ such that

- (i) φ satisfies (2.16) in Ω and (2.15) on $\partial\Omega$;
- (ii) $\varphi \in C^{2,\alpha}(\overline{\Omega^+})$ is a solution of (3.6) in $\Omega^+ := \{\varphi(x) < \varphi^-(x)\} \cap \Omega$, the non-coincidence set;
- (iii) the free boundary $S = \partial\Omega^+ \cap \Omega$ is given by the equation $x_n = f(x')$ for $x' \in (0, a)^{n-1}$ so that $\Omega^+ = \{x_n > f(x')\} \cap \Omega$, where $f \in C^{2,\alpha}([0, a]^{n-1})$ and $D_{x'}f = 0$ on $\partial((0, a)^{n-1} \times [-N_1, N_2])$;
- (iv) the free boundary condition (3.7) holds on S.

Remark 3.1. *Introduce the function*

$$u := \varphi^- - \varphi,$$

and rewrite Problem C in terms of the function $u(x)$. Then, by (2.16)*, the problem is to find a non-negative $u \in C(\overline{\Omega})$, with boundary conditions determined by* (2.15) *and φ^-, satisfying* (1.5) *and* (1.6) *with*

$$\begin{aligned}
&A(x, P) = \tilde{\rho}(|D\varphi^-(x) - P|^2)(D\varphi^-(x) - P) - \tilde{\rho}(|D\varphi^-(x)|^2)D\varphi^-(x), \quad P \in \mathbf{R}^n,\\
&f(x) = -div\,(\tilde{\rho}(|D\varphi^-(x)|^2)D\varphi^-(x)),\\
&G(x, \nu) = (\rho(|D\varphi^-(x)|^2) - \tilde{\rho}(|D\varphi^-(x)|^2))D\varphi^-(x) \cdot \nu.
\end{aligned}$$

3.2. **Extension to the Domain $\Omega_e = \mathbf{T}^{n-1} \times (-N_1, N_2)$.** We now extend the domain Ω of the free boundary problem to the domain Ω_e to overcome the difficulties for the study of the free boundary up to the fixed boundary.

Observe that, if a function $\phi \in C^{2,\alpha}(\overline{\Omega})$ with $\Omega := (0, a)^{n-1} \times (-N_1, N_2)$ and

$$\phi_\nu = 0 \qquad \text{on} \ \partial(0, a)^{n-1} \times (-N_1, N_2), \tag{3.8}$$

then ϕ can be extended to $\mathbf{R}^{n-1} \times [-N_1, N_2]$ so that the extension (still denoted) ϕ satisfies

$$\phi \in C^{2,\alpha}(\mathbf{R}^{n-1} \times [-N_1, N_2]),$$

and, for every $m = 1, \cdots, n-1$ and $k = 0, \pm 1, \pm 2, \cdots$,

$$\phi(x_1, \cdots, x_{m-1}, ka - z, x_{m+1}, \cdots, x_n) = \phi(x_1, \cdots, x_{m-1}, ka + z, x_{m+1}, \cdots, x_n), \tag{3.9}$$

that is, ϕ is symmetric with respect to every hyperplane $\{x_m = ka\}$. Indeed, for $\mathbf{k} = (k_1, \cdots, k_{n-1}, 0)$ with $k_1, \cdots, k_{n-1}$ integers, we define

$$\phi(x + a\mathbf{k}) = \phi(\eta(x_1, k_1), \cdots, \eta(x_{n-1}, k_{n-1}), x_n) \qquad \text{for} \quad x \in (0, a)^{n-1} \times [-N_1, N_2],$$

with

$$\eta(t, k) = \begin{cases} t & \text{if } k \text{ is even,} \\ a - t & \text{if } k \text{ is odd.} \end{cases}$$

It follows from (3.9) that $\phi(x', x_n)$ is $2a$-periodic in the directions $x_1, \cdots, x_{n-1}$:

$$\phi(x + 2ae_m) = \phi(x), \qquad \text{for } x \in \mathbf{R}^{n-1} \times [-N_1, N_2],\ m = 1, \cdots, n-1,$$

where e_m is the unit vector in the direction of x_m.

Thus, with respect to this $2a$-periodicity, we can consider ϕ as a function on $\Omega_e := \mathbf{T}^{n-1} \times [-N_1, N_2]$, where $\mathbf{T}^{n-1}$ is an $(n-1)$-dimensional flat torus with its coordinates given by the cube $(0, 2a)^{n-1}$. Note that (3.9) represents an extra symmetry condition, in addition to $\phi \in C^{2,\alpha}(\mathbf{T}^{n-1} \times [-N_1, N_2])$, and (3.9) implies (3.8).

Thus we can extend φ^- in this way, that is, $\varphi^- \in C^{2,\alpha}(\Omega_e)$ satisfies (3.9). Also, $\varphi_0^\pm$ can be also considered as the functions in Ω_e satisfying (3.9), since $\varphi_0^\pm(x) = q^\pm x_n$ in $\mathbf{R}^{n-1} \times [-N_1, N_2]$, which are independent of x'.

Then we focus on the free boundary problem, Problem C, on Ω_e.

4. Existence of Solutions

In this section, we develop a nonlinear approach to prove the existence of solutions of the free boundary problem. Our approach is an iteration scheme, which is based on the non-degeneracy of the free boundary condition: the jump of the normal derivative of solutions across the free boundary has a strict lower bound. The iteration procedure in §4.1 has no additional compactness effect, which is different from that in [8]; the elliptic estimates in §4.2 alone do not produce what we require to get the existence of a fixed point. We use certain cancellations in order to get (4.20) and thus to close the argument for the existence of a fixed point.

4.1. **Iteration Procedure.** Let $M \geq 1$. We set

$$\mathcal{K}_M := \{\psi \in C^{2,\alpha}(\overline{\Omega_e}) \quad : \quad \|\psi - \varphi_0^+\|_{2,\alpha,\Omega_e} \leq M\sigma, \quad \psi \text{ satisfies (3.9)}\}, \tag{4.1}$$

where $\varphi_0^+(x) = q^+ x_n$. According to the definition, $\mathcal{K}_M$ is convex and compact in $C^{2,\beta}(\overline{\Omega_e}), 0 < \beta < \alpha$.

Let $\psi \in \mathcal{K}_M$. Since $q^- > q^+$, it follows that, if

$$\sigma \leq \frac{q^- - q^+}{C(M+1)}, \tag{4.2}$$

with large C depending only on n, then (4.1) and (2.13) imply

$$(\varphi^- - \psi)_{x_n}(x) \geq \frac{q^- - q^+}{2} > 0. \tag{4.3}$$

Then, by the implicit function theorem, the set $\Omega^+(\psi) := \{\psi(x) < \varphi^-(x)\} \cap \Omega_e$ has the form:

$$\Omega^+(\psi) = \{x_n > f(x')\} \cap \Omega_e, \qquad \|f\|_{2,\alpha,\mathbf{T}^{n-1}} \leq CM\sigma, \tag{4.4}$$

with C depending upon $q^- - q^+$. The corresponding unit normal vector

$$\nu(x') = \frac{(-D_{x'}f(x'), 1)}{\sqrt{1 + |D_{x'}f(x')|^2}} \in C^{1,\alpha}(\mathbf{T}^{n-1}; \mathbf{S}^{n-1}),$$

and

$$\|\nu - \nu_0\|_{1,\alpha,\mathbf{R}^{n-1}} \leq CM\sigma, \tag{4.5}$$

where ν_0 is defined by

$$\nu_0 := \frac{D\varphi_0^+}{|D\varphi_0^+|} = (0, \cdots, 0, 1)^\top. \tag{4.6}$$

Also, $\nu(\cdot)$ can be considered as a function on $S_\psi := \{x_n = f(x')\}$. From the definition of $f(x')$, it follows that, for $x \in S_\psi$,

$$\nu(x) = \frac{D\varphi^-(x) - D\psi(x)}{|D\varphi^-(x) - D\psi(x)|}. \tag{4.7}$$

By the definition of $\mathcal{K}_M$ and (4.2), the formula (4.7) defines ν on Ω_e, and

$$\|\nu - \nu_0\|_{1,\alpha,\Omega_e} \le CM\sigma, \qquad C = C(q^+, q^-). \tag{4.8}$$

Motivated by the free boundary condition (3.7), we define the function G_ψ on Ω_e:

$$G_\psi(x) := \rho(|D\varphi^-(x)|^2) D\varphi^-(x) \cdot \nu(x), \tag{4.9}$$

where $\nu(\cdot)$ is defined by (4.7).

We now solve the following fixed-boundary value problem in the domain $\Omega^+(\psi)$:

$$\operatorname{div}(\tilde{\rho}(|D\varphi|^2) D\varphi) = 0 \qquad \text{in} \quad \Omega^+(\psi), \tag{4.10}$$

$$\tilde{\rho}(|D\varphi|^2)\varphi_\nu = G_\psi \quad \text{on} \quad S_\psi, \tag{4.11}$$

$$\varphi = N_2 q^+ \quad \text{on} \quad \{x_n = N_2\} = \partial\Omega^+(\psi) \setminus S_\psi, \tag{4.12}$$

and show that the solution φ can be extended to the whole domain Ω_e so that $\varphi \in \mathcal{K}_M$.

4.2. **Existence and Uniqueness of the Solution for the Fixed Boundary Value Problem** (4.10)–(4.12)**.** Now we show the existence and uniqueness of the solution φ for the problem (4.10)–(4.12) and show that φ is close in $C^{2,\alpha}(\overline{\Omega^+(\psi)})$ to the unperturbed subsonic solution φ_0^+.

Proposition 4.1. *Let $M \ge 1$. There exists $\sigma_0 > 0$, depending only on the n, γ, q^+, Ω, and M such that, if $\sigma \in (0, \sigma_0)$ so that φ^- satisfies* (2.13) *and $\psi \in \mathcal{K}_M$, then there exists a unique solution $\varphi \in C^{2,\alpha}(\overline{\Omega^+(\psi)})$ of the problem* (4.10)–(4.12) *that satisfies* (3.9) *and*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+(\psi)} \le C\sigma, \tag{4.13}$$

where C depends only on n, γ, q^+, and Ω, and is independent of M, $\psi \in \mathcal{K}_M$, and $\sigma \in (0, \sigma_0)$.

Proof. In the argument below, the constants C and C_1 depend only on n, γ, q^+, and Ω, and are independent of M, $\psi \in \mathcal{K}_M$, and $\sigma \in (0, \sigma_0)$, unless other dependence is specified. We divide the proof into four steps.

Step 1. We first rewrite the problem (4.10)–(4.12) in terms of the function $v := \varphi - \varphi_0^+$. The problem then takes the form:

$$\operatorname{div} A(Dv) = 0 \quad \text{in} \quad \Omega^+(\psi), \tag{4.14}$$

$$A(Dv) \cdot \nu = g_\psi \quad \text{on} \quad S_\psi, \tag{4.15}$$

$$v = 0 \quad \text{on} \quad \Gamma_1 := \mathbf{T}^{n-1} \times \{N_2\} \equiv \partial\Omega^+(\psi) \setminus S_\psi, \tag{4.16}$$

where

$$A(P) = \tilde{\rho}(|P + q^+\nu_0|^2)(P + q^+\nu_0) - \rho((q^+)^2)q^+\nu_0 \quad \text{for} \;\; P \in \mathbf{R}^n,$$
$$g_\psi(x) = G_\psi(x) - \rho((q^+)^2)q^+\,\nu(x) \cdot \nu_0.$$

Thus, $v(x)$ satisfies the uniformly elliptic equation with the same ellipticity constants as those in (3.6). Note that

$$A(0) = 0. \tag{4.17}$$

Now we show the crucial (but simple) estimate of $g_\psi(x)$, based on a cancellation. We first note that

$$g_\psi(x) = \big[\rho(|D\varphi^-(x)|^2)D\varphi^-(x) - \rho(|D\varphi_0^+(x)|^2)D\varphi_0^+(x)\big] \cdot \nu(x). \tag{4.18}$$

Furthermore,

$$\rho(|D\varphi_0^+|^2)D\varphi_0^+ = \rho(|D\varphi_0^-|^2)D\varphi_0^- \tag{4.19}$$

in Ω_e, since both sides of (4.19) are equal to $\rho((q^+)^2)q^+\nu_0$. Thus,

$$g_\psi(x) = \big(\rho(|D\varphi^-(x)|^2)D\varphi^-(x) - \rho(|D\varphi_0^-(x)|^2)D\varphi_0^-(x)\big) \cdot \nu(x).$$

Using (2.13) and (4.8), we obtain

$$\|g_\psi\|_{1,\alpha,\Omega_e} \le C\|D\varphi^- - D\varphi_0^-\|_{1,\alpha,\Omega_e}\|\nu\|_{1,\alpha,\Omega_e} \le C\sigma(1 + M\sigma).$$

Therefore, choosing $\sigma \le \frac{1}{M}$, we have

$$\|g_\psi\|_{1,\alpha,\Omega_e} \le C\sigma. \tag{4.20}$$

2. In order to study the problem (4.14)–(4.16), we now consider the corresponding linear problem:

$$\begin{cases} \overline{\mathcal{L}}[u] := \displaystyle\sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} = r(x) \quad \text{in} \quad \Omega^+(\psi), \\ \displaystyle\sum_{i=1}^n b_i(x)u_{x_i} = g(x) \quad \text{on} \quad S_\psi, \\ u = 0 \quad \text{on} \quad \Gamma_1. \end{cases} \tag{4.21}$$

Here $a_{ij} \in C^\alpha(\overline{\Omega_e})$ satisfy the ellipticity condition:

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in \Omega_e,\ \xi \in \mathbf{R}^n,$$

with $0 < \lambda < \Lambda$, and $b_i \in C^{1,\alpha}(\overline{\Omega_e})$ satisfy

$$|b_i(x)| \le \Lambda$$

and the strict obliqueness condition:

$$b(x) \cdot \nu(x) \ge \kappa > 0 \quad \text{for any } x \in S_\psi,$$

where $b(x)$ denotes the vector $(b_1, \cdots, b_n)(x)$.

Lemma 4.2. *Let $M \ge 1$ and $\psi \in \mathcal{K}_M$. Then there exists σ_0 depending only on n, γ, q^+, Ω, and M such that, if $\sigma \in (0, \sigma_0)$ so that φ^- satisfy (2.13) and $u \in C^1(\overline{\Omega^+(\psi)}) \cap C^2(\Omega^+(\psi))$ is a solution of (4.21) with $r \in C^\alpha(\overline{\Omega_e})$, then*

(i) *there exists C depending only on λ, Λ, κ, and Ω such that*

$$\|u\|_{0,\Omega^+(\psi)} \le C(\|r\|_{0,\Omega^+(\psi)} + \|g\|_{0,\Omega^+(\psi)}); \tag{4.22}$$

(ii) *there exists C depending only on λ, Λ, κ, $\|a_{ij}\|_{0,\alpha,\Omega_e}$, $\|b_i\|_{1,\alpha,\Omega_e}$, and Ω such that*

$$\|u\|_{2,\alpha,\Omega^+(\psi)} \le C(\|r\|_{0,\alpha,\Omega^+(\psi)} + \|g\|_{1,\alpha,\Omega^+(\psi)}). \tag{4.23}$$

We now prove this lemma. Consider the functions

$$\underline{u}(x) := K\left(e^{x_n} - e^{N_2}\right), \qquad \overline{u}(x) := -K\left(e^{x_n} - e^{N_2}\right), \tag{4.24}$$

with $K = C_1(\|g\|_{0,\Omega^+(\psi)} + \|r\|_{0,\Omega^+(\psi)})$, where C_1 will be chosen below depending only on the data as in (i). Then, using (4.4) and the ellipticity of $\overline{\mathcal{L}}$, we get

$$\overline{\mathcal{L}}[\underline{u}] = a_{nn}Ke^{x_n} \ge C_1\lambda e^{-CM\sigma}\|r\|_{0,\Omega^+(\psi)} \ge \|r\|_{0,\Omega^+(\psi)},$$

if $M\sigma \le 1$ and $C_1 \ge e^C$. Similarly, for such σ and C_1,

$$\overline{\mathcal{L}}[\overline{u}] \le -\|r\|_{0,\Omega^+(\psi)}.$$

In addition,

$$\underline{u} = \overline{u} = 0 \quad \text{on} \quad \Gamma_1.$$

Let $w := \underline{u} - u$. Then

$$\overline{\mathcal{L}}w \ge \|r\|_{0,\Omega^+(\psi)} - r \ge 0 \qquad \text{in } \Omega^+(\psi),$$

i.e. w is a subsolution of $\overline{\mathcal{L}}$. Thus, by the strong maximum principle, if the maximum of w is achieved in the interior of $\Omega^+(\psi)$, then $w = const.$ in $\Omega^+(\psi)$, and thus $w \equiv 0$ in $\Omega^+(\psi)$ since $w|_{\Gamma_1} = 0$.

Let the maximum of w be attained at $x_0 \in S_\psi$. Then $w_\tau(x_0) = 0$ and thus, since $b(x_0)\cdot\nu(x_0) > 0$, we get

$$w_\nu(x_0) = \frac{1}{b(x_0)\cdot\nu(x_0)}b(x_0)\cdot Dw(x_0) = \frac{1}{b(x_0)\cdot\nu(x_0)}\left(b(x_0)\cdot D\underline{u}(x_0) - g(x_0)\right).$$

Using (4.8) and choosing σ small, depending only on n, γ, q^+, Ω, and M, and choosing C_1 sufficiently large, depending only upon the data as in (i), we obtain, on S_ψ,

$$\begin{aligned} b\cdot D\underline{u} &= Ke^{x_n}b\cdot\nu_0 \ge Ke^{x_n}\left(b\cdot\nu + b\cdot(\nu_0-\nu)\right) \\ &\ge C_1e^{-CM\sigma}(\kappa - \Lambda CM\sigma)\|g\|_{0,\Omega^+(\psi)} \ge \|g\|_{0,\Omega^+(\psi)}. \end{aligned}$$

Then

$$w_\nu(x_0) \ge \frac{1}{b(x_0)\cdot\nu(x_0)}\left(\|g\|_{0,\Omega^+(\psi)} - g(x_0)\right) \ge 0,$$

which contradicts the Hopf Lemma. Thus,

$$\sup_{\Omega^+(\psi)}(\underline{u} - u) = \sup_{\Gamma_1}(\underline{u} - u) = 0.$$

This and similar argument for $\overline{u}$ imply

$$\underline{u}(x) \le u(x) \le \overline{u}(x), \qquad \text{for any } x \in \Omega^+(\psi),$$

which implies (4.22). This also implies the uniqueness of classical solutions of (4.21).

Furthermore, let

$$\Omega_0^+ := \mathbf{T}^{n-1}\times(0,1).$$

Define a flattening map $\Phi : \Omega^+(\psi) \to \Omega_0^+$, which maps S_ψ to $\{x_n = 0\}$, by

$$\Phi(x', x_n) = \left(x', \frac{x_n - f(x')}{N_2 - f(x')}\right) \tag{4.25}$$

for the function $f(x')$ in (4.4). Note that, from the estimate for $f(x')$ in (4.4),

$$\|\Phi - Id\|_{2,\alpha,\Omega^+(\psi)},\ \|\Phi^{-1} - Id\|_{2,\alpha,\Omega_0^+} \le \frac{1}{10}, \tag{4.26}$$

if σ is small. Now, consider $u(x)$ as a periodic solution with respect to the x' variables in $\mathbf{R}^{n-1} \times [-N_1, N_2] \cap \{x_n > f(x')\}$. Then, (4.23) follows from the Schauder estimates (see Theorem 6.2, Corollary 6.7, and Lemma 6.29 in [17]) and from (4.22); the fact that C is independent of M, σ, and ψ follows from (4.26). This completes the proof of Lemma 4.2.

From Lemma 4.2, we easily have

Lemma 4.3. *Let σ be sufficiently small, depending only on n, γ, q^+, Ω, and M. Then there exists a unique solution $u \in C^{2,\alpha}(\overline{\Omega^+(\psi)})$ of (4.21) satisfying (4.23) with C depending only on λ, Λ, κ, $\|a_{ij}\|_{0,\alpha,\Omega_e}$, and $\|b_i\|_{1,\alpha,\Omega_e}$.*

The proof is based on some standard existence argument for elliptic equations; we sketch it in Appendix A.

3. Now we turn to the nonlinear problem (4.14)–(4.16). Rewrite the equation (4.14) in the non-divergence form:

$$\mathcal{N}[v] := \sum_{i,j=1}^{n} A^{ij}(Dv) v_{x_i x_j} = 0, \tag{4.27}$$

where $A_{ij}(P) := A^i_{P_j}(P),\ P \in \mathbf{R}^n$. From the definition of $A(P)$, (3.3), and (3.4),

$$A_{ij}(P) = A_{ji}(P), \tag{4.28}$$

$$(1 + |P|)|D_P A_{ij}(P)| \le \hat{C}, \tag{4.29}$$

for any $P \in \mathbf{R}^n$, $i, j = 1, \cdots, n$.

The unique solvability of the linear problem (4.21) allows to use the nonlinear method of continuity to solve (4.27), (4.15), and (4.16). Namely, in order to show the existence and uniqueness of a solution $v \in C^{2,\alpha}(\overline{\Omega^+(\psi)})$ of (4.27), (4.15), and (4.16), by Lemma 17.29 and Theorem 17.7 in [17], it suffices to verify the following estimate.

Lemma 4.4. *For any solution $u \in C^{2,\alpha}(\overline{\Omega^+(\psi)})$ of the problem:*

$$\sum_{i,j=1}^{n} A_{ij}(Du(x)) u_{x_i x_j} = 0 \quad in \quad \Omega^+(\psi), \tag{4.30}$$

$$\sum_{i=1}^{n} A^i(Du)\nu^i = t g_\psi(x) \quad on \quad S_\psi, \tag{4.31}$$

$$u = 0 \quad on \quad \Gamma_1, \tag{4.32}$$

the estimate

$$\|u\|_{1,\delta,\Omega^+(\psi)} \le C \tag{4.33}$$

holds with some $\delta > 0$ for any $t \in [0, 1]$, where C and δ depend only on n, λ, Λ, the constant $\hat{C}$ from (4.29), $\|g\|_{1,\alpha,\Omega^+(\psi)}$, S_ψ, and Ω, but are independent of t.

Proof of Lemma 4.4 follows from some well-known estimates for nonlinear elliptic equations of second-order; we outline the proof in Appendix B.

Note that the constants C and δ in (4.33) and (4.34) are independent of M and $\psi \in \mathcal{K}_M$, if $M\sigma$ is small. Indeed, the dependence of C and δ in (4.33) on S_ψ is through the estimates of derivatives of the regularized distance function $\rho(x)$ described in [24, page 522]. These are estimated in terms of the $C^{2,\alpha}$-norm of S_ψ [24, page 522], which, in our case, is determined by the $C^{2,\alpha}$ norm of $f(x')$ from (4.4) and estimated by $CM\sigma$. Thus, if $\psi \in \mathcal{K}_M$ and $M\sigma \le 1$, then C and δ in (4.33) depends only on the n, γ, q^+, Ω, and $\|g_\psi\|_{1,\alpha,\Omega_e}$.

Moreover, it follows from [23, Theorem 2] that the solution $v \in C^2(\overline{\Omega^+(\psi)})$ of (4.27), (4.15), and (4.16) satisfies $v \in C^{2,\alpha}(\overline{\Omega^+(\psi)})$, and

$$\|v\|_{2,\alpha,\Omega^+(\psi)} \le C, \tag{4.34}$$

where C depends only on the same quantities as the constants in (4.33), i.e., on n, γ, q^+, Ω and $\|g_\psi\|_{1,\alpha,\Omega_e}$.

By (4.17), the solution $v(x)$ of (4.27), (4.15), and (4.16) satisfies a linear problem of the form (4.21) with

$$\begin{aligned} &a_{ij}(x) = A^i_{P_j}(Dv(x)),\\ &b_i(x) = \sum_{j=1}^n \int_0^1 A^j_{P_i}(sDv(x))ds\, \nu^j(x),\\ &r(x) \equiv 0. \end{aligned} \tag{4.35}$$

Then, using (4.20) and (4.34), we conclude that the functions defined by (4.35) satisfy $\|a_{ij}\|_{1,\alpha,\Omega^+(\psi)} + \|b_i\|_{1,\alpha,\Omega^+(\psi)} \le C$, where C depends only on n, γ, q^+, and Ω if $M\sigma \le 1$. Thus, using (4.23) for $v(x)$, with $r(x) = 0$ and $g = g_\psi$, and recalling (4.20), we obtain

$$\|v\|_{2,\alpha,\Omega^+(\psi)} \le C\sigma.$$

Thus, $\varphi(x) = v(x) + \varphi_0^+(x)$ is the unique solution of (4.10)–(4.12) and satisfies (4.13).

4. Now we show that $\varphi(x)$ satisfies (3.9). Since $\psi(x)$ satisfies (3.9), it follows that $G_\psi(x)$ and $\Omega^+(\psi)$, i.e., the function $f(x)$ in (4.4), satisfy (3.9). Fix any $m \in \{1, \cdots, n-1\}$ and $k \in \{0, \pm 1, \pm 2, \cdots\}$, and let

$$v(x) := \varphi(x_1, \cdots, x_{m-1}, 2ka - x_m, x_{m+1}, \cdots x_n).$$

Then $v(x)$ is a solution of (4.10)–(4.12): Indeed, since $G_\psi(x)$ and $\Omega^+(\psi)$ satisfy (3.9), the only fact we should check is that, if $\varphi(x)$ is a solution of (4.10), then $v(x)$ also satisfies (4.10). This follows from the structure of (4.10) and is readily checked by a direct calculation. Thus, by the uniqueness of solutions of the problem (4.10)–(4.12), we obtain $\varphi(x) \equiv v(x)$, and so $\varphi(x)$ satisfies (3.9). $\square$

4.3. **Construction and Continuity of the Iteration Map.** Now we first construct the iteration map by an extension of the unique solution of (4.10)–(4.12) satisfying (4.13). Then we show the continuity of the iteration map.

Proposition 4.5. *Let $M \geq 1$. Let $\sigma > 0$, φ^-, and ψ be as those in Proposition 4.1. Let φ be a solution of the problem (4.10)–(4.12) in the domain $\Omega^+(\psi)$. Then φ can be extended to the whole domain Ω_e, and this extension, denoted by $\mathcal{P}_\psi\varphi$, satisfies the following two properties:*

(i) *There exists $C_0 > 0$, which depends only on n, γ, q^+, and Ω, and is independent of M, σ, and ψ, such that*

$$\|\mathcal{P}_\psi\varphi - \varphi_0^+\|_{2,\alpha,\Omega_e} \leq C_0\sigma. \tag{4.36}$$

(ii) *Let $\beta \in (0,\alpha)$. Let a sequence $\psi_j \in \mathcal{K}_M$ converge in $C^{2,\beta}(\overline{\Omega_e})$ to $\psi \in \mathcal{K}_M$. Let $\varphi_j \in C^{2,\alpha}(\overline{\Omega^+(\psi_j)})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega^+(\psi)})$ be the solutions of the problems (4.10)–(4.12) for ψ_j and ψ, respectively. Then $\mathcal{P}_{\psi_j}\varphi_j \to \mathcal{P}_\psi\varphi$ in $C^{2,\beta}(\overline{\Omega_e})$.*

Proof. We divide the proof into four steps.

Step 1. Let

$$\kappa = \frac{N_1}{N_2}.$$

Denote

$$\Omega_0 := \mathbf{T}^{n-1} \times (-2\kappa, 1), \qquad \Omega_0^+ := \mathbf{T}^{n-1} \times (0,1).$$

We first employ the extension map in [17, pp. 136–137] to define an extension operator $\mathcal{E}_2 : C^{2,\beta}(\overline{\Omega_0^+}) \to C^{2,\beta}(\overline{\Omega_0})$ for any $\beta \in (0,1)$. Let $v \in C^{2,\beta}(\overline{\Omega_0^+})$. Define $\mathcal{E}_2 v = v$ in Ω_0^+. For $(x', x_n) \in \mathbf{T}^{n-1} \times (-2\kappa, 0)$, define

$$\mathcal{E}_2 v(x', x_n) = \sum_{i=1}^{3} c_i v(x', -\frac{x_n}{2\kappa i}),$$

where c_1, c_2, and c_3 are constants determined by the system of equations

$$\sum_{i=1}^{3} c_i \left(-\frac{1}{2\kappa i}\right)^m = 1, \quad m = 0, 1, 2.$$

It is easy to see that $\mathcal{E}_2 v \in C^{2,\beta}(\overline{\Omega_0})$ and

$$\|\mathcal{E}_2 v\|_{2,\beta,\Omega_0} \leq C\|v\|_{2,\beta,\Omega_0^+}, \tag{4.37}$$

with a uniform constant C. Since $\mathcal{E}_2$ is a linear operator, it follows that $\mathcal{E}_2 : C^{2,\beta}(\overline{\Omega_0^+}) \to C^{2,\beta}(\overline{\Omega_0})$ is continuous.

The extension map $\mathcal{E}_2$ has also the following properties:

(1) If v satisfies (3.9), then so does $\mathcal{E}_2 v$;

(2) Let $v_j \in C^{2,\beta}(\overline{\Omega_0^+})$ and $\mathcal{E}_2 v_j \to v$ in $C^{2,\beta}(\overline{\Omega_0})$ as $j \to \infty$. Then

$$v = \mathcal{E}_2\left(v|_{\Omega_0^+}\right) \quad \text{in} \quad \Omega_0.$$

Assertion (1) follows directly from the definition of $\mathcal{E}_2$. Assertion (2) follows from the continuity of $\mathcal{E}_2$. Indeed, since $v_j \to v|_{\Omega_0^+}$, then $\mathcal{E}_2 v_j \to \mathcal{E}_2\left(v|_{\Omega_0^+}\right)$, which implies (2).

Step 2. We first point out the following elementary fact, whose proof can be readily obtained by explicit calculations.

Lemma 4.6. *Let $\Omega_1, \Omega_2 \subset \mathbf{R}^n$ be bounded, open sets. Let $f \in C^{2,\beta}(\overline{\Omega}_2)$ and $\Phi : \Omega_1 \to \Omega_2$ satisfy $\Phi \in C^{2,\beta}(\overline{\Omega}_1; \mathbf{R}^n)$. Then $f \circ \Phi \in C^{2,\beta}(\overline{\Omega}_1)$ and*

$$\|f \circ \Phi\|_{2,\beta,\Omega_1} \le C\|f\|_{2,\beta,\Omega_2},$$

where $C = C(n, \|\Phi\|_{2,\beta,\Omega_1})$. Also, if $f_j \to f$ in $C^{2,\beta}(\overline{\Omega}_2)$ and $\Phi_j \to \Phi$ in $C^{2,\beta}(\overline{\Omega}_1; \mathbf{R}^n)$, then

$$f_j \circ \Phi_j \to f \circ \Phi \qquad in \quad C^{2,\beta}(\overline{\Omega}_1).$$

Now let $\psi \in \mathcal{K}_M$. We define the extension operator $\mathcal{E}_\psi : C^{2,\beta}(\overline{\Omega^+(\psi)}) \to C^{2,\beta}(\overline{\Omega_e})$ for any $\beta \in (0, \alpha]$ as following. Let $\Phi : \Omega^+(\psi) \to \Omega_0^+$ be a map that flattens S_ψ and is defined by (4.25), which in fact defines $\Phi : \Omega_e \to \mathbf{T}^{n-1} \times \mathbf{R}$. This map is $C^{2,\alpha}$ and, if $M\sigma$ is small enough, then

$$\mathbf{T}^{n-1} \times [-\frac{\kappa}{2}, 1] \subset \Phi(\Omega_e) \subset \mathbf{T}^{n-1} \times [-\frac{3\kappa}{2}, 1]. \tag{4.38}$$

Also, if $M\sigma$ is small, then the inverse $\Phi^{-1} : \Phi(\Omega_e) \to \Omega_e$ exists and, similar to (4.26),

$$\|\Phi - Id\|_{2,\alpha,\Omega_e},\ \|\Phi^{-1} - Id\|_{2,\alpha,\Phi(\Omega_e)} \le \frac{1}{10}. \tag{4.39}$$

Then, for $v \in C^{2,\beta}(\overline{\Omega^+(\psi)})$, we define

$$\mathcal{E}_\psi v = \left[\mathcal{E}_2(v \circ \Phi^{-1})\right] \circ \Phi. \tag{4.40}$$

This is well-defined by the right inclusion in (4.38). From (4.37), (4.39), and Lemma 4.6, we obtain that, if σ is so small that (4.38) and (4.39) hold, then, for any $\psi \in \mathcal{K}_M$ and $v \in C^{2,\beta}(\overline{\Omega_e})$,

$$\|\mathcal{E}_\psi v\|_{2,\beta,\Omega_e} \le C\|v\|_{2,\beta,\Omega^+(\psi)} \tag{4.41}$$

with C depending only on n and Ω. Since the map $\mathcal{E}_\psi : C^{2,\beta}(\overline{\Omega^+(\psi)}) \to C^{2,\beta}(\overline{\Omega_e})$ is linear, it follows that this map is continuous.

Step 3. The map $\mathcal{E}_\psi$ satisfies the following properties.

Lemma 4.7. *Assume that $M\sigma$ is small and $\beta \in (0, \alpha]$.*

(i) *If $\psi \in \mathcal{K}_M$ and $v \in C^{2,\beta}(\overline{\Omega^+(\psi)})$ satisfies (3.9), then $\mathcal{E}_\psi v$ also satisfies (3.9);*

(ii) *Let $\psi_j \in \mathcal{K}_M$ and $\psi_j \to \psi$ in $C^{2,\beta}(\overline{\Omega_e})$ as $j \to \infty$. Let $v_j \in C^{2,\beta}(\overline{\Omega^+(\psi_j)})$ and $\mathcal{E}_{\psi_j} v_j \to v$ in $C^{2,\beta}(\overline{\Omega_e})$ as $j \to \infty$. Then*

$$v = \mathcal{E}_\psi\left(v|_{\Omega^+(\psi)}\right) \qquad in \quad \Omega_e.$$

These can be seen as follows. Assertion (i) follows from the definition of $\mathcal{E}_\psi$ and the property (i) of $\mathcal{E}_2$ in Step 1, since the maps Φ and Φ^{-1} satisfy (3.9), which follows from (4.25) because ψ satisfies (3.9).

Now we prove (ii). Since $\psi_j \in \mathcal{K}_M$ and $\psi_j \to \psi$ in $C^{2,\beta}$, then $\psi \in \mathcal{K}_M$. Let f_j and f be the functions from (4.4) for ψ_j and ψ, respectively. Since $\psi_j \to \psi$ in $C^{2,\beta}$ and (4.3) holds, then

$$f_j \to f \quad \text{in} \quad C^{2,\beta}(\mathbf{T}^{n-1}). \tag{4.42}$$

Let $\Phi_j : \Omega_e \to \mathbf{T}^{n-1} \times \mathbf{R}$ be the flattening map (4.25) corresponding to ψ_j. It follows that

$$\Phi_j \to \Phi \qquad \text{in} \quad C^{2,\beta}(\Omega_e; \mathbf{T}^{n-1} \times \mathbf{R}).$$

Also, by (4.38), it follows that

$$\Phi_j^{-1} \to \Phi^{-1} \qquad \text{in} \quad C^{2,\beta}(\mathbf{T}^{n-1} \times [-\frac{\kappa}{2}, 1]; \Omega_e).$$

Let $w \in C^{2,\beta}(\overline{\Phi(\Omega_e)})$ and $w_j \in C^{2,\beta}(\overline{\Omega_0^+})$ be defined by $w = v \circ \Phi^{-1}$ and $w_j = v_j \circ \Phi_j^{-1}$, which is well-defined since $\Phi_j(\Omega^+(\psi_j)) = \Omega_0^+$. Then, using the first inclusion in (4.38) and the second assertion of Lemma 4.6, we have

$$w_j \to w \qquad \text{in} \quad C^{2,\beta}(\overline{\Omega_0^+}).$$

Thus, using the continuity of $\mathcal{E}_2$, we have

$$\mathcal{E}_2 w_j \to \mathcal{E}_2(w|_{\Omega_0^+}) \qquad \text{in} \quad C^{2,\beta}(\overline{\Omega_0}).$$

Using the second inclusion in (4.38) for Φ and Φ_j, we get

$$(\mathcal{E}_2 w_j) \circ \Phi_j \to \mathcal{E}_2(w|_{\Omega_0^+}) \circ \Phi \qquad \text{in} \quad C^{2,\beta}(\overline{\Omega_e}),$$

which is

$$\mathcal{E}_{\psi_j} v_j \to \mathcal{E}_\psi(v|_{\Omega^+(\psi)}) \qquad \text{in} \quad C^{2,\beta}(\overline{\Omega_e}).$$

This implies (ii). □

Step 4. Finally, we define the following extension operator $\mathcal{P}_\psi : C^{2,\beta}(\overline{\Omega^+(\psi)}) \to C^{2,\beta}(\overline{\Omega_e})$ for $\psi \in \mathcal{K}_M$ and $\beta \in (0, \alpha]$:

$$\mathcal{P}_\psi v = \mathcal{E}_\psi(v - \varphi_0^+) + \varphi_0^+, \qquad \text{where } v \in C^{2,\alpha}(\overline{\Omega^+(\psi)}). \tag{4.43}$$

The estimate (4.36) follows from (4.13), (4.41), and (4.43).

Now we prove the assertion (ii) of Proposition 4.5. Let $0 < \beta < \alpha$. By (4.36),

$$\|\mathcal{P}_{\psi_j} \varphi_j\|_{2,\alpha,\Omega_e} \le C.$$

Thus, there exists a subsequence (still denoted) φ_j such that

$$\mathcal{P}_{\psi_j} \varphi_j \to v \qquad \text{in} \quad C^{2,\beta}(\overline{\Omega_e}), \tag{4.44}$$

for some $v \in C^{2,\alpha}(\overline{\Omega_e})$.

Denoting f_j as the function from (4.4) for ψ_j as above and using (4.42) yield that v satisfies the equation (4.10) in $\Omega^+(\psi)$. Also, v obviously satisfies the condition (4.12). Now we show that $v(x)$ also satisfies the condition (4.11) on S_ψ. Denote ν_j as the function (4.7) corresponding to ψ_j. Then

$$\nu_j \to \nu \qquad \text{in} \quad C^{1,\beta}(\Omega_e). \tag{4.45}$$

Let $x' \in \mathbf{T}^{n-1}$. Denote $x_j = (x', f_j(x'))$ and $x = (x', f(x'))$. By (4.42), $x_j \to x$. Since φ_j satisfies (4.11) on S_{ψ_j}, we have

$$\tilde\rho(|D\varphi_j(x_j)|^2) D\varphi_j(x_j) \cdot \nu_j(x_j) = G_{\psi_j}(x_j).$$

By (4.9) and (4.45),

$$G_{\psi_j} \to G_\psi \qquad \text{in} \quad C^{1,\beta}(\Omega_e). \tag{4.46}$$

Then we have

$$\tilde\rho(|D\varphi(x)|^2) Dv(x) \cdot \nu(x) = \lim_{j\to\infty} \tilde\rho(|D\varphi_j(x_j)|^2) D\varphi_j(x_j) \cdot \nu_j(x_j) = \lim_{j\to\infty} G_{\psi_j}(x_j) = G_\psi(x).$$

Since $S_\psi = \{(x', f(x')) \;:\; x' \in \mathbf{T}^{n-1}\}$, we conclude that v satisfies (4.11).

Thus, v is a solution of (4.10)–(4.12) in $\Omega^+(\psi)$. By the uniqueness in Proposition 4.1,

$$v|_{\Omega^+(\psi)} = \varphi.$$

Then, by (4.43), (4.44), and Lemma 4.7(ii), we have

$$v = \mathcal{E}_\psi\left((v - \varphi_0^+)|_{\Omega^+(\psi)}\right) + \varphi_0^+ = \mathcal{E}_\psi(\varphi - \varphi_0^+) + \varphi_0^+ = \mathcal{P}_\psi\varphi \qquad \text{in} \quad \Omega_e.$$

We have thus proved that a subsequence of $\mathcal{P}_{\psi_j}\varphi_j$ converges to $\mathcal{P}_\psi\varphi$ in $C^{2,\beta}(\overline{\Omega_e})$. Moreover, by the same argument, from any subsequence of $\mathcal{P}_{\psi_j}\varphi_j$ we can extract a further subsequence, converging in $C^{2,\beta}(\overline{\Omega_e})$ to the same limit $\mathcal{P}_\psi\varphi$. Thus the whole sequence $\mathcal{P}_{\psi_j}\varphi_j$ converges to $\mathcal{P}_\psi\varphi$. Proposition 4.5 is proved.

4.4. **Existence of Solutions of the Free Boundary Problem.** With Sections 4.1–4.3, we can now prove the existence of solutions of the free boundary problem.

Define the iteration map $J : \mathcal{K}_M \to C^{2,\alpha}(\overline{\Omega_e})$ by

$$J\psi := \mathcal{P}_\psi\varphi, \tag{4.47}$$

where φ is the unique solution of the problem (4.10)–(4.12) for ψ. By Proposition 4.5(ii), J is continuous in the $C^{2,\beta}(\overline{\Omega_e})$-norm for any positive $\beta < \alpha$.

Now we denote by φ both the function φ in $\Omega^+(\psi)$ and its extension $\mathcal{P}_\psi\varphi$.

Choose M to be the constant C_0 from (4.36). Then, for $\psi \in \mathcal{K}_M$, we have from Proposition 4.5(i) that $\varphi := J\psi \in \mathcal{K}_M$ if $\sigma > 0$ is sufficiently small depending only on n, γ, q^+, and Ω, since M is now fixed. Thus, (4.47) defines the iteration map $J : \mathcal{K}_M \to \mathcal{K}_M$ and, from Proposition 4.5(ii), J is continuous on $\mathcal{K}_M$ in the $C^{2,\beta}(\overline{\Omega_e})$-norm for any positive $\beta < \alpha$.

In order to find a classical solution of Problem C, we seek a fixed point of the map J. We use the Schauder Fixed Point Theorem (cf. [17, Theorem 11.1]) in the following setting:

Let $\sigma > 0$ satisfy the conditions of Proposition 4.5. Let $\beta \in (0, \alpha)$. Since Ω_e is a compact manifold with boundary, the set $\mathcal{K}_M$ is a compact convex subset of $C^{2,\beta}(\overline{\Omega_e})$. We have shown that $J(\mathcal{K}_M) \subset \mathcal{K}_M$, and J is continuous in the $C^{2,\beta}(\overline{\Omega_e})$-norm. Then, by the Schauder Fixed Point Theorem, J has a fixed point $\varphi \in \mathcal{K}_M$.

If φ is such a fixed point, then

$$\tilde{\varphi}(x) := \min(\varphi^-(x), \varphi(x))$$

is a classical solution of Problem C, and S_φ is its free boundary.

It follows that $\tilde{\varphi}$ is a solution of Problem B, provided that σ is small enough so that (4.13) implies that $|D\varphi| < p^1_{sonic} - \varepsilon$, where ε is defined by (3.2). Indeed, then (3.5) implies that φ lies in the non-truncated region for the equation (3.6).

For such values of σ, the function $\tilde{\varphi}$ is a solution of Problem A. Indeed, $|D\tilde{\varphi}| < p^1_{sonic} - \varepsilon$ on $\Omega^+(\tilde{\varphi}) := \{\tilde{\varphi}(x) < \varphi^-(x)\}$ since $\tilde{\varphi} = \varphi$ on $\Omega^+(\tilde{\varphi})$.

This completes the existence proof for Theorems 2.1 and 2.2.

5. Uniqueness of Solutions

In this section, we prove the uniqueness of solutions of Problem B (hence, Problem A) we have constructed in Section 4.

Theorem 5.1. *Let q^+, q^-, and φ^- be as those in Theorem 2.1. Let $M > 0$. If $\sigma > 0$ is a sufficiently small constant depending only on M, n, q^+, γ, and Ω, then there exists at most one solution φ of Problem B satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+(\varphi)} \le M\sigma. \tag{5.1}$$

Proof. In this proof, the constants σ, C, and c depend only on M, n, q^+, γ, and Ω, unless other dependence is specified. The proof consists of seven steps.

Step 1. We consider Problem B extended to the domain Ω_e. Let $\varphi \neq \hat{\varphi}$ be two solutions of Problem B satisfying (5.1). Define

$$u := \varphi^- - \varphi, \qquad \hat{u} := \varphi^- - \hat{\varphi} \qquad \text{in } \Omega_e.$$

Then

$$u \ge 0, \qquad \hat{u} \ge 0 \qquad \text{in } \Omega_e,$$

and

$$\Omega^+(\varphi) = \{u(x) > 0\} \cap \Omega_e, \qquad \Omega^+(\hat{\varphi}) = \{\hat{u}(x) > 0\} \cap \Omega_e.$$

Below $\Omega^+(u)$, $S(u)$, $\Omega^+(\hat{u})$, and $S(\hat{u})$ stand for $\Omega^+(\varphi)$, $S(\varphi)$, $\Omega^+(\hat{\varphi})$, and $S(\hat{\varphi})$, respectively. Note that

$$S(u) = \partial\{u(x) > 0\} \cap \Omega_e, \qquad S(\hat{u}) = \partial\{\hat{u}(x) > 0\} \cap \Omega_e.$$

The definition of u and $\hat{u}$ with (5.1) implies

$$\|u - (q^- - q^+)x_n\|_{2,\alpha,\Omega^+(u)} \le M\sigma, \quad \|\hat{u} - (q^- - q^+)x_n\|_{2,\alpha,\Omega^+(\hat{u})} \le M\sigma. \tag{5.2}$$

If (4.2) holds with large enough C, then the regions $\Omega^+(u)$ and $\Omega^+(\hat{u})$ have the form (4.4) with the functions f and $\hat{f}$, respectively. To see this, we use (5.2) and extend u from $\Omega^+(u)$, which we consider now as a subset of $\mathbf{R}^n$, into $\mathbf{R}^n$ so that the extension $\mathcal{E}u$ satisfies

$$\|\mathcal{E}u - (q^- - q^+)x_n\|_{2,\alpha,\mathbf{R}^n} \le CM\sigma, \tag{5.3}$$

where C depends only on n (see e.g. [31, Chap. 6, Theorem 4]). Then, by (4.2),

$$(\mathcal{E}u)_{x_n}(x) \ge q^- - q^+ - CM\sigma \ge \frac{q^- - q^+}{2} \ge 0.$$

Thus, $\Omega^+(u) = \{\mathcal{E}u(x) > 0\} \cap \Omega_e$, and (4.4) holds for u by the implicit function theorem; and the corresponding results hold for $\hat{u}$.

Step 2. Rewrite Problem (4.10)–(4.12) in terms of $u = \varphi^- - \varphi$. It follows from (3.5) and (5.2) that, if $\sigma > 0$ is sufficiently small, then u in $\Omega^+(u)$ is the solution of the following problem:

$$\operatorname{div} A(x, Du) = 0 \quad \text{in } \ \Omega^+(u), \tag{5.4}$$

$$u_\nu = G_f \quad \text{on } \ S(u), \tag{5.5}$$

$$u = \varphi^- - q^+ N_2 \qquad \text{on } \ \Gamma_1 := \mathbf{T}^{n-1} \times \{N_2\}, \tag{5.6}$$

where

$$A(y, P) = \tilde{\rho}(|D\varphi^-(y) - P|^2)(D\varphi^-(y) - P) \qquad \text{for } \ y \in \Omega_e, \ \ P \in \mathbf{R}^n, \tag{5.7}$$

with $\tilde{\rho}$ defined by (3.4).

The function $G_f(x')$ is defined as following. Using (2.2), we can rewrite (2.5) as

$$\rho(|\varphi_\tau^-|^2 + (\varphi_\nu^+)^2)\varphi_\nu^+ = \rho(|D\varphi^-|^2)\varphi_\nu^- \qquad \text{on } \ S(u). \tag{5.8}$$

We intend to solve (5.8) for φ_ν^+. Note that, by (2.13),

$$\big|D\varphi^- - q^-\nu_0\big| < \sigma \qquad \text{in } \Omega,$$

and, by (5.3), the unit normal vector $\nu(x)$ to $S(u)$ satisfies

$$|\nu(x) - \nu_0| < C\sigma. \tag{5.9}$$

Thus, noting that

$$\varphi_\nu^- = D\varphi^- \cdot \nu, \quad |\varphi_\tau^-|^2 = |D\varphi^-|^2 - (\varphi_\nu^-)^2 = |D\varphi^-|^2 - (D\varphi^- \cdot \nu)^2, \tag{5.10}$$

we get

$$|\varphi_\nu^- - q^-| \le C\sigma, \qquad |\varphi_\tau|^2 \le C\sigma.$$

Denoting

$$K := 1 - \theta|\varphi_\tau^-|^2, \tag{5.11}$$

we rewrite (5.8) as the equation:

$$\Phi_K(p) = \Phi_K(D\varphi^- \cdot \nu), \tag{5.12}$$

where $\Phi_K(\cdot)$ is the function in Lemma 2.1. Now we solve (5.12) for p. For this purpose, based on (5.10) and (5.11), we define

$$e_\nu = e \cdot \nu, \qquad |e_\tau|^2 = |e|^2 - (e_\nu)^2 = |e|^2 - (e \cdot \nu)^2,$$

and

$$K(s,t) := 1 - \theta(s - t^2), \qquad s, t \in \mathbf{R}.$$

Then

$$K(|e|^2, e\cdot\nu) = 1 - \theta(|e|^2 - (e\cdot\nu)^2) = 1 - \theta|e_\tau|^2 \qquad \text{for} \quad e \in \mathbf{R}^n,\ \nu \in \mathbf{S}^{n-1}.$$

Now, (5.12) can be written as

$$F(p;\, |D\varphi^-|^2, D\varphi^- \cdot \nu) = 0,$$

where $F \in C^\infty$ in the neighborhood of $(q^+; (q^-)^2, q^-)$ under consideration is defined by

$$F(p; s, t) := \Phi_{K(s,t)}(p) - \Phi_{K(s,t)}(t), \qquad p, s, t \in \mathbf{R}.$$

Moreover, by the definition of q^+ and q^- and Lemma 2.1(iii), we have

$$F(q^+; (q^-)^2, q^-) = 0, \qquad F_p(q^+; (q^-)^2, q^-) = \Phi_1'(q^+) > 0.$$

Thus, by the Implicit Function Theorem, there exists a C^∞ function $G(s,t)$ defined on a neighborhood:

$$\mathcal{O}_{r_0}(q^-, q^-) = \{(s,t) \ : \ |s - (q^-)^2| < \delta_0,\ |t - q^-| < \delta_0\},$$

where δ_0 depends only on q^+, q^-, and γ such that

$$G((q^-)^2, q^-) = q^+,$$

and

$$F(G(s,t); s, t) = 0.$$

This means that (5.8) can be rewritten as

$$\varphi_\nu^+ = G(|D\varphi^-|^2, D\varphi^- \cdot \nu),$$

provided that, on $S(u)$,

$$|D\varphi^- \cdot \nu - q^-| < \delta_0, \qquad \big||D\varphi^-|^2 - (q^-)^2\big| < \delta_0,$$

which can be achieved by choosing σ small depending only on n, γ, q^+, Ω, and M.

Thus, for $u = \varphi^- - \varphi$, (5.5) holds with

$$G_f(x') := D\varphi^-(x) \cdot \nu_f(x) - G(|D\varphi^-(x)|^2, D\varphi^-(x) \cdot \nu_f(x)), \tag{5.13}$$

for $x' \in \mathbf{T}^{n-1}, x = (x', f(x')) \in S(u)$, and $\nu_f(x)$ is the inward unit normal vector to $S(u)$ at x.

Note that (5.4) is uniformly elliptic with the same ellipticity constants as those for (3.6).

The function $\hat{u}$ is a solution of the similar problem in $\Omega^+(\hat{u})$, i.e., $\hat{u}$ satisfies the equation (5.4) in $\Omega^+(\hat{u})$, (5.6) on Γ_1, and

$$\hat{u}_\nu = G_{\hat{f}} \qquad \text{on} \quad S(\hat{u}),$$

where

$$G_{\hat{f}}(x') = D\varphi^-(x) \cdot \nu_{\hat{f}}(x) - G(|D\varphi^-(x)|^2, D\varphi^-(x) \cdot \nu_{\hat{f}}(x)),$$

for $x' \in \mathbf{T}^{n-1}, x = (x', \hat{f}(x')) \in S(\hat{u})$, and the inward unit normal vector $\nu_{\hat{f}}(x)$ to $S(\hat{u})$ at x.

Step 3. We may assume $f \neq \hat{f}$; otherwise $\varphi = \hat{\varphi}$ by the uniqueness of solutions to the problem (4.10)–(4.12) in the domain $\Omega^+(\varphi) = \Omega^+(u)$ by Proposition 4.1. Thus we may assume that $\hat{f}(\hat{x'}) > f(\hat{x'})$ for some $\hat{x'} \in \mathbf{R}^{n-1}$, since the opposite inequality can be handled similarly.

We shift the domain $\Omega^+(\hat{u})$ in the direction $-\nu_0$ by a distance $\delta > 0$ so that the resulting domain $\mathcal{B}$ contains $\Omega^+(u) \cap \{x_n < N_2 - \delta\}$ and $\partial\mathcal{B} \cap S(u) \neq \emptyset$. Precisely, for positive $y < N_2$, define $v_y : \Omega_e \cap \{x_n < N_2 - y\} \to \mathbf{R}$ by

$$v_y(x', x_n) = \hat{u}(x', x_n + y).$$

Let

$$\delta = \sup\{y \geq 0 \ : \ S(u) \cap S(v_y) \neq \emptyset\}.$$

By the above assumption, $\delta > 0$. Applying (4.4) to both $f(x')$ and $\hat{f}(x')$, we have $\delta \leq CM\sigma$. We assume $CM\sigma < N_2/10$.

We denote $v_\delta(x)$ by $v(x)$ and denote $h(x') := \hat{f}(x') - \delta$. Clearly,

$$\Omega^+(v) = \{(x', x_n) \ : \ h(x') < x_n < N_2 - \delta\} \cap \Omega_e.$$

It follows that $\Omega^+(u) \cap \{x_n < N_2 - \delta\} \subset \Omega^+(v)$. By construction, $f(x') \geq h(x')$, and there exists $x'_* \in \mathbf{T}^{n-1}$ such that $f(x'_*) = h(x'_*)$. Denote $x_* := (x'_*, f(x'_*)) \in \mathbf{T}^{n-1} \times \mathbf{R}$. Then the smooth surface $S(u)$ touches the smooth surface $S(v)$ at x_*. Denote the common unit normal vector to $S(u)$ and $S(v)$ at x_* in the direction of $\Omega^+(v)$ by $\nu(x_*)$. Since $S(\hat{u}) = S(v) + \delta\nu_0$, it follows that the inward unit normal vector $\nu_{\hat{f}}(x_* + \delta\nu_0)$ to $S(\hat{u})$ at $x_* + \delta\nu_0 = (x', f(x') + \delta)$ is equal to $\nu(x_*)$. Then, from the definition of G_f and $G_{\hat{f}}$,

$$|G_f(x'_*) - G_{\hat{f}}(x'_*)| \leq C|D\varphi^-(x'_*, f(x'_*)) - D\varphi^-(x'_*, f(x'_*) + \delta)| \leq C\delta\sigma,$$

where we used (2.13) in the last inequality. Also, since $\hat{u}(x)$ satisfies the free boundary condition $\hat{u}_\nu(x', \hat{f}(x')) = G_{\hat{f}}(x')$ and $v(x) = \hat{u}(x + \delta\nu_0)$ for any x, we have

$$v_\nu(x_*) := Dv(x_*) \cdot \nu(x_*) = D\hat{u}(x_* + \delta\nu_0) \cdot \nu_{\hat{f}}(x_* + \delta\nu_0) = G_{\hat{f}}(x'_*).$$

Since $u(x)$ satisfies $u_\nu(x_*) := Du(x_*) \cdot \nu(x_*) = G_f(x'_*)$, we have

$$|v_\nu(x_*) - u_\nu(x_*)| = |G_f(x'_*) - G_{\hat{f}}(x'_*)| \leq C\sigma\delta. \tag{5.14}$$

We will come to a contradiction for small σ by showing that $v_\nu(x_*) - u_\nu(x_*) \geq c\delta$ with $c > 0$.

Step 4. Denote $\mathcal{D} := \Omega^+(u) \cap \{x_n < N_2 - \delta\}$. Then

$$\partial\mathcal{D} = S(u) \cup \{x_n = N_2 - \delta\} = \{x_n = f(x')\} \cup \{x_n = N_2 - \delta\}. \tag{5.15}$$

Then $x_* \in \partial\mathcal{D}$, and $\nu(x_*)$ is the inward normal vector to $\partial\mathcal{D}$ at x_*. Also

$$v|_{\partial\mathcal{D}} \geq u|_{\partial\mathcal{D}}. \tag{5.16}$$

Indeed, $v \geq 0 = u$ on $S(u)$ from the definition of $\mathcal{D}$ and v. On $\{x_n = N_2 - \delta\}$, we apply (5.2) to u and use the fact that $v|_{\{x_n = N_2-\delta\}} = u|_{\{x_n=N_2\}}$ to have

$$v \geq u + (q^- - q^+ - M\sigma)\delta \geq u + \frac{q^- - q^+}{2}\delta \qquad \text{on } \{x_n = N_2 - \delta\}, \tag{5.17}$$

if σ is sufficiently small.

Since $\hat{u}$ satisfies (5.4) in $\Omega^+(\hat{u})$, then v satisfies

$$\operatorname{div} A(x + \delta\nu_0, Dv) = 0 \qquad \text{in } \mathcal{D}.$$

We write this equation in the form:

$$\operatorname{div} A(x, Dv) = \operatorname{div}\psi(x) \qquad \text{in } \mathcal{D}, \tag{5.18}$$

where $A(y, P)$ is the function (5.7) and

$$\psi(x) := A(x, Dv(x)) - A(x + \delta\nu_0, Dv(x)) = -\delta\int_0^1 D_y A(x + \delta t\nu_0, Dv(x)) \cdot \nu_0\, dt.$$

By (5.7) and (2.13), for any $L_0 > 0$,

$$\sup_{|P|\leq L_0} \|D_y A(\cdot, P)\|_{0,\alpha,\Omega_e} \leq C(L_0)\sigma, \qquad \sup_{|P|\leq L_0} \|D^2_{yP} A(\cdot, P)\|_{0,0,\Omega_e} \leq C(L_0)\sigma.$$

From this, we use $|Dv| \leq q^- - q^+ + CM\sigma$ with $M\sigma \leq 1$ to conclude

$$|\psi| \leq C\delta\sigma \qquad \text{in } \Omega_e,$$

and

$$\begin{aligned}
|\psi(x) - \psi(\hat{x})| &\leq \delta\left|\int_0^1 (D_y A(x + \delta s\nu_0, Dv(x)) - D_y A(\hat{x} + \delta s\nu_0, Dv(x))) \cdot \nu_0\, ds\right| \\
&\quad + \delta\left|\int_0^1 (D_y A(\hat{x} + \delta s\nu_0, Dv(x)) - D_y A(\hat{x} + \delta s\nu_0, Dv(\hat{x}))) \cdot \nu_0\, ds\right| \\
&\leq C\delta\sigma|x - \hat{x}|^\alpha + C\delta|Dv(x) - Dv(\hat{x})| \\
&\leq CM\delta\sigma|x - \hat{x}|^\alpha,
\end{aligned}$$

where we used (5.2) and $v(x) = \hat{u}(x + \delta\nu_0)$ in the last inequality. Thus, we have

$$\|\psi\|_{0,\alpha,\Omega_e} \leq CM\delta\sigma. \tag{5.19}$$

Then, denoting $w := v - u$,

$$\operatorname{div}(A(x, Dv) - A(x, Du)) = \operatorname{div}\psi(x) \quad \text{in } \mathcal{D}$$

can be rewritten as

$$\sum_{i,j=1}^n D_i(a_{ij}(x) D_j w) = \operatorname{div}\psi(x) \quad \text{in } \mathcal{D}, \tag{5.20}$$

where $a_{ij}(x) = \int_0^1 A^i_{P_j}(x, sDv(x) + (1-s)Du(x))ds$. Thus, the equation (5.20) is uniformly elliptic with the ellipticity constants and norms of $a_{ij} \in C^{1,\alpha}(\overline{\mathcal{D}})$ depending only on n, q^+, γ, and Ω.

Step 5. We write $w := w_1 + w_2$, where w_1 and w_2 are the solutions of

$$\begin{aligned} \sum_{i,j=1}^n D_i(a_{ij}(x)D_j w_1) &= 0 \qquad \text{in } \mathcal{D}, \\ w_1 &= w \qquad \text{on } \partial\mathcal{D}, \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} \sum_{i,j=1}^n D_i(a_{ij}(x)D_j w_2) &= \operatorname{div}\psi(x) \qquad \text{in } \mathcal{D}, \\ w_2 &= 0 \qquad \text{on } \partial\mathcal{D}. \end{aligned} \tag{5.22}$$

Indeed, we obtain the solution $w_1 \in C^{2,\alpha}(\overline{\mathcal{D}})$ of (5.21) in the periodic case $\mathcal{D} \subset \mathbf{T}^{n-1} \times [-N_1, N_2]$ by the argument similar to the proof of Step 2 in §4.2. Note that $\partial\Omega_e \in C^\infty$, $w = v - u \in C^{2,\alpha}$, and the maximum principle:

$$\sup_{\mathcal{D}} w_1 = \sup_{\partial\mathcal{D}} w_1,$$

which follows from the ellipticity and structure of the equation (5.21). As long as w_1 is obtained, $w_2 = w - w_1$ is a solution of (5.22).

By Theorems 8.16, 8.32, and 8.33 in [17], which can be adapted to the periodic case $\mathcal{D} \subset \mathbf{T}^{n-1} \times [-N_1, N_2]$ without change in the proofs, we have

$$\|w_2\|_{1,\alpha,\mathcal{D}} \le C\|\psi\|_{0,\alpha,\Omega_e},$$

where C depends only on the ellipticity constants and $\mathcal{D}$. Furthermore, using (5.15) and $\|f\|_{2,\alpha,\mathbf{T}^{n-1}} \le CM\sigma$ and choosing $\sigma \le \min(N_2/100, 1)$ yield that the dependence of C on $\mathcal{D}$ becomes the dependence only on N_2. Thus, we use (5.19) to obtain

$$\|w_2\|_{1,\alpha,\mathcal{D}} \le CM\delta\sigma. \tag{5.23}$$

Step 6. Now we estimate $(w_1)_\nu(x_*)$ from below. By (5.16), $w_1 \ge 0$ on $\partial\mathcal{D}$. Thus, $w_1 \ge 0$ in $\mathcal{D}$ by the maximum principle. Also, $w_1(x_*) = 0$. Moreover, by (5.17),

$$w_1 \ge \frac{q^- - q^+}{2}\delta \qquad \text{on } \{x_n = N_2 - \delta\}.$$

We first show that

$$w_1(0, \frac{N_2}{2}) \ge c(q^- - q^+)\delta, \tag{5.24}$$

where $c > 0$ depends only on the ellipticity constants λ and Λ of the equation (5.21), i.e., on q^+, q^-, γ, n, and Ω.

Since $a_{ij} \in C^{1,\alpha}(\overline{\mathcal{D}})$, the equation (5.21) can be rewritten in the nondivergence form:

$$\sum_{i,j=1}^n a_{ij} D_{ij} w_1 + \sum_{i=1}^n b_i D_i w_1 = 0 \quad \text{in } \mathcal{D}, \tag{5.25}$$

where $b_i := \sum_{j=1}^n a_{ij,x_i} \in C^\alpha(\overline{\mathcal{D}})$, and $B := \left(\sum_{i=1}^n \|b_i\|^2_{L^\infty(\mathcal{D})}\right)^{1/2}$ depends only on q^+, γ, n, Ω, and M. In the rest of the proof, we consider $\mathcal{D}$ as a subset of $\mathbf{R}^{n-1} \times [-N_1, N_2]$ and the functions w_1, a_{ij}, and b_i in (5.25) as the functions on $\mathcal{D}$, $2a$-periodic with respect to $x_1, \cdots, x_{n-1}$. Define a domain $\mathcal{D}_0 := (-3a, 3a)^{n-1} \times (\frac{N_2}{4}, N_2 - \delta) \subset \mathcal{D}$, and let $W \in C^{2,\alpha}(\mathcal{D}_0) \cap C(\overline{\mathcal{D}_0})$ be a solution of

$$
\begin{aligned}
&\mathcal{P}^-_{\Lambda,\lambda}(D^2 W) - B|DW| = 0 \qquad \text{in } \mathcal{D}_0, \\
&W|_{\partial\mathcal{D}_0} = \phi,
\end{aligned}
\tag{5.26}
$$

where $\mathcal{P}^-_{\Lambda,\lambda}$ is the extremal Pucci operator (e.g. [7]), and $\phi \in C^\alpha(\overline{\mathcal{D}}_0)$ is a function satisfying

$$
\begin{aligned}
&0 \le \phi \le 1 \qquad \text{on } \partial\mathcal{D}_0, \\
&\phi = 0 \qquad \text{on } \partial\mathcal{D}_0 \setminus \{x_n = N_2 - \delta\}, \\
&\phi = 1 \qquad \text{on } \{x_n = N_2 - \delta, |x'| \le a\}.
\end{aligned}
$$

The existence and regularity of W follows from Chapter 9 in [7], with standard modifications, to take into account the dependence on DW in the equation. Also, $W > 0$ in $\mathcal{D}_0$, by the strong maximum principle.

Now w_1, as a solution of (5.25), satisfies

$$\mathcal{P}^-_{\Lambda,\lambda}(D^2 w_1) - B|Dw_1| \le 0 \quad \text{in } \mathcal{D},$$

i.e., w_1 is a supersolution. Also, $w_1 \ge \frac{q^- - q^+}{2} W\delta$ on $\partial\mathcal{D}_0$. Thus, by the homogeneity of the equation (5.26) and the maximum principle, $w_1 \ge \frac{q^- - q^+}{2} W\delta$ in $\mathcal{D}_0$. Thus, (5.24) is proved with $c = W(\frac{N_2}{2}, 0)/2 > 0$ depending only on λ, Λ, n, and Ω.

Step 7. By (5.24) and the interior Harnack inequality applied in $\overline{\mathcal{D}_0} \subset \mathcal{D}$, there exists $c > 0$ such that

$$w_1 \ge c\delta \qquad \text{in } \mathcal{D}_0 \cap \{N_2/4 \le x_n \le 3N_2/4\}.$$

By the periodicity of w_1 with respect to $x_1, \cdots, x_{n-1}$, we conclude that the above inequality holds in $\mathcal{D} \cap \{N_2/4 \le x_n \le 3N_2/4\}$.

Since $S(u) = \{x_n = f(x')\}$ and $\|f\|_{2,\alpha,\mathbf{T}^{n-1}} \le CM\sigma$, it follows that, if $M\sigma$ is small, every point of $S(u)$ has a tangent ball with radius $R \ge N_2/4$ and center within $\{N_2/4 \le x_n \le 3N_2/4\}$. Let x_0 be the center of such ball, tangent to $S(u)$ at x_*, i.e., $B_R(x_0) \subset \mathcal{D}$ and $x_* \in \partial B_R(x_0) \cap \partial\mathcal{D}$. Then, $x_0 \in \{N_2/4 \le x_n \le 3N_2/4\}$, and thus

$$w_1(x_0) \ge c\delta.$$

Now the Harnack inequality, applied to w_1 in $B_R(x_0)$, implies

$$\inf_{B_{R/2}(x_0)} w_1 \ge c\delta.$$

By the proof of the Hopf Lemma in [17, Lemma 3.4],

$$(w_1)_\nu(x_*) \ge c_1 \inf_{B_{R/2}(x_0)} w_1,$$

where $c_1 > 0$ depends only on $R > 0, \Lambda, \lambda$, and the L^∞-norm of the coefficients of (5.25).

Thus,

$$(w_1)_\nu(x_*) \geq c\delta, \tag{5.27}$$

where $c > 0$ depends only on n, γ, q^+, Ω, and M.

Combining (5.27) with (5.23), we obtain

$$(v-u)_\nu(x_*) = \partial_\nu(w_1 + w_2)(x_*) \geq (c - CM\sigma)\delta \geq \frac{c}{2}\delta,$$

if $M\sigma$ is small. If $\delta > 0$ and σ is small, this contradicts (5.14). Thus $\delta = 0$, which implies Theorem 5.1. □

6. Stability of Free Boundaries

As a consequence of the uniqueness, non-degeneracy, and regularity of solutions of the free boundary problem, we have the following stability theorem.

Theorem 6.1. *Let $M > 0$. There exist a non-negative function $\Psi \in C(\overline{\mathbf{R}^+})$, satisfying $\Psi(0) = 0$, and $\sigma_0 > 0$, depending only on M, q^+, q^-, γ, and Ω, such that, if $\sigma < \sigma_0$, φ^- satisfies (2.13), and $\hat{\varphi}^-$ satisfies*

$$\|\varphi^- - \hat{\varphi}^-\|_{2,\alpha,\Omega_e} \leq \kappa, \tag{6.1}$$

with $\kappa < \sigma$, then the unique solutions φ and $\hat{\varphi}$ of Problem B with (5.1) for φ^- and $\hat{\varphi}^-$, respectively, satisfy

$$\|f_\varphi - f_{\hat{\varphi}}\|_{2,\alpha,\mathbf{T}^{n-1}} \leq \Psi(\kappa), \tag{6.2}$$

where f_φ and $f_{\hat{\varphi}}$ are the "free boundary" functions in (4.4) with φ and $\hat{\varphi}$, respectively.

Proof. Let σ_0 be such that Theorem 5.1 holds for $2\sigma_0$. If the assertion is false, then there exist φ_k^- and $\hat{\varphi}_k^-$ for $k = 1, \cdots$, such that

$$\begin{aligned}
&\varphi_k^- \ \text{ satisfy (2.13) with } \ \sigma \leq \sigma_0;\\
&\|\varphi_k^- - \hat{\varphi}_k^-\|_{2,\alpha,\Omega_e} \leq \frac{1}{k};\\
&\|f_k - \hat{f}_k\|_{2,\alpha,\mathbf{T}^{n-1}} \geq \epsilon > 0.
\end{aligned}$$

Here f_k and $\hat{f}_k$ are the "free boundary" functions in (4.4) for φ_k and $\hat{\varphi}_k$, respectively, where φ_k and $\hat{\varphi}_k$ are the unique solutions of Problem B for φ_k^- and $\hat{\varphi}_k^-$, respectively, satisfying (5.1).

By selecting a subsequence (for which we do not change notation), we have

$$\varphi_k^- \to \varphi^- \quad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\Omega_e}), \tag{6.3}$$

$$\hat{\varphi}_k^- \to \varphi^- \quad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\Omega_e}), \tag{6.4}$$

$$f_k \to f \quad \text{in } C^{2,\frac{\alpha}{2}}(\mathbf{T}^{n-1}), \tag{6.5}$$

$$\hat{f}_k \to \hat{f} \quad \text{in } C^{2,\frac{\alpha}{2}}(\mathbf{T}^{n-1}), \tag{6.6}$$

$$f \neq \hat{f}, \tag{6.7}$$

and $\varphi^- \in C^{2,\alpha}(\overline{\Omega_e})$ satisfies (2.13) with $\sigma \leq \sigma_0$ and $f,\ \hat{f} \in C^{2,\alpha}(\mathbf{T}^{n-1})$.

Also, the argument similar to the proof of Proposition 4.5(ii) yield that

$$\mathcal{P}_{\varphi_k}\left(\varphi_k|_{\Omega^+(\varphi_k)}\right) \to \mathcal{P}_{\varphi}\left(\varphi|_{\Omega^+(\varphi)}\right) \qquad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\Omega_e}),$$
$$\mathcal{P}_{\hat\varphi_k}\left(\hat\varphi_k|_{\Omega^+(\hat\varphi_k)}\right) \to \mathcal{P}_{\hat\varphi}\left(\hat\varphi|_{\Omega^+(\hat\varphi)}\right) \qquad \text{in } C^{2,\frac{\alpha}{2}}(\overline{\Omega_e}).$$

Here $\varphi \in C(\Omega_e) \cap C^{2,\alpha}(\overline{\Omega^+(\varphi)})$ is a solution of Problem B for the limiting function φ^- in (6.3) and (6.4), and the "free boundary" function of φ is the limiting function f in (6.5); it also follows that φ satisfies (5.1). Similarly, $\hat\varphi \in C(\Omega_e) \cap C^{2,\alpha}(\overline{\Omega^+(\varphi)})$ is a solution of Problem B for the limiting function φ^- in (6.3) and (6.4), and the "free boundary" function of $\hat\varphi$ is the limiting function $\hat f$ in (6.6); it also follows that $\hat\varphi$ satisfies (5.1). By (6.7), this contradicts the uniqueness result of Theorem 5.1 for φ^-. ☐

7. Multidimensional transonic shocks near spherical and circular shocks

In this section we are concerned with applications of our approach to the construction of multidimensional transonic shocks with more complex geometries. As an example, we focus on multidimensional transonic shocks near spherical ($n \geq 3$) or circular ($n = 2$) transonic shocks.

We first show the existence of spherical and circular transonic shocks. That is, choosing any

$$0 < R_1 < R_0 < R_2 < \infty,$$

we consider the domain $\Omega = \{x \in \mathbf{R}^n \ : \ R_1 < |x| < R_2\}$ and show that there exists a weak solution $\varphi_0 \in W^{1,\infty}(\Omega)$ of (1.1) and (1.2) in the sense of (2.1) with $\varphi_0(x) = w(|x|)$ for some $w : \mathbf{R} \to \mathbf{R}$ such that $\varphi_0 \in C^\infty(\overline{\Omega_0^\pm})$ and

$$\Omega_0^- = \{x \in \mathbf{R}^n \ : \ R_1 < |x| < R_0\}, \text{ and } \Omega_0^+ = \{x \in \mathbf{R}^n \ : \ R_0 < |x| < R_2\},$$

are respectively supersonic and subsonic regions of φ_0, i.e.,

$$\begin{aligned} &|D\varphi_0| > p^1_{sonic} \qquad \text{in } \Omega_0^-,\\ &|D\varphi_0| < p^1_{sonic} \qquad \text{in } \Omega_0^+, \end{aligned} \tag{7.1}$$

and $|D\varphi_0|$ has a jump across $S_0 := \{|x| = R_0\}$.

It is easy to see that a function $\varphi_0(x) = w(|x|)$ satisfies (2.1) in Ω if and only if, for any $\zeta \in C_0^\infty(R_1, R_2)$,

$$\int_{R_1}^{R_2} \rho(|w'(r)|^2) w'(r) \zeta'(r) r^{n-1} dr = 0.$$

It follows that $w(|x|)$ is a smooth solution of (1.1) if

$$\left(\rho(|w'(r)|^2) w'(r) r^{n-1}\right)' = 0.$$

Thus,

$$\rho(|w'(r)|^2) w'(r) = \frac{c}{r^{n-1}},$$

which can be written as

$$\Phi_1(w'(r)) = \frac{c}{r^{n-1}}, \tag{7.2}$$

where $\Phi_1(\cdot)$ is defined in Lemma 2.1. From Lemma 2.1, it follows that there exist smooth functions

$$\Phi_+^{-1} : (0, \Phi_1(p^1_{sonic})) \to (0, p^1_{sonic}),$$
$$\Phi_-^{-1} : (0, \Phi_1(p^1_{sonic})) \to (p^1_{sonic}, \sqrt{1/\theta}),$$

which are the inverse functions of $\Phi_1(\cdot)$ in the sense that $\Phi_1(\Phi_\pm^{-1}(\tau)) = \tau$ for any $\tau \in (0, \Phi_1(p^1_{sonic}))$, such that $(\Phi_+^{-1})'(\tau) > 0$ and $(\Phi_-^{-1})'(\tau) < 0$ for $\tau \in (0, \Phi_1(p^1_{sonic}))$. Thus, in order to satisfy (7.1) and (7.2), we have to choose $c > 0$ such that

$$0 < \frac{c}{R_2^{n-1}} < \frac{c}{R_0^{n-1}} < \frac{c}{R_1^{n-1}} < \Phi_1(p^1_{sonic}),$$

and set

$$w'(r) = \Phi_-^{-1}\left(\frac{c}{r^{n-1}}\right) \quad \text{for } R_1 < r < R_0,$$
$$w'(r) = \Phi_+^{-1}\left(\frac{c}{r^{n-1}}\right) \quad \text{for } R_0 < r < R_2.$$

Thus, we obtain a weak solution $\varphi_0(x) = w(|x|)$ of (1.1) in Ω, satisfying (7.1), by setting

$$w(r) = -\int_r^{R_0} \Phi_-^{-1}\left(\frac{c}{\tau^{n-1}}\right) d\tau < 0 \quad \text{for } R_1 < r < R_0,$$
$$w(r) = \int_{R_0}^{r} \Phi_+^{-1}\left(\frac{c}{\tau^{n-1}}\right) d\tau > 0 \quad \text{for } R_0 < r < R_2.$$

On the other hand, we can switch subsonic and supersonic regions of the solution with a spherical or circular shock. Precisely, we can construct another weak solution $\tilde\varphi_0(x) = \tilde w(|x|)$ of (1.1) in Ω such that

$$\begin{aligned} &|D\tilde\varphi_0| > p^1_{sonic} \quad \text{in } \tilde\Omega_0^- := \{x \in \mathbf{R}^n \; : \; R_0 < |x| < R_2\}, \\ &|D\tilde\varphi_0| < p^1_{sonic} \quad \text{in } \tilde\Omega_0^+ := \{x \in \mathbf{R}^n \; : \; R_1 < |x| < R_0\}. \end{aligned} \tag{7.3}$$

Indeed, extend Φ_1 to $(-\infty, 0]$ by $\Phi_1(-p) = -\Phi_1(p)$ for $p > 0$. Then we have the inverse functions

$$\tilde\Phi_+^{-1} : \; (-\Phi_1(p^1_{sonic}), 0) \to (-p^1_{sonic}, 0),$$
$$\tilde\Phi_-^{-1} : \; (-\Phi_1(p^1_{sonic}), 0) \to (-\sqrt{1/\theta}, -p^1_{sonic});$$

and, following the above argument, we can obtain a solution satisfying (7.3) by defining

$$\tilde w(r) = \int_{R_0}^{r} \tilde\Phi_-^{-1}\left(-\frac{c}{\tau^{n-1}}\right) d\tau < 0 \quad \text{for } R_0 < r < R_2,$$
$$\tilde w(r) = -\int_r^{R_0} \tilde\Phi_+^{-1}\left(-\frac{c}{\tau^{n-1}}\right) d\tau > 0 \quad \text{for } R_1 < r < R_0,$$

where $c > 0$ is as above.

We can express the function φ_0 as

$$\varphi_0(x) = \min(\varphi_0^+(x), \varphi_0^-(x)) \quad \text{for } x \in \bar\Omega,$$

where $\varphi_0^\pm \in C^\infty(\overline{\Omega})$ are defined as

$$\varphi_0^\pm(x) = w_\pm(|x|), \quad w_\pm(r) = \int_{R_0}^{r} \Phi_\pm^{-1}\left(\frac{c}{\tau^{n-1}}\right) d\tau, \qquad \text{for } R_1 < r < R_2. \tag{7.4}$$

The solution φ_0 containing the spherical or circular transonic shock satisfies the entropy condition:

$$\rho(|D\varphi_0^-|^2) < \rho(|D\varphi_0^+|^2),$$

across the transonic shock from the hyperbolic phase to the elliptic phase, which is the direction of fluid motions.

The function $\tilde{\varphi}_0$ has the same properties as φ_0.

We now state our results on the existence and stability of multidimensional transonic shocks that are close to the solution φ_0; similar results for $\tilde{\varphi}_0$ hold.

Problem A'. Given a supersonic solution φ^- of (1.1) in Ω, which is a $C^{2,\alpha}$ perturbation of φ_0^-, for some $\alpha > 0$:

$$\|\varphi^- - \varphi_0^-\|_{C^{2,\alpha}(\Omega)} \le \sigma, \tag{7.5}$$

with $\sigma > 0$ small, find a transonic shock solution φ in Ω such that $\varphi = \varphi^-$ in Ω^-, where Ω^- is the supersonic region of φ, and

$$\begin{aligned} \varphi &= \varphi^- \qquad \text{on} \quad \{|x| = R_1\}, \\ \varphi &= \varphi_0^+ \qquad \text{on} \quad \{|x| = R_2\}. \end{aligned} \tag{7.6}$$

Theorem 7.1. *There exist positive constants σ_0, C_1, and C_2, depending only on n, γ, c, and Ω, such that, for every $\sigma \le \sigma_0$ and any supersonic solution φ^- of (1.1) satisfying the conditions stated in Problem A', there exists a unique solution φ of Problem A' satisfying*

$$\|\varphi - \varphi_0^+\|_{2,\alpha,\Omega^+} \le C_1\sigma,$$

and $\Omega^+ = \{|x| > f(\frac{x}{|x|})\} \cap \Omega$, where $f : S^{n-1} \to \mathbf{R}$, and

$$\|f - R_0\|_{2,\alpha,S^{n-1}} \le C_2\sigma.$$

The proof of Theorem 7.1 follows closely the proof of Theorem 2.1. We will only point out some details.

First, we reformulate Problem A' into a free boundary problem. Following the heuristic discussion preceding the statement of Problem B and taking into account the geometry of the present case, we expect that the solution of Problem A' satisfies

$$\varphi(x) \le \varphi^-(x), \qquad \text{for } x \in \Omega. \tag{7.7}$$

Then our free boundary problem is

Problem B'. Find $\varphi \in C(\overline{\Omega})$ such that

(i) φ satisfies (7.7) in Ω and (7.6) on $\partial\Omega$;
(ii) $\varphi \in C^{2,\alpha}(\overline{\Omega^+})$ is a solution of (1.1) in $\Omega^+ = \{\varphi(x) < \varphi^-(x)\} \cap \Omega$, the non-coincidence set;
(iii) the free boundary $S = \partial\Omega^+ \cap \Omega$ is given by the equation $|x| = f(\frac{x}{|x|})$ so that $\Omega^+ = \{|x| > f(\frac{x}{|x|})\} \cap \Omega$, where $f : S^{n-1} \to \mathbf{R}$ satisfies $f \in C^{2,\alpha}(S^{n-1})$;
(iv) the free boundary conditions (2.3) and (2.4) hold on S.

We now solve Problem B', similar to Problem B above. Namely, we consider the problem with the truncated equation (3.6) and the free boundary condition (3.7). We do not need to extend our domain Ω here, since we expect that the free boundary lies in the interior of Ω.

Similar to (4.6), we define

$$\nu_0(x) := \frac{D\varphi_0^+(x)}{|D\varphi_0^+(x)|} = \frac{x}{|x|}. \tag{7.8}$$

Now we can follow the argument of Section 4. We clarify the following three points.

First, the non-degeneracy estimate (4.3) is now

$$(\varphi^- - \psi)_{\nu_0}(x) \geq \frac{1}{2}\left(\Phi_-^{-1}\left(\frac{c}{R_2^{n-1}}\right) - \Phi_+^{-1}\left(\frac{c}{R_1^{n-1}}\right)\right) > 0,$$

if σ is sufficiently small which depends only on n, q^+, γ, and Ω. We obtain this estimate since, for any $x \in \Omega$,

$$(\varphi_0^-)_{\nu_0}(x) = w_-'(|x|) \geq \Phi_-^{-1}\left(\frac{c}{R_2^{n-1}}\right) > p_{sonic}^1 > \Phi_+^{-1}\left(\frac{c}{R_1^{n-1}}\right) \geq w_+'(|x|) = (\varphi_0^+)_{\nu_0}(x),$$

where we used that $\Phi_+^{-1}(\cdot)$ is increasing and $\Phi_-^{-1}(\cdot)$ decreasing, and $R_1 \leq |x| \leq R_2$.

Second, $A(P)$ in the argument of Section 4.2 is now replaced by $A(x,P)$ with the formula:

$$A(x,P) = \tilde{\rho}(|P+D\varphi_0^+(x)|^2)(P+q^+D\varphi_0^+(x)) - \rho(|D\varphi_0^+(x)|^2)D\varphi_0^+(x) \quad \text{for } P \in \mathbf{R}^n,$$

and $g_\psi(x)$ is defined by (4.18), where $\varphi_0^+(x)$ is now defined by (7.4).

Third, (4.19) still holds in Ω. Indeed, since $D\varphi_0^\pm(x) = w_\pm'(|x|)\nu_0(x)$, we use the function $\Phi_1(\cdot)$ from Lemma 2.1 to obtain

$$\rho(|D\varphi_0^\pm(x)|^2)D\varphi_0^\pm(x) = \Phi_1(|w_\pm'(|x|))\nu_0(x) = \Phi_1\left(\Phi_\pm^{-1}\left(\frac{c}{|x|^{n-1}}\right)\right)\frac{x}{|x|} = c\frac{x}{|x|^n}.$$

The uniqueness and stability results, similar to Theorem 6.1, can also be established (see [9]).

Appendix A. Proof of Lemma 4.3

Consider first the case for the Laplace equation with the Neumann condition on S_ψ:

$$\begin{cases} \Delta u = r & \text{in } \Omega^+(\psi), \\ u_\nu = g & \text{on } S_\psi, \\ u = 0 & \text{on } \Gamma_1. \end{cases} \tag{A.1}$$

Let $\psi \in C^\infty(\Omega_e) \cap \mathcal{K}_M$ and $r, g \in C^\infty(\overline{\Omega^+(\psi)})$. We can easily construct $\tilde{g} \in C^\infty(\overline{\Omega^+(\psi)})$ satisfying $\tilde{g} = g$ near S_ψ and $\tilde{g} = 0$ on Γ_1. Then the function $v := u - \tilde{g}$ satisfies

$$\begin{cases} \Delta v = h := r - \Delta\tilde{g} & \text{in } \Omega^+(\psi), \\ v_\nu = 0 & \text{on } S_\psi, \\ v = 0 & \text{on } \Gamma_1. \end{cases} \tag{A.2}$$

We construct a variational solution of (A.2), that is, we minimize a functional

$$I[w] = \int_{\Omega^+(\psi)} (|\nabla w|^2 - hw)dx$$

over the set $\{w \in H^1(\Omega^+(\psi)) \ : \ w|_{\Gamma_1} = 0\}$, where $\Omega^+(\psi) \subset \mathbf{T}^{n-1} \times [-N_1, N_2]$ is a compact Riemannian manifold with boundary, and $H^1(\Omega^+(\psi))$ is the Sobolev space, and the condition $w|_{\Gamma_1} = 0$ is understood in the sense of traces. The existence of a solution v follows from the argument similar to the proof of Theorem 4.8 in [4], with the aid of Lemma 4.2. Now a standard regularity theory implies that $v \in C^\infty(\overline{\Omega^+(\psi)})$ and v satisfies (A.2): Indeed, since we can consider $v(x)$ as a periodic solution with respect to the x' variables in $\mathbf{R}^{n-1}\times[-N_1, N_2]\cap\{x_n > f(x')\}$, then the interior regularity follows from Theorem 8.10 in [17] and the regularity near Γ_1 from Section 8.4 in [17]; and the regularity near S_ψ follows from the interior regularity by the local flattening of S_ψ and then by the reflection.

Now $u = v+\tilde{g}$ is a C^∞ solution of (A.1). Then u satisfies (4.23) with C depending only on the domain Ω.

If $\psi \in \mathcal{K}_M$, $r \in C^\alpha(\overline{\Omega^+(\psi)})$, and $g \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$, we approximate them by smooth functions ψ_j, r_j, and g_i, and use (4.23) to pass the limit.

Thus we have shown that (A.1) is uniquely solvable for $r \in C^\alpha(\overline{\Omega^+(\psi)})$ and $g \in C^{1,\alpha}(\overline{\Omega^+(\psi)})$, and the solution satisfies (4.23).

The proof of the existence of a solution for the general problem (4.21) can be achieved via the method of continuity by considering the following problem for $t \in [0,1]$:

$$\begin{cases} \overline{\mathcal{L}}_t[u] = t\overline{\mathcal{L}}u + (1-t)\Delta u = r \quad \text{in} \quad \Omega^+(\psi), \\ tb\cdot Du + (1-t)u_\nu = g \quad \text{on} \quad S_\psi, \\ v = 0 \quad \text{on} \quad \Gamma_1, \end{cases}$$

which has the structure of (4.21) with the same ellipticity and obliqueness constants. The argument just repeats the proof of Theorem 6.8 in [17], with the aid of Lemma 4.2 and (4.23).

Appendix B. Proof of Lemma 4.4

We now prove Lemma 4.4 for the estimate (4.33) in several steps. Denote $g := tg_\psi$.

First, we note that $u(x)$ is a solution of the problem (4.21) with $r(x) \equiv 0$, $a_{ij}(x) = A_{ij}(Du(x))$, and $b_i(x) = \sum_{j=1}^n \int_0^1 A^j_{P_i}(sDu(x))ds\,\nu^j(x)$, where we used that $A(0) = 0$. Then $a_{ij}, b_i \in C^{1,\alpha}$. Also the uniform ellipticity of $a_{ij}(x)$ is obvious, and the strict obliqueness of $b_i(x)$ follows from

$$b(x)\cdot\nu(x) = \sum_{i,j=1}^n \int_0^1 A^j_{P_i}(sDu(x))\nu^i(x)\nu^j(x)ds \ge \lambda > 0,$$

where λ and Λ are the ellipticity constants of (3.6). Then, by Lemma 4.2(i),

$$\|u\|_{0,\Omega^+(\psi)} \le C\|g\|_{0,\Omega^+(\psi)}, \tag{B.1}$$

where C depends only on λ, Λ, and Ω. Noting also that $\underline{u}(x)$ and $\overline{u}(x)$ constructed in the proof of Lemma 4.2 are the barriers for $u(x)$ on Γ_1, we have

$$|Du| \le C\|g\|_{0,\Omega^+(\psi)} \quad \text{on} \quad \Gamma_1. \tag{B.2}$$

Thus we can combine the estimates for the oblique derivative problem with the estimates for the Dirichlet problem from Chapters 8 and 14 in [17] to prove (4.33).

For the oblique derivative problem, we use the estimates from Sections 2–4 in [24]. Note that (4.29) is the structure condition which allows to apply these estimates.

Also note that the conormal boundary condition on S_ψ satisfies the nonlinear strict obliqueness property of [24]: Indeed, denoting $G(x,P) := \sum_{i=1}^n A^i(P)\nu_i(x)$ for $x \in S_\psi$ and $P \in \mathbf{R}^n$, we obtain $\chi := G_P \cdot \nu = \sum_{i,j=1}^n A^i_{P_j}\nu_i\nu_j \geq \lambda > 0$, and this with the properties (4.17) and (4.29) implies that $G(x,P)$ satisfies the conditions (G2) and (G3) of [24]. Thus, using the regularized distance to S_ψ described in [24, page 522], and using Lemmas 3.1 and 3.3 in [24] (with modifications outlined on pages 522–523) and Theorem 4.1 (with the aid of Lemma 2.2), we obtain (4.33) in a boundary neighborhood of S_ψ, where $\delta > 0$ is from Theorem 4.1 in [24]. The interior gradient bound follows from Lemma 3.1 in [24].

These estimates combined with (B.1), (B.2), and the estimates of Chapter 13 in [17] yield (4.33) with C depending only on $\|g\|_{1,\alpha,\Omega^+(\psi)}$, S_ψ, and Ω.

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ABSTRACT. We establish the existence and stability of multidimensional transonic shocks for the Euler equations for steady potential compressible fluids. The Euler equations, consisting of the conservation law of mass and the Bernoulli law for the velocity, can be written as a second-order, nonlinear elliptic-hyperbolic equation of mixed type for the velocity potential. The transonic shock problem can be formulated into the following free boundary problem: The free boundary is the location of the transonic shock which divides the two regions of smooth flow, and the equation is hyperbolic in the upstream region where the smooth perturbed flow is supersonic. We develop a nonlinear approach to deal with such a free boundary problem in order to solve the transonic shock problem. Our results indicate that there exists a unique solution of the free boundary problem such that the equation is always elliptic in the downstream region and the free boundary is smooth, provided that the hyperbolic phase is close to a uniform flow. We prove that the free boundary is stable under the steady perturbation of the hyperbolic phase. We also establish the existence and stability of multidimensional transonic shocks near spherical or circular transonic shocks.

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