

ON THE SEMI-GEOSTROPHIC SYSTEM IN PHYSICAL SPACE WITH GENERAL INITIAL DATA

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Abstract. In order to accommodate general initial data, an appropriately relaxed notion of renormalized Lagrangian solutions for the Semi-Geostrophic system in physical space is introduced. This is shown to be consistent with previous notions, generalizing them. A weak stability result is obtained first, followed by a general existence result whose proof employs said stability and approximating solutions with regular initial data. The renormalization property ensures the return from physical to dual space; as consequences we get conservation of Hamiltonian energy and some weak time-regularity of solutions.

Key words. Semi-Geostrophic system, flows of maps, optimal mass transport, Wasserstein metric, optimal maps, absolutely continuous curves.

AMS subject classifications. 35D05, 35G25, 76B03

1. Introduction. The Semi-Geostrophic (abbreviated SG in this paper) system is a model of large scale atmosphere/ocean flows, where “large-scale” means that the flow is rotation-dominated. Physically interesting solutions of the SG system must satisfy the convexity principle, introduced by Cullen and Purser, which requires that the modified pressure function be convex. All previous works on existence of solutions for the SG system require some additional conditions/restrictions, which are essentially some form of strict convexity of the initial data. In this paper, we prove existence of solutions for all convex initial data, i.e. in the most general class of physically admissible solutions. The solutions which we construct are necessarily weak, in the sense defined below. Specifically, in this paper we introduce and prove existence for a relaxed version of the weak Lagrangian solutions constructed by Cullen and Feldman in [9] for the SG system in a fixed three-dimensional domain. Some results in this direction were obtained in [15], yet the decision to maintain flow maps/streamlines in physical space as building blocks for those solutions prevented us from reaching a satisfactorily general conclusion. Loss of incompressibility was also exhibited by those solutions. Here we fix these issues by means of the introduction of a relaxed notion of solution, for which the physical space volume-preserving flow map is replaced by a “volume-preserving measure”. This relaxation procedure is reminiscent of Kantorovich’s relaxation of Monge’s Optimal Transport problem. Thus, it is not so surprising, given the close connection between Optimal Transport and SG solutions satisfying the Cullen-Purser stability condition.

Throughout the entire paper, $\Omega \subset \mathbb{R}^3$ is a given open, bounded set with

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C^1 boundary, and $T \in (0, \infty)$ is fixed. A version of the 3D Semi-Geostrophic system [6, 9] is

$$\begin{aligned} D_t X &= J(X - x), \\ X &= \nabla P, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u} \cdot \nu &= 0, \quad \text{on } [0, T) \times \partial\Omega, \\ P(0, \cdot) &= P_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where ∇ stands for the spatial gradient, $\nabla \cdot$ the spatial divergence and $D_t := \partial_t + \mathbf{u} \cdot \nabla$. One looks for solutions $(P, \mathbf{u})$ such that $P : [0, T) \times \Omega \rightarrow \mathbb{R}$, $\mathbf{u} : [0, T) \times \Omega \rightarrow \mathbb{R}^3$ which satisfy the Cullen-Purser stability condition (see, e.g., [10]). This amounts to imposing that $P_t(\cdot) := P(t, \cdot)$ be convex for all $t \in [0, T)$. Here,

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Henceforth, we shall assume without loss of generality that $\mathcal{L}^3(\Omega) = 1$ (otherwise all the measures considered will have total mass equal to the volume of Ω) and denote

$$\chi := \mathcal{L}^3|_{\Omega}.$$

If $\nabla P_{t\#}\chi =: \alpha_t$ are all absolutely continuous with respect to $\mathcal{L}^3$, then one can use the Legendre-Fenchel transforms P_t^* of P_t to formally rewrite (1.1) as the so-called SG in dual variables

$$\partial_t \alpha + \nabla \cdot (U \alpha) = 0 \quad \text{in } [0, T) \times \mathbb{R}^3, \tag{1.2}$$

$$\nabla P(t, \cdot)_{\#}\chi = \alpha(t, \cdot) \quad \text{for any } t \in [0, T); \tag{1.3}$$

$$U(t, X) = J[X - \bar{\gamma}(t, X)], \tag{1.4}$$

$$\alpha(0, X) = \alpha_0(X) \quad \text{for a.e. } X \in \mathbb{R}^3, \tag{1.5}$$

where $\bar{\gamma}(t, X)$ (or $\bar{\gamma}_t(X)$ as alternate notation) denotes the barycentric projection onto α_t of the optimal transfer plan between χ and α_t . It is a map defined by (see [4], [5])

$$\int_{\mathbb{R}^3} \xi(X) \cdot \bar{\gamma}(X) d\alpha(X) = \iint_{\mathbb{R}^3 \times \Omega} \xi(X) \cdot y d\gamma(X, y) \tag{1.6}$$

for all continuous $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of at most quadratic growth, where γ is the (unique) optimal plan between α and χ . Since χ is absolutely continuous with respect to $\mathcal{L}^3$, we deduce

$$\int_{\mathbb{R}^3} \xi(X) \cdot \bar{\gamma}_t(X) d\alpha_t(X) = \int_{\Omega} \xi(\nabla P_t(x)) \cdot x dx. \tag{1.7}$$

If $\alpha_t \ll \mathcal{L}^3$ for a.e. $t \in (0, T)$, the dual velocity field $U(t, X) = J[X - \nabla P^*(t, X)]$ may be utilized instead of the one from the above display, which

gives rise to the well-known dual-space formulation for SG. A detailed explanation of why ∇P^* should be replaced by $\bar{\gamma}$ in the general case can be found in the introduction of [15]. Same paper gives a comprehensive bibliography of results on SG in dual space. These solutions are not known to be regular enough to be translated into Eulerian solutions of the problem in physical space. Cullen and Feldman [9] constructed weak Lagrangian solutions in the physical space, for initial data P_0 satisfying $\nabla P_{0\#}\chi =: \alpha_0 \in L^p(\mathbb{R}^3)$, $p > 1$. We note that the condition $\nabla P_{0\#}\chi \in L^p(\mathbb{R}^3)$ is a quantitative version of the condition of strict convexity of P_0 . Indeed, “flat spots” in P_0 produce singular parts in the measure $\nabla P_{0\#}\chi$: for example, if P_0 is linear or affine, with $\nabla P_0 = a \in \mathbb{R}^3$, then $\nabla P_{0\#}\chi = \mathcal{L}^3(\Omega)\delta_a$ (volume of Ω concentrated at a).

Recently, existence of Eulerian solutions for a class of initial data, where the conditions include the requirement that the support of $\alpha_0 = \nabla P_{0\#}\chi$ in the dual space is the whole space, was obtained by L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli [2, 3] based on the results of G. De Philippis and A. Figalli [13] on regularity of solutions for the Monge-Ampere equation. Existence of Eulerian solutions in physical space for more general initial data, when the support of α_0 in the dual space may have a non-empty boundary, is presently not known.

Observations show that the atmosphere contains significant regions where the potential temperature and absolute momentum of the atmosphere are well-mixed, representing a state of neutrality to parcel displacements (yielding “flat spots” in ∇P_t). Such states commonly arise as a result of atmospheric forcing either through surface heating or latent heat release. These considerations show that the singular case, as studied in this work, is physically relevant [8]. In [15] we considered the case of singular initial data, i.e. when $\alpha_0 = \nabla P_{0\#}\chi$ is a singular measure. Our motivation originates with the work of Cullen and Feldman [9] where Lagrangian solutions in physical space were constructed in the case $\alpha_0 \in L^p(\mathbb{R}^3)$ for $p > 1$; the question left open in its generalization (to $p \geq 1$) by Faria et al. [14] regards the even more general case, i.e. the case in which α_0 may be a singular probability measure. The “pathological” case discussed in [14] aims to emphasize that the physical space streamlines are not stable to small perturbations in the singular case (as opposed to the L^1 -case). However, as we show in the present paper (see also [15] for a different approach), this can be fixed by redefining the weak Lagrangian solutions in the singular case by taking into account the correct formulation of the problem in dual space (which requires replacing ∇P^* by $\bar{\gamma}$, as mentioned above).

Existence of a solution for the dual problem (1.2), (1.3), (1.4), (1.5) for any probability measure α_0 with finite second moment was proved in [4].

In [15] we defined weak Lagrangian solutions in physical space for any convex initial data P_0 (i.e. any initial probability measure $\alpha_0 = \nabla P_{0\#}\chi$), and proved existence of such solutions in the case of discrete measures. Moreover, we showed conservation of geostrophic energy along the Lagrangian trajectories in the physical space. One stand-out feature of those Lagrangian solutions

in physical space in the case of singular initial data was the absence of the Lebesgue measure preserving property for the flow map F_t . The solution of the dual problem (1.2), (1.3), (1.4), (1.5) does not provide enough information to separate the particle paths (in physical space) within the subgradient set $\partial P_t^*(X) \subset \Omega$ for $X \in \text{supp } \alpha_t$, so we defined the flow map F_t by collapsing the set $\partial P_t^*(X)$ to one point, its “ α_t -barycenter” $\bar{\gamma}_t(X) \in \partial P_t^*(X)$; the χ -measure carried by the set $\partial P_t^*(X)$ should now be concentrated at the point $\bar{\gamma}_t(X)$. Thus, the measure preserving property $F_{t\#}\chi = \chi$ was replaced by

$$F_{t\#}\chi = \mu_t \text{ for } t > 0, \quad \text{where } \mu_t := (\bar{\gamma}_t \circ \nabla P_t)_{\#}\chi. \quad (1.8)$$

If α_t is absolutely continuous with respect to the Lebesgue measure, we have $(\bar{\gamma}_t(X) \circ \nabla P_t)(x) = (\nabla P_t^* \circ \nabla P_t)(x) = x$ for a.e. $x \in \Omega$, and thus we recover the Lebesgue measure preserving property of F_t , as required in [9]. Furthermore, in the case of Lagrangian solutions with absolutely continuous initial measure $\alpha_0 = \nabla P_0_{\#}\chi$, a stability property was proved in [14]. We did not manage to prove a stability result for the solutions we built in [15] with singular initial data α_0 , and that has remained an open problem.

However, an example in [15] (inspired from [14]) suggests that some stability properties may be expected in this case too. Furthermore, the existence result in [15] is hardly satisfactory as it only covers one of the limiting cases (i.e. α_t 's are convex combinations of point masses; the other limiting case, $\alpha_t \ll \mathcal{L}^3$, being settled in [9], [14]). Even if the elusive dual space Lagrangian flow were obtained in the general case (which is the main issue preventing a general existence result in [15]), it is physically unsatisfactory to only produce solutions which are not measure preserving in any sense, as that “destroys” the incompressibility of the flow. Thus, in this paper we propose a more natural notion of solution, which resolves all of the above issues. It involves a relaxation of the nonlinear interaction between the physical space maps ∇P_t and F_t by replacing maps $F_t : \Omega \rightarrow \Omega$ by the corresponding transport plans, which are measures on $\Omega \times \Omega$: it is, as discussed earlier, perhaps not so surprising that this approach is successful, in light of the intimate connection between SG system and Optimal Transport.

In Section 2 we introduce the concept of relaxed Lagrangian solutions. By means of an example first discussed in [14], we show that the time-continuity these solutions exhibit in the regular case (i.e. absolutely continuous dual space solutions) cannot be expected in general. Furthermore, we show that under additional regularity a relaxed solution determines a classical solution of SG system (1.1), and that the new notion of solutions is compatible/consistent with the previous notion of weak Lagrangian solutions in [9]. In Section 3 we collect some results on the stability of solutions, and optimal maps, in dual space. These are used in Section 4 to construct the relaxed Lagrangian solutions in physical space; here we obtain an existence and stability result for such solutions. We first show that these solutions give rise to weak distributional solutions of the problem in dual space, so the results of Section 3 can

be used here. We conclude Section 4 (and the paper) by proving some weak time-continuity of our relaxed Lagrangian solutions.

2. Relaxed Lagrangian solutions in physical space. The starting point will be the weak Lagrangian solutions in the physical space. Such solutions were introduced by Cullen & Feldman [9], and existence of Lagrangian solutions of (1.1) was shown in [9] for any $\alpha_0 = \nabla P_{0\#}\chi \in L^q(\Omega)$ for $q > 1$, on the basis of Ambrosio's theory on transport equations and flows of BV vector fields [1]. These results were extended to the case $q = 1$ in [14]. The restriction $\alpha_0 \ll \mathcal{L}^3$ amounts to requiring some *strict convexity* property of the potentials P_t , which is certainly more than what Cullen and Purser's stability condition imposes (which is simply the convexity of P_t). The definition of weak Lagrangian solutions in the physical space is following (where we slightly modify the definition given in [9] by relaxing assumptions on P).

Definition 2.1. *Let $P_0 \in H^1(\Omega)$ be convex, $F : [0, T) \times \Omega \rightarrow \Omega$ be a Borel map such that $F \in C([0, T); L^2(\Omega; \mathbb{R}^3))$, and let $P \in C([0, T); H^1(\Omega))$. Assume $P(t, \cdot)$ is convex in Ω for each $t \in [0, T)$. Then the pair (P, F) is called a weak Lagrangian solution of (1.1) in $[0, T) \times \Omega$ if*

- i. $F(0, x) = x$, $P(0, x) = P_0(x)$ for a.e. $x \in \Omega$,
- ii. for any $t > 0$ the mapping $F_t = F(t, \cdot) : \Omega \rightarrow \Omega$ is Lebesgue measure preserving, in the sense that $F_{t\#}\chi = \chi$;
- iii. There exists a Borel map $F^* : [0, T) \times \Omega \rightarrow \Omega$ such that for every $t \in (0, T)$ the map $F_t^* = F^*(t, \cdot) : \Omega \rightarrow \Omega$ is Lebesgue measure preserving: $F_{t\#}^*\chi = \chi$, and satisfies $F_t^* \circ F_t(x) = x$ and $F_t \circ F_t^*(x) = x$ for a.e. $x \in \Omega$;
- iv. The function

$$Z(t, x) = \nabla P(t, F_t(x)) \quad (2.1)$$

is a distributional solution of

$$\begin{aligned} \partial_t Z(t, x) &= J[Z(t, x) - F(t, x)] && \text{in } [0, T) \times \Omega, \\ Z(0, x) &= \nabla P_0(x) && \text{in } \Omega. \end{aligned} \quad (2.2)$$

Note that the sense in which (2.2) must be satisfied is

$$\int_0^T \int_{\Omega} [Z \cdot \partial_t \varphi + J(Z - F) \cdot \varphi] dx dt + \int_{\Omega} \nabla P_0(x) \cdot \varphi(0, x) dx = 0 \quad (2.3)$$

for any $\varphi \in C_c^1([0, T) \times \Omega; \mathbb{R}^3)$. This implies (as observed in [9] in a similar case) $t \mapsto Z(t, x) \in H^1(0, T; \mathbb{R}^3)$ for $\mathcal{L}^3$ -a.e. $x \in \Omega$, and the differential equation in (2.2) is satisfied in the $\mathcal{L}^4$ -a.e. in $(0, T) \times \Omega$. Thus, for $\xi \in C_b^1(\mathbb{R}^3)$ (C^1 with bounded gradient) the map $t \mapsto \xi(Z(t, x))$ is absolutely continuous for χ -a.e. $x \in \Omega$ and

$$\frac{d}{dt} \xi(Z(t, x)) = \nabla \xi(Z(t, x)) \cdot J[Z(t, x) - F(t, x)] \text{ for a.e. } t \in [0, T).$$

$0 < T < \infty$. Consequently, a more general, “renormalized” version of (2.3) is available in the form

$$\int_0^T \int_{\Omega} \left\{ \xi(Z(t, x)) \partial_t \zeta(t, x) + \nabla \xi(Z(t, x)) \cdot J[F(t, x) - Z(t, x)] \zeta(t, x) \right\} dx dt + \int_{\Omega} \xi(\nabla P_0(x)) \zeta(0, x) dx = 0, \quad (2.4)$$

for any $\xi \in C_b^1(\mathbb{R}^3)$, $\zeta \in C_c^1([0, T] \times \Omega)$.

Remark 2.2. *The use of arbitrary $\xi \in C_b^1(\mathbb{R}^3)$ instead of the obvious first choices $\xi_k(x) = x_k$ for $k = 1, 2, 3$ is enabled by the fact that, as discussed above, equation (2.2) is satisfied not only in the sense of distributions, but pointwise a.e. as well. This amounts to solutions defined through this general choice of test functions having some sort of “renormalization” property. One of consequences of this property is that it allows to prove that the relaxed solutions, defined below, give rise to distributional solutions in dual space, which in turn allows to show conservation the Hamiltonian energy and weak time-continuity of relaxed solutions.*

Our goal is to define relaxed Lagrangian solutions in the physical space, replacing maps $F_t : \Omega \rightarrow \Omega$ by the corresponding transport plans, which are measures σ_t on $\Omega \times \Omega$. Thus we base the definition of relaxed Lagrangian solution on the pair (P, σ) , and try to capture all the features of (P, F) from Definition 2.1 (for which $\sigma_t = (\text{Id} \times F(t, \cdot))_{\#} \chi$).

More precisely, assuming existence of Lagrangian solution (P, F) in the sense of Definition 2.1, we define the measure σ on $(0, T) \times \Omega \times \Omega$ by

$$\int_0^T \int_{\Omega} \int_{\Omega} \xi(t, x, y) \sigma(dt, dx, dy) = \int_0^T \int_{\Omega} \xi(t, x, F(t, x)) dx dt, \quad (2.5)$$

for all $\xi \in C_b((0, T) \times \Omega \times \Omega)$. We notice first that the volume-preserving property of $F(t, \cdot)$ shows that σ disintegrates as

$$\int_0^T \int_{\Omega} \int_{\Omega} \xi(t, x, y) \sigma(dt, dx, dy) = \int_0^T \left(\int_{\Omega} \int_{\Omega} \xi(t, x, y) \sigma_t(dx, dy) \right) dt \quad (2.6)$$

for all $\xi \in C_b((0, T) \times \Omega \times \Omega)$, where $[0, T] \ni t \mapsto \sigma_t = (\text{Id} \times F_t)_{\#} \chi$ is a Borel family of measures on $\Omega \times \Omega$ such that

$$\text{proj}_x \sigma_t = \text{proj}_y \sigma_t = \chi \quad \text{for all } t \in [0, T]. \quad (2.7)$$

Furthermore, (2.4) becomes, in light of (2.1),

$$\int_0^T \iint_{\Omega \times \Omega} \left\{ \xi(\nabla P_t(y)) \partial_t \zeta(t, x) + \nabla \xi(\nabla P_t(y)) \cdot J[y - \nabla P_t(y)] \zeta(t, x) \right\} \sigma(dt, dx, dy) + \int_{\Omega} \xi(\nabla P_0(x)) \zeta(0, x) dx = 0 \quad (2.8)$$

for all $\xi \in C_b^1(\mathbb{R}^3)$ and all $\zeta \in C_c^1([0, T] \times \Omega)$. Let us, from now on, use the notation $\int_{\Omega^2}$ for $\int_{\Omega} \int_{\Omega}$.

Lemma 2.3. *Let $P : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R} \cup \{\infty\}$ be a Borel function such that $P(t, \cdot)$ is convex for all $t \in [0, T]$. If $\nabla P \in L^2(\Omega; \mathbb{R}^3)$ and σ is a nonnegative, finite Borel measure on $(0, T) \times \Omega \times \Omega$ satisfying (2.6) and (2.7), then the left-hand-side of the equation (2.8) is well-defined for all $\xi \in C^1(\mathbb{R}^3) \cap \text{Lip}(\mathbb{R}^3)$ and all $\zeta \in C_c^1([0, T] \times \Omega)$.*

Proof. Take any sequence $\{\zeta_n\}_n \subset C_c^\infty((0, T) \times \Omega; \mathbb{R}^3)$ which converges to ∇P in $L^2((0, T) \times \Omega; \mathbb{R}^3)$ and $\mathcal{L}^4$ -a.e. in $(0, T) \times \Omega$. By (2.6) and (2.7),

$$\begin{aligned} \int_0^T \int_{\Omega^2} |\xi \circ \zeta_m(t, y) - \xi \circ \zeta_n(t, y)|^2 \sigma(dt, dx, dy) = \\ \int_0^T \int_{\Omega} |\xi \circ \zeta_m(t, y) - \xi \circ \zeta_n(t, y)|^2 dy dt \end{aligned}$$

and since ξ is Lipschitz, we deduce that $(0, T) \times \Omega \ni (t, y) \mapsto \xi \circ \zeta_n(t, y)$ is a Cauchy sequence in $L^2(\sigma; \mathbb{R}^3)$. Thus, it is convergent in this space, and, by a standard argument, the limiting function will not depend on the choice of the sequence $\{\zeta_n\}_n$. This gives unequivocal meaning to $(0, T) \times \Omega \ni (t, y) \mapsto \xi \circ \nabla P_t(y)$ as an element of $L^2(\sigma; \mathbb{R}^3)$.

Next, note that

$$\begin{aligned} \int_0^T \int_{\Omega^2} |y|^2 |\nabla \xi \circ \zeta_m(t, y) - \nabla \xi \circ \zeta_n(t, y)|^2 \sigma(dt, dx, dy) \\ \leq \sup_{y \in \Omega} |y|^2 \int_0^T \int_{\Omega} |\nabla \xi \circ \zeta_m(t, y) - \nabla \xi \circ \zeta_n(t, y)|^2 dy dt. \end{aligned} \tag{2.9}$$

But

$$\int_0^T \int_{\Omega} |\nabla \xi \circ \nabla P_t(y) - \nabla \xi \circ \zeta_n(t, y)|^2 dy dt \xrightarrow{n \rightarrow \infty} 0$$

due to the fact that the integrand converges to zero pointwise $\mathcal{L}^4$ -a.e. in $(0, T) \times \Omega$ and is bounded a.e. by $4\|\nabla \xi\|_\infty^2 \in L^1((0, T) \times \Omega)$. Thus, $\nabla \xi \circ \zeta_n$ converges to $\nabla \xi \circ \nabla P$ in $L^2((0, T) \times \Omega; \mathbb{R}^3)$. This implies $\{\nabla \xi \circ \zeta_n\}_n$ is Cauchy in $L^2((0, T) \times \Omega; \mathbb{R}^3)$, which, by (2.9), shows that $(0, T) \times \Omega \ni (t, y) \mapsto \{\nabla \xi \circ \zeta_n(t, y) \cdot y\}_n$ is Cauchy in $L^2(\sigma; \mathbb{R}^3)$. Since, as observed above, its limit in $L^2(\sigma; \mathbb{R}^3)$ will be independent of the chosen sequence $\{\zeta_n\}_n$, we now have a meaning for $(0, T) \times \Omega \ni (t, y) \mapsto \nabla \xi \circ \nabla P_t(y) \cdot y$ as an element of $L^2(\sigma; \mathbb{R}^3)$.

Finally, to show that $(0, T) \times \Omega \ni (t, y) \mapsto \nabla \xi \circ \nabla P_t(y) \cdot \nabla P_t(y) \in L^2(\sigma; \mathbb{R}^3)$, we first show that $\nabla \xi \circ \zeta_n \cdot \zeta_n$ converges to $\nabla \xi \circ \nabla P \cdot \nabla P$ in $L^2((0, T) \times \Omega; \mathbb{R}^3)$.

Indeed,

$$\begin{aligned}
& \int_0^T \int_{\Omega} |\nabla \xi \circ \zeta_n(t, y) \cdot \zeta_n(t, y) - \nabla \xi \circ \nabla P_t(y) \cdot \nabla P_t(y)|^2 dy dt \\
& \leq 2 \int_0^T \int_{\Omega} |\nabla \xi \circ \zeta_n(y)|^2 |\zeta_n(t, y) - \nabla P_t(y)|^2 dy dt \\
& \quad + 2 \int_0^T \int_{\Omega} |\nabla P_t(y)|^2 |\nabla \xi(\zeta_n(t, y)) - \nabla \xi(\nabla P_t(y))|^2 dy dt \\
& =: I_1 + I_2.
\end{aligned}$$

Now, $0 \leq I_1 \leq 2 \|\nabla \xi\|_{L^\infty(\mathbb{R}^3)}^2 \|\zeta_n - \nabla P\|_{L^2((0,T) \times \Omega; \mathbb{R}^3)}^2 \rightarrow 0$. Also, $I_2 \rightarrow 0$ by Dominated Convergence, since $|\nabla P|^2 \in L^1((0, T) \times \Omega)$. It follows that the sequence $\nabla \xi \circ \zeta_n \cdot \zeta_n$ converges in $L^2((0, T) \times \Omega; \mathbb{R}^3)$, thus it is Cauchy there, which leads to its being Cauchy in $L^2(\sigma; \mathbb{R}^3)$ (again, by (2.7)). $\square$

We now have all the ingredients for our new definition.

Definition 2.4. Let $P_0 \in H^1(\Omega)$ be convex. Consider a family of convex functions $P \in L^\infty(0, T; H^1(\Omega))$ and a Borel family of measures $[0, T) \ni t \mapsto \sigma_t \in \mathcal{M}(\Omega \times \Omega)$. Let σ be given by $d\sigma = d\sigma_t dt$ as in (2.6). We say that (P, σ) is a relaxed Lagrangian solution for the SG system (1.1) with initial data P_0 if (2.7)–(2.8) are satisfied.

Note that this definition takes into account all features of Definition 2.1. Indeed, the measure-preserving property $F(t, \cdot)_{\#} \chi = \chi$ for all $t \in [0, T]$ is accounted for by (2.7), and (2.3) follows from (2.4) which is accounted for by (2.8). Also, from Definition 2.4 we will show that $P \in C([0, T]; H^1(\Omega))$ with $P(\cdot, 0) = P_0(\cdot)$, and derive some form of time-continuity $t \rightarrow \sigma_t$ with the initial data $\sigma_0 = \delta_{\{x=y\}} := (\text{Id} \times \text{Id})_{\#} \chi$ (understood in an appropriately weak sense), which corresponds to the condition $F(0, \cdot) \equiv \text{Id}_\Omega$ in Definition 2.1. But the time-continuity of σ_t and the sense in which the initial condition $\sigma_0 = \delta_{\{x=y\}}$ is taken are necessarily sufficiently weak, as follows from the next remark.

Remark 2.5. The $C([0, T]; L^2(\Omega; \mathbb{R}^3))$ continuity of F , in light of the $L^p(\Omega; \mathbb{R}^3)$ -norm of $F(t, \cdot)$ being time-independent (for all $1 \leq p < \infty$), is equivalent to $F \in C([0, T]; L_{weak}^2(\Omega; \mathbb{R}^3))$, i.e. for all $t_0 \in [0, T)$ we have (the limit below is to be understood as $t \rightarrow t_0^+$ if $t_0 = 0$)

$$\lim_{t \rightarrow t_0} \int_{\Omega} \zeta(x) \cdot F(t, x) dx = \int_{\Omega} \zeta(x) \cdot F(t_0, x) dx \text{ for all } \zeta \in C_c(\Omega; \mathbb{R}^3).$$

In terms of σ , this reads

$$\lim_{t \rightarrow t_0} \int_{\Omega^2} \zeta(x) \cdot y \sigma_t(dx, dy) = \int_{\Omega^2} \zeta(x) \cdot y \sigma_{t_0}(dx, dy) \text{ for all } \zeta \in C_c(\Omega; \mathbb{R}^3). \quad (2.10)$$

Since σ_t is only unambiguously defined (from σ) for a.e. $t \in [0, T]$, the above equality should be understood in the sense that there exists a Borel disintegration $d\sigma = d\sigma_t dt$ for which (2.10) holds. Consequently, the condition

$F(0, \cdot) \equiv \text{Id}_\Omega$ translates to

$$\int_{\Omega^2} \zeta(x) \cdot y \sigma_0(dx, dy) = \int_{\Omega} \zeta(x) \cdot x dx \text{ for all } \zeta \in C_c(\Omega; \mathbb{R}^3), \quad (2.11)$$

which implies that $\sigma_0 = \delta_{\{x=y\}}$. If we denote by $\bar{\sigma}_t$ the barycentric projection of σ_t as defined in (1.6), then (2.10) and (2.11) combine into

$$[0, T] \ni t \mapsto \bar{\sigma}_t \in L^2_{\text{weak}}(\Omega; \mathbb{R}^3) \text{ is continuous and } \bar{\sigma}_0 \equiv \text{Id}_\Omega. \quad (2.12)$$

However, as the example in subsection 2.2 clearly shows, this property cannot be expected from our relaxed solutions in general. In fact, in [9] the property $F \in C([0, T]; L^2(\Omega; \mathbb{R}^3))$ and $F(0, \cdot) \equiv \text{Id}_\Omega$ was obtained from the particular way of constructing those weak Lagrangian solutions; thus, it should be regarded as a regularity property specific to the case $\alpha_t \ll \mathcal{L}^3$. For the general case, we will prove a weaker time-continuity property for σ_t , while giving a weaker meaning to the initial condition σ_0 (see subsection 4.3).

Definition 2.4 will produce a solution (P, F) exactly as in Definition 2.1 provided that we had (2.10) and (2.11), $P \in C([0, T], H^1(\Omega))$, and we knew that for a.e. $t \in (0, T)$ we have that σ_t is supported on the graph of some $F(t, \cdot) : \Omega \rightarrow \Omega$ such that $Z := \nabla P \circ F \in H^1(0, T; L^2(\Omega; \mathbb{R}^3))$ (see Theorem 2.6 below).

Our approach to proving existence of such solutions is the following: first we approximate $\alpha_0 := \nabla P_{0\#}\chi$ by measures $\nabla P_{0\#}^n\chi =: \alpha_0^n \ll \mathcal{L}^3$, for which solutions in the sense of the above definition can easily be constructed as shown above from the stronger solutions of Cullen & Feldman [9] in the sense of Definition 2.1. Then we wish to pass to the limit as $n \rightarrow \infty$ by using Theorem 3.2 and Proposition 3.4 in order to obtain a solution with initial data α_0 . All this will be achieved in Section 4.

2.1. Consistency. As announced, we shall argue that relaxed Lagrangian solutions with some extra regularity determine classical solution of SG system (1.1). We start by showing that relaxed Lagrangian solutions with σ_t supported on graphs and satisfying (2.10) and (2.11) give rise to weak Lagrangian solutions as in [9], except the almost everywhere invertibility of the measure preserving maps may fail.

Theorem 2.6. *Let (P, σ) be a relaxed Lagrangian solution for SG. Suppose (2.10) and (2.11) are satisfied, $P \in C([0, T], H^1(\Omega))$, and $\sigma_t := (\text{Id} \times F_t)_\# \chi$ for all $t \in [0, T]$ and some Borel map $F : [0, T] \times \Omega \rightarrow \Omega$. Then (P, F) is a weak Lagrangian solution as in Definition 2.1 except possibly (iii).*

Proof. Note that the volume preserving property of F_t is a direct consequence of (2.7). Next, we have that (2.10) implies the weak convergence of F_t to F_{t_0} as $t \rightarrow t_0$. Due to volume preservation, the maps F_t share the same L^p -norm for all $1 \leq p < \infty$, so we conclude that F_t converges to F_{t_0} strongly in $L^p(\Omega; \mathbb{R}^3)$ for all $1 < p < \infty$. Then (2.11) yields $F_0 \equiv \text{Id}$ a.e. in Ω .

Finally, let $\varphi \in C_c^1([0, T] \times \Omega; \mathbb{R}^3)$. Denote $Z(t, x) := \nabla P(t, F(t, x))$ and let $\xi(X) = X_i$, $\zeta(t, x) := \varphi_i(t, x)$ for $i = 1, 2, 3$ in (2.8). Then add the resulting equations to obtain (2.3). $\square$

As a result, we have (as in [9, Lemma 2.8], see it for a detailed proof) the following:

Corollary 2.7. *Under the same hypotheses from Theorem 2.6 and the extra assumptions that $\partial_t F \in L^\infty((0, T) \times \Omega)$ and F^* as in Definition 2.1 exists, let us define $\mathbf{u}(t, \cdot) := \partial_t F(t, F^*(t, \cdot))$. Then $(P, \mathbf{u})$ is a weak (Eulerian) solution for SG in physical space, i.e. for (1.1), in the following sense:*

$$\begin{aligned} & \int_0^T \int_\Omega \{ \nabla P(t, x) \cdot [\partial_t \phi(t, x) + \nabla \phi(t, x) \mathbf{u}(t, x)] \\ & \quad + J[\nabla P(t, x) - x] \cdot \phi(t, x) \} dx dt + \int_\Omega \nabla P_0(x) \cdot \phi(0, x) dx = 0; \\ & \int_0^T \int_\Omega \mathbf{u}(t, x) \cdot \nabla \psi(t, x) dx dt = 0 \end{aligned}$$

for any $\phi \in C_c^1([0, T] \times \Omega; \mathbb{R}^3)$, $\psi \in C_c^1([0, T] \times \overline{\Omega})$.

Furthermore, if $(F, F^*, P) \in C^2([0, T] \times \overline{\Omega})$, then the function $\mathbf{u}(t, \cdot) := \partial_t F(t, F^*(t, \cdot))$ satisfies $\mathbf{u} \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^3)$, and $(P, \mathbf{u})$ is a classical solution of (1.1) in $[0, T] \times \Omega$.

2.2. Weak time-continuity, and under-determinacy on level sets of the potential: an example. Next example shows in particular that (2.10) and (2.11) cannot hold for general relaxed Lagrangian solutions.

We discuss the case $\alpha_0 = \pi \delta_{z_0}$ for some $z_0 \in \mathbb{R}^2$ by considering a sequence of approximations

$$\alpha_0^\varepsilon = \frac{1}{\varepsilon^2} \chi_{B_{z_0} \varepsilon}, \quad \text{where } \varepsilon > 0,$$

where χ_A denotes the indicator function of the set A . This is a version of the counterexample in [14], and it can easily be modified to accommodate the third dimension (see [15]). For notational simplicity, we drop the requirements that α_t be probability measures and that $\mathcal{L}^2(\Omega) = 1$; thus, α_t are measures of total mass equal to $\mathcal{L}^2(\Omega)$. We fix the domain in physical space to be the ball $\Omega = B_1(0) \subset \mathbb{R}^2$, which means $\chi = \mathcal{L}^2|_{B_1(0)}$. It is easy to see that $\alpha_0 = \nabla P_{0\#} \chi$ and $\alpha_0^\varepsilon = \nabla P_{0\#}^\varepsilon \chi$, where

$$\begin{aligned} P_0(x) &= z_0 \cdot x, \\ P_0^\varepsilon(x) &= z_0 \cdot x + \varepsilon \frac{|x|^2}{2}. \end{aligned} \tag{2.13}$$

Following the calculations in [14], we find that for each $\varepsilon > 0$, a Lagrangian solution $(P^\varepsilon, F^\varepsilon)$ in the sense of Definition 2.1 with initial data P_0^ε , can be obtained as follows. Let $z(t)$ be determined by solving $\dot{z}(t) = Jz(t)$, $z(0) = z_0$;

thus, $z(t) = M_1(t)z_0$, where

$$M_c(t) = \begin{pmatrix} \cos ct & -\sin ct \\ \sin ct & \cos ct \end{pmatrix} \quad \text{for } c \in \mathbb{R}. \quad (2.14)$$

Then

$$\begin{aligned} P_t^\varepsilon(x) &= z(t) \cdot x + \varepsilon \frac{|x|^2}{2}, \\ F_t^\varepsilon(x) &= M_c(t)x, \text{ with } c = 1 - \frac{1}{\varepsilon}. \end{aligned} \quad (2.15)$$

We have used that the Legendre transform (over Ω) of P_t^ε is

$$(P_t^\varepsilon)^*(X) = \begin{cases} \frac{|X - z(t)|^2}{2\varepsilon}, & \text{if } X \in B_{z(t)}(\varepsilon); \\ |X - z(t)| - \frac{\varepsilon}{2}, & \text{if } X \notin B_{z(t)}(\varepsilon), \end{cases}$$

and the flow map in the dual variables is

$$\Phi_t^\varepsilon(X) = z(t) + M_c(t)(X - z_0), \text{ with } c = 1 - \frac{1}{\varepsilon}.$$

Expression (2.15) was obtained using $(P_t^\varepsilon)^*$, Φ_t^ε given above. Clearly,

$$\nabla P_t^\varepsilon \rightarrow \nabla P_t \quad \text{uniformly in } \Omega \quad \text{for all } t \geq 0, \quad (2.16)$$

where

$$P_t(x) = z(t) \cdot x \quad \text{for } x \in \Omega, t \geq 0, \quad \alpha_t := \nabla P_t \# \chi = \pi \delta_{z(t)} \quad (2.17)$$

and the Kantorovich optimal plan $\gamma_t := (\nabla P_t \times \text{Id})_{\#} \chi$ is defined by $\gamma_t(A \times B) = \delta_{z(t)}(A) \chi(B)$ for all Borel $A \subset \mathbb{R}^2$, $B \subset \Omega$. Then (1.6) implies that the barycentric projection of γ_t onto α_t is

$$\bar{\gamma}_t(z(t)) = 0, \quad (2.18)$$

which defines $\bar{\gamma}_t(X)$ for α_t -a.e. $X \in \mathbb{R}^3$. Note that by taking $t = 0$ in (2.17) we obtain the function P_0 in (2.13). On the other hand, as far as F_t^ε is concerned, we see that $F_0^\varepsilon(x) = x$ for all $x \in \Omega$, $\varepsilon > 0$.

Let us now proceed with defining σ^ε by (2.5), where F is taken to be F^ε from (2.15). In fact, in order to stick with the notation from previous sections, let us consider the sequence $\varepsilon_n := 1/(n+1)$ for all integers $n \geq 1$, then set $\sigma^n := \sigma^{\varepsilon_n}$.

Lemma 2.8. *The sequence $\{\sigma^n\}_n$ defined above converges in the weak $\star$ topology of measures on $\mathcal{Q} := (0, 2\pi) \times \Omega \times \Omega$ to σ given by*

$$\begin{aligned} & \int_{\mathcal{Q}} \xi(t, x, y) \sigma(dt, dx, dy) \\ &= \int_0^{2\pi} \int_0^1 \int_0^{2\pi} \int_0^{2\pi} r \xi(t, r \cos \theta, r \sin \theta, r \cos \vartheta, r \sin \vartheta) d\vartheta d\theta dr dt, \end{aligned} \quad (2.19)$$

for any $\xi \in C(\bar{\mathcal{Q}})$.

Proof. By a standard density argument, it is enough to work with $\xi(t, x, y) = f(t)\zeta(x, y)$. We use (2.14) for $c := 1 - (1/\varepsilon_n) = -n$ and the definition of σ^n to write (by switching to polar coordinates and changing the integration order for the second equality)

$$\begin{aligned} \int_{\mathcal{Q}} \xi d\sigma^n &= \int_0^{2\pi} f(t) \int_{\Omega} \zeta(x, M_n(t)x) dx dt \\ &= \int_0^1 r \int_0^{2\pi} \int_0^{2\pi} f(t) \zeta(r \cos \theta, r \sin \theta, r \cos(\theta - nt), r \sin(\theta - nt)) dt d\theta dr, \end{aligned} \quad (2.20)$$

where M_n denotes the matrix defined at (2.14) with $c = -n$. To see that the innermost integral has the anticipated limit, we let $s = nt$ to rewrite it (after repeated changes of variables and the use of the periodicity of the integrand) as

$$\begin{aligned} I_n &:= \frac{1}{n} \int_0^{2n\pi} f\left(\frac{s}{n}\right) \zeta(r \cos \theta, r \sin \theta, r \cos(\theta - s), r \sin(\theta - s)) ds \\ &= \int_0^{2\pi} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{s + 2k\pi}{n}\right) \zeta(r \cos \theta, r \sin \theta, r \cos(\theta - s), r \sin(\theta - s)) ds. \end{aligned}$$

Now, since $f \in C[0, 2\pi]$, we use the convergence of its Riemann sums to get

$$\lim_{n \rightarrow \infty} I_n = \int_0^{2\pi} f(t) dt \int_0^{2\pi} \zeta(r \cos \theta, r \sin \theta, r \cos(\theta - s), r \sin(\theta - s)) ds$$

which, yet by another change of variables $\vartheta = \theta - s$ and one more use of periodicity, yields, in view of (2.20), equation (2.19) for ξ as defined at the start of this proof. $\square$

Remark 2.9. *Note carefully that we have not proved that the measures σ_t^n arising for each $t \in [0, 2\pi]$ by the disintegration (2.6) converge for each such t to the time-independent measure $\tilde{\sigma}$ on $\Omega \times \Omega$ given, for all $\zeta \in C(\bar{\Omega} \times \bar{\Omega})$, by*

$$\begin{aligned} \int_{\Omega^2} \zeta d\tilde{\sigma} &= \int_0^1 \int_0^{2\pi} \int_0^{2\pi} r \zeta(r \cos \theta, r \sin \theta, r \cos \vartheta, r \sin \vartheta) d\vartheta d\theta dr \\ &= \int_{\mathcal{S}} \frac{\zeta(\mathbf{x})}{|\mathbf{x}|} dS(\mathbf{x}), \end{aligned} \quad (2.21)$$

where $\mathcal{S}$ is the hypersurface in $\Omega \times \Omega$ of equation $x^2 = y^2$ for $\mathbf{x} := (x, y) \in \Omega \times \Omega$. In fact, we will see below that these measures σ_t^n do not converge in any reasonable sense, unless $t = 0$ (the latter due to $F^n(0, \cdot) := F^{\varepsilon_n}(0, \cdot) \equiv \text{Id}_{\Omega}$). The above proof of convergence holds for the full measures σ^n converging to σ . However, as it is now obvious, the time-disintegration of the limiting measure σ defined in (2.19) does produce the time-independent measure $\tilde{\sigma}$ defined in (2.21).

Remark 2.10. Fix some $T \in (0, 2\pi]$. It is straightforward to check that $\tilde{\sigma}$ satisfies (2.6)–(2.8) for $P^n := P^{\varepsilon_n}$. By the constancy of the disintegration (i.e. $\sigma_t = \tilde{\sigma}$ for a.e. t), we see that the only choice of $t \mapsto \sigma_t$ which is $L^2_{\text{weak}}(\Omega; \mathbb{R}^3)$ –continuous in $(0, T)$ is $\sigma_t = \tilde{\sigma}$ for all $t \in (0, T)$. For this, we have that $\tilde{\sigma}_t \equiv 0$ in Ω for all $t \in (0, T)$, which means (2.12) is violated. This, as anticipated in Remark 2.5, is strong evidence that (2.12) is not to be expected in general, which is why it cannot be part of the requirements on σ in Definition 2.4.

These observations are in connection with the fact that the maps $F^n(t, \cdot)$ do not converge weakly to $F^n(t_0, \cdot)$ uniformly with respect to n for any $t_0 \in (0, 2\pi]$. This is in spite of the measures $\alpha_t^{\varepsilon_n}$ being compactly supported uniformly in n, t . In fact, any measure σ with $\sigma_t = \tilde{\sigma}$ stationary with barycenter at the origin, i.e.

$$\int_{\Omega^2} \zeta(x)y \tilde{\sigma}(dx, dy) = \mathbf{0} \in \mathbb{R}^3,$$

satisfies (2.7)–(2.10), yet violates (2.11). For example, one such measure is $\mathcal{L}^1|_{(0, 2\pi)} \otimes \mathcal{L}^3|_{\Omega} \otimes \mathcal{L}^3|_{\Omega}$, i.e. none other than the 7-dimensional Lebesgue measure restricted to $\mathcal{Q}$.

3. Solutions in dual space. Let us recall [5] the definition of the set of absolutely continuous paths/curves $AC^p(0, T; \mathcal{P}_2(\mathbb{R}^3))$ (for $1 \leq p \leq \infty$) as the set of all paths $\mu : [0, T) \ni t \rightarrow \mu_t \in \mathcal{P}_2(\mathbb{R}^3)$ for which there exists $\beta \in L^p(0, T)$ such that

$$W_2(\mu_s, \mu_t) \leq \int_s^t \beta(\tau) d\tau \text{ for all } 0 \leq s \leq t \leq T,$$

where W_2 is the quadratic Wasserstein distance [21].

We first prove the following:

Lemma 3.1. Suppose $\alpha \in \mathcal{P}_2(\mathbb{R}^3)$ and γ is the optimal plan between α and χ for the quadratic cost. Then

$$W_2(\alpha, \chi) \geq \|\bar{\gamma} - \text{Id}\|_{L^2(\alpha; \mathbb{R}^3)}, \quad (3.1)$$

where $\bar{\gamma}$ is the barycentric projection of γ onto α (see (1.6)).

Proof. From (1.6) applied to α and γ , and by using the properties

$$\text{proj}_X \gamma = \alpha, \quad \text{proj}_Y \gamma = \chi$$

(which hold since $\gamma(X, y)$ is a transportation plan between α and χ), we get

$$\int_{\mathbb{R}^3} \xi(X) \cdot \bar{\gamma}(X) \alpha(dX) \leq \left(\int_{\mathbb{R}^3} |\xi(X)|^2 \alpha(dX) \right)^{1/2} \left(\int_{\Omega} y^2 dy \right)^{1/2},$$

for all $\xi \in C_c(\mathbb{R}^3; \mathbb{R}^3)$, which implies that $\|\text{Id}\|_{L^2(\Omega; \mathbb{R}^3)} \geq \|\bar{\gamma}\|_{L^2(\alpha; \mathbb{R}^3)}$. This leads to

$$\begin{aligned} W_2^2(\alpha, \chi) &= \int_{\mathbb{R}^3} |x|^2 \alpha(dx) + \int_{\Omega} |y|^2 dy - 2\langle \text{Id}, \bar{\gamma} \rangle_{L^2(\alpha; \mathbb{R}^3)} \\ &\geq \|\bar{\gamma} - \text{Id}\|_{L^2(\alpha; \mathbb{R}^3)}^2, \end{aligned}$$

i.e. the desired thesis. $\square$

Now we prove the stability of dual-space solutions that are Lipschitz in time (i.e., lying in $AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$). Since the convex functions P_t in (1.3) and the optimal transport plans $\gamma_t = (\nabla P_t \times \text{Id})_\# \chi$ between α_t and χ are unambiguously defined by α_t (by solving the Monge-Kantorovich problem), we will discuss the dual problem (1.2) – (1.5) in terms of α_t only.

Theorem 3.2. *Let $\{\alpha_0\} \cup \{\alpha_0^n\}_{n \geq 1} \subset \mathcal{P}_2(\mathbb{R}^3)$ be such that*

$$\alpha_0^n \rightharpoonup \alpha_0 \text{ weakly } \star \text{ as measures, and } \sup_{n \geq 1} W_2(\alpha_0^n, \alpha_0) < \infty. \quad (3.2)$$

For all integers $n \geq 1$, let $\alpha^n \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ be solutions of (1.2)–(1.5), corresponding to the initial data α_0^n , respectively. Then a subsequence of α^n converges to a solution $\alpha \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ with initial data α_0 . More precisely, we have

(i) $\alpha_t^n \rightarrow \alpha_t$ in $\mathcal{P}_p(\mathbb{R}^3)$ for all $t \in [0, T)$ and all $p \in [1, 2)$.

(ii) $\alpha \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ is a solution of (1.2)–(1.5) with initial data α_0 .

Proof. First, note that we can use (1.2) satisfied by α^n , Theorem 8.3.1 [5] and (3.1), to write

$$W_2(\alpha_s^n, \alpha_t^n) \leq \int_s^t \|J[\text{Id} - \bar{\gamma}_\tau^n]\|_{L^2(\alpha_\tau^n; \mathbb{R}^3)} d\tau \leq \int_s^t W_2(\alpha_\tau^n, \chi) d\tau \quad (3.3)$$

for all $0 \leq s \leq t \leq T$. Thus

$$W_2(\alpha_t^n, \chi) \leq W_2(\alpha_s^n, \chi) + \int_s^t W_2(\alpha_\tau^n, \chi) d\tau$$

for all $0 \leq s \leq t \leq T$. By Gronwall's Lemma we get

$$W_2(\alpha_t^n, \chi) \leq e^t W_2(\alpha_0^n, \chi) \text{ for all } n \geq 1 \text{ and } t \in [0, T].$$

But (3.2) ensures that $W_2(\alpha_0^n, \mu)$ is bounded for any $\mu \in \mathcal{P}_2(\mathbb{R}^3)$, uniformly in n . Thus,

$$W_2(\alpha_t^n, \chi) \text{ is bounded uniformly in } n \geq 1 \text{ and } t \in [0, T]. \quad (3.4)$$

Since

$$\int_{\mathbb{R}^3} |X|^2 \alpha_t^n(dX) = W_2^2(\alpha_t^n, \delta_0) \leq 2W_2^2(\alpha_t^n, \chi) + 2 \int_{\Omega} |x|^2 dx,$$

we infer that

$$\alpha_t^n \text{ has bounded second moment, uniformly in } t \in [0, T], \ n \geq 1. \quad (3.5)$$

(Note that (3.4) and (3.5) are equivalent; however, we record them both since further reference may prefer one form over the other.) Fix $p \in [1, 2)$. By a standard diagonalization argument, a subsequence (not relabeled) can be extracted

such that $\alpha_t^n \rightarrow \alpha_t$ in $\mathcal{P}_p(\mathbb{R}^3)$ for every $t \in [0, T] \cap \mathbb{Q}$ and for some Borel probabilities α_t . Again by diagonalization (and, possibly, a further subsequence extraction), one can ensure that this convergence holds for all $p \in [1, 2)$, thus

$$\alpha_t^n \rightarrow \alpha_t \text{ in } \mathcal{P}_p(\mathbb{R}^3) \text{ for every } t \in [0, T] \cap \mathbb{Q}, p \in [1, 2). \quad (3.6)$$

Another look at (3.3) and (3.4) reveals

$$W_2(\alpha_s^n, \alpha_t^n) \leq C|s - t|, \text{ with } C \in \mathbb{R} \text{ independent of } n \geq 1, s, t \in [0, T]. \quad (3.7)$$

This leads to

$$W_p(\alpha_s, \alpha_t) \leq W_p(\alpha_s^n, \alpha_s) + W_p(\alpha_t^n, \alpha_t) + C|s - t|$$

for all $s, t \in [0, T] \cap \mathbb{Q}$, $n \geq 1$, $p \in [1, 2)$. Let $n \rightarrow \infty$ and use the completeness of $\mathcal{P}_p(\mathbb{R}^3)$ to infer that there exists α_t for *all* times $t \in [0, T]$ such that

$$W_p(\alpha_s, \alpha_t) \leq C|s - t| \text{ for all } p \in [1, 2) \text{ and all } s, t \in [0, T]. \quad (3.8)$$

Then, the inequality

$$W_p(\alpha_t^n, \alpha_t) \leq W_p(\alpha_s^n, \alpha_s) + W_p(\alpha_s^n, \alpha_t^n) + W_p(\alpha_s, \alpha_t)$$

yields, in light of (3.6), (3.7) and (3.8), that α_t^n converges to α_t in $\mathcal{P}_p(\mathbb{R}^3)$ for *all* $t \in [0, T]$ and all $1 \leq p < 2$. Obviously, α_t has bounded second moment uniformly in $t \in [0, T]$; indeed, this follows from (3.5) and the lower-semicontinuity property

$$\int_{\mathbb{R}^3} |X|^2 \alpha_t(dX) \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |X|^2 \alpha_t^n(dX), \quad (3.9)$$

which holds due to the narrow convergence of α_t^n to α_t (see [5], subsection 5.1.1). Thus, we can let $p \rightarrow 2^-$ in (3.8) (or, alternatively, use the lower-semicontinuity of W_2 with respect to narrow convergence) to get

$$W_2(\alpha_s, \alpha_t) \leq C|s - t| \text{ for all } s, t \in [0, T]. \quad (3.10)$$

So, $\alpha \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$. From (1.6) we get

$$\int_{\mathbb{R}^3} (\bar{\gamma}_t^n(X) - X) \cdot \xi(X) \alpha_t^n(dX) = \int \int_{\mathbb{R}^3 \times \Omega} (y - X) \cdot \xi(X) \gamma_t^n(dX, dy)$$

for all $\xi \in C_c(\mathbb{R}^3; \mathbb{R}^3)$, where $\gamma_t^n = (\nabla P_t^n \times \text{Id})_\# \chi$ is the optimal plan between α_t^n and χ . Due to (3.4) and the narrow convergence of α_t^n to α_t , we can use Proposition 7.1.3 [5] (stability of optimal plans with respect to narrow convergence) and the uniqueness of the optimal plan $\gamma_t = (\nabla P_t \times \text{Id})_\# \chi$ between α_t and χ to deduce that the left hand side of the last displayed equation converges

to $\langle \bar{\gamma}_t - \text{Id}, \xi \rangle_{L^2(\alpha_t; \mathbb{R}^3)}$. The equation (1.2) (with initial α_0^n) satisfied by α^n in the sense of distributions reads

$$\int_0^T \int_{\mathbb{R}^3} \{ \partial_t \zeta(t, X) + J[X - \bar{\gamma}_t^n(X)] \cdot \nabla \zeta(t, X) \} \alpha_t^n(dX) dt + \int_{\mathbb{R}^3} \zeta(0, X) \alpha_0^n(dX) = 0$$

for all $\zeta \in C_c^\infty([0, T] \times \mathbb{R}^3)$. According to the above considerations, we can pass to the limit (use Dominated Convergence for the time integral in view of (3.1) and (3.4)) to obtain that α is a distributional solution for (1.2)–(1.5) with initial measure α_0 . $\square$

Remark 3.3. *Conservation of Hamiltonian energy $t \mapsto -W_2^2(\alpha_t, \chi)/2$ follows by Theorem 5.2 [4] in light of both $\alpha \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ and the convexity properties of this particular Hamiltonian (Proposition 4.3 [4]). Since the same is true for $t \mapsto W_2(\alpha_t^n, \chi)$, all the Lipschitz constants and second moments bounds in the proof of Theorem 3.2 can be taken independent of T ; thus, the above theorem holds for $T = \infty$ as well.*

Proposition 3.4. *Let $\Omega \in \mathbb{R}^d$ be open and bounded and consider the Borel measures $\{\mu_n\}_{n \geq 1}$, μ in $\mathbb{R}^d$ with total mass equal to $\mathcal{L}^d(\Omega)$ and with finite second moments. Denote by M_n , M the optimal maps for pushing $\mathcal{L} := \mathcal{L}^d|_\Omega$ forward to μ_n , μ for the quadratic cost (i.e., gradients of convex functions such that $M_n \# \mathcal{L} = \mu_n$, $M \# \mathcal{L} = \mu$). Then, for any $1 < p \leq 2$, we have*

$$\lim_{n \rightarrow \infty} W_p(\mu_n, \mu) = 0 \text{ is equivalent to } \lim_{n \rightarrow \infty} \|M_n - M\|_{L^p(\Omega; \mathbb{R}^d)} = 0. \quad (3.11)$$

Proof. Since the measure $(M \times M_n) \# \mathcal{L}$ has marginals μ and μ_n , the reverse implication is obvious.

Consider now convex functions Φ_n , Φ defined on $\mathbb{R}^d$ and restricted to Ω such that $M_n = \nabla \Phi_n$ and $M = \nabla \Phi$. It is well known [21, Theorem 7.12] that $W_p(\mu_n, \mu) \rightarrow 0$ is equivalent to

$$\mu_n \rightharpoonup \mu \text{ weakly } \star \text{ as measures, and } \int_{\mathbb{R}^d} |x|^p \mu_n(dx) \rightarrow \int_{\mathbb{R}^d} |x|^p \mu(dx).$$

That is,

$$\int_\Omega \varphi \circ M_n(x) dx \rightarrow \int_\Omega \varphi \circ M(x) dx \text{ for all } \varphi \in C_b(\mathbb{R}^d), \quad (3.12)$$

and

$$\lim_{n \rightarrow \infty} \|M_n\|_{L^p(\Omega; \mathbb{R}^d)} = \|M\|_{L^p(\Omega; \mathbb{R}^d)}. \quad (3.13)$$

Define, for sufficiently small $r > 0$, the set $\Omega_r := \{x \in \Omega : \text{dist}(x, \partial\Omega) > r\}$. The restrictions of Φ_n to $\bar{\Omega}_r$ are continuous (since Φ_n , Φ are convex and Ω is in the domain of their a.e. gradients), therefore, bounded on $\bar{\Omega}_r$ and we can assume $\Phi_n(x_n) = 0 = \min \Phi_n$ over $\bar{\Omega}_r$. Thus,

$$0 \leq \Phi_n(x) \leq M_n(x) \cdot (x - x_n) \leq \text{diam}(\Omega) |M_n(x)| \text{ for a.e. } x \in \Omega_r,$$

which by (3.13) leads to the fact that $\{\Phi_n\}_n$ is bounded in $W^{1,p}(\Omega_r)$. Thus, there exists a (convex) $\Phi \in W^{1,p}(\Omega_r)$ such that, up to a subsequence,

$$\Phi_{n_k} \rightarrow \Phi \text{ weakly in } W^{1,p}(\Omega_r) \text{ and strongly in } L^p(\Omega_r) \text{ as } k \rightarrow \infty. \quad (3.14)$$

We can also assume $\Phi_{n_k} \rightarrow \Phi$ a.e. in Ω_r . Now let $A \subset \Omega_r$ be the set of full $\mathcal{L}$ -measure where all gradients M_n , $\nabla\Phi$ exist, and also where Φ_{n_k} converges pointwise to Φ . Note that since Φ is convex and it belongs to $L^p(\Omega_r)$, it is also continuous and bounded on $\bar{\Omega}_{2r}$ (to be used for (3.16) below). Fix $x \in A \cap \Omega_{3r}$ and for each $i = 1, \dots, d$ choose some vectors $\mathbf{e}_i^+$, $\mathbf{e}_i^-$ such that $|\mathbf{e}_i - \mathbf{e}_i^\pm| \leq \min\{r, 1/(2d)\}$ and such that $x \pm r\mathbf{e}_i^\pm \in A$, where $\mathbf{e}_i$ is the i^{th} standard basis vector for $\mathbb{R}^d$. We have

$$\frac{1}{r} [\Phi_{n_k}(x \pm r\mathbf{e}_i^\pm) - \Phi_{n_k}(x)] \geq \pm M_{n_k}(x) \cdot \mathbf{e}_i^\pm \geq \pm (M_{n_k}(x))_i - \frac{1}{2d} |M_{n_k}(x)| \quad (3.15)$$

for all $i = 1, \dots, d$, which implies

$$\limsup_{k \rightarrow \infty} \left[|(M_{n_k}(x))_i| - \frac{1}{2d} |M_{n_k}(x)| \right] \leq \frac{2}{r} \max_{\Omega_{2r}} |\Phi| \text{ for all } i = 1, \dots, d. \quad (3.16)$$

Summation by $i = 1, \dots, d$ yields that $\{M_{n_k}(x)\}_k$ is bounded for each $x \in A \cap \Omega_{3r}$. It is easy to see that any accumulation point must necessarily be $\nabla\Phi(x)$ (by the “continuity” of the sub gradient; see, e.g., [21]), so the entire sequence $M_{n_k}x$ must converge to $\nabla\Phi(x)$. Thus, M_{n_k} converges pointwise to $\nabla\Phi$ a.e. in Ω_{3r} . Note that if we take $0 < \tilde{r} < r$, we can extract a further subsequence (not relabeled) such M_{n_k} converges pointwise to $\nabla\tilde{\Phi}$ a.e. in $\Omega_{3\tilde{r}}$ for some $\tilde{\Phi} \in W^{1,p}(\Omega_{\tilde{r}})$, which necessarily satisfies $\nabla\tilde{\Phi} \equiv \nabla\Phi$ a.e. in Ω_{3r} . Ultimately, let $\{r_j\}_{j \geq 1}$ be decreasing and converging to zero from the right. Repeating the argument above with $r = r_j$ and $\tilde{r} = r_{j+1}$ and using induction by j , we get a subsequence (not relabeled) M_{n_k} and a convex function $\Phi \in W_{loc}^{1,p}(\Omega)$ such that $M_{n_k} \rightarrow \nabla\Phi$ a.e. in Ω . Furthermore, by Dominated Convergence, we conclude

$$\int_{\Omega} \varphi \circ M_{n_k}(x) dx \rightarrow \int_{\Omega} \varphi \circ \nabla\Phi(x) dx \text{ for all } \varphi \in C_b(\mathbb{R}^d),$$

which, by (3.12), yields $\nabla\Phi \equiv M$ a.e. in Ω (by uniqueness of the optimal transport from $\mathcal{L}$ to μ with quadratic cost, since they are both gradients of convex functions). Thus, we use (3.14) to deduce that M_n converges (the whole sequence, by the uniqueness of the limit) weakly in $L^p(\Omega_r; \mathbb{R}^d)$ to M for any sufficiently small $r > 0$. So, $M_n \rightharpoonup M$ weakly in $L^p(\Omega; \mathbb{R}^d)$. The convergence (3.13) finishes the proof of the direct implication. $\square$

Remark 3.5. Note that Theorem 3.2 (i) and Proposition 3.4 imply

$$\lim_{n \rightarrow \infty} \|\nabla P_t^n - \nabla P_t\|_{L^p(\Omega; \mathbb{R}^3)} = 0 \quad (3.17)$$

for all $1 \leq p < 2$ and all $t \geq 0$. Furthermore, due to the uniform bounds on the $L^2(\Omega; \mathbb{R}^3)$ -norms of all the gradients involved, (3.17) also implies (by Dominated convergence)

$$\lim_{n \rightarrow \infty} \|\nabla P^n - \nabla P\|_{L^p((0,T) \times \Omega; \mathbb{R}^3)} = 0 \quad (3.18)$$

for all $1 \leq p < 2$ and all $0 \leq T < \infty$.

Our plan is to use this stability to obtain weak solutions in the physical space $[0, T) \times \Omega$.

4. Main results: existence and stability of relaxed Lagrangian solutions in physical space. Here we state and prove the main results of the paper. In the first subsection we make the connection with dual space solutions. Then we move on to the stability and existence results on the relaxed Lagrangian solutions in physical space.

4.1. Return to dual space. We will begin with a very helpful proposition, which shows that relaxed Lagrangian solutions in physical space produce dual space solutions as in the statement of Theorem 3.2; this is due to the renormalization property. As discussed in Remark 3.3, Theorem 4.1 (ii) below guarantees conservation of the Hamiltonian energy; however, this fact is unnecessary to our subsequent argument.

Theorem 4.1. *Let (P, σ) be a relaxed Lagrangian solution as in Definition 2.4 and let $\alpha_t := \nabla P_{t\#} \chi$. Then:*

(i) α is a distributional solution of the initial-value problem (1.2)–(1.5) with $\alpha_0 = \nabla P_{0\#} \chi$ in dual space, i.e.

$$\int_0^T \int_{\mathbb{R}^3} \{ \partial_t \zeta(t, X) + J[X - \bar{\gamma}_t(X)] \cdot \nabla \zeta(t, X) \} \alpha_t(dX) dt + \int_{\mathbb{R}^3} \zeta(0, X) \alpha_0(dX) = 0$$

for all $\zeta \in C_c^\infty([0, T) \times \mathbb{R}^3)$;

(ii) $P \in C([0, T]; H^1(\Omega))$;

(iii) $\alpha \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$.

Proof. Pick arbitrary $\varphi \in C_c^1[0, T)$ and $\xi \in C_c^1(\mathbb{R}^3)$. For the test function $\zeta(t, X) = \varphi(t)\xi(X)$ in dual space, we have, by using (2.7),

$$\begin{aligned} I &:= \int_0^T \int_{\mathbb{R}^3} \dot{\varphi}(t) \xi(X) \alpha_t(dX) dt = \int_0^T \int_{\Omega} \dot{\varphi}(t) \xi(\nabla P_t(y)) dy dt \\ &= \int_0^T \int_{\Omega \times \Omega} \dot{\varphi}(t) \xi(\nabla P_t(y)) \sigma(dt, dx, dy). \end{aligned}$$

Furthermore, by 2.8,

$$\begin{aligned}
I &= - \int_0^T \int_{\Omega \times \Omega} \varphi(t) \nabla \xi(\nabla P_t(y)) \cdot J \left(\nabla P_t(y) - y \right) \sigma(dt, dx, dy) \\
&\quad - \int_{\Omega} \varphi(0) \xi(\nabla P_0(x)) dx \\
&= - \int_0^T \int_{\Omega} \varphi(t) \nabla \xi(\nabla P_t(y)) \cdot J \left(\nabla P_t(y) - y \right) dy dt - \int_{\mathbb{R}^3} \varphi(0) \xi(X) \alpha_0(dX) \\
&= -I_1 - I_2 - \int_{\mathbb{R}^3} \varphi(0) \xi(X) \alpha_0(dX).
\end{aligned}$$

Then,

$$\begin{aligned}
I_1 &= \int_0^T \int_{\Omega} \varphi(t) \nabla \xi(\nabla P_t(y)) \cdot J \nabla P_t(y) dy dt \\
&= \int_0^T \int_{\mathbb{R}^3} \varphi(t) \nabla \xi(X) \cdot J X \alpha_t(dX) dt.
\end{aligned}$$

Let $\gamma_t := (\text{Id} \times \nabla P_t)_{\#} \chi$ to get

$$\begin{aligned}
I_2 &= - \int_0^T \int_{\Omega} \varphi(t) \nabla \xi(\nabla P_t(y)) \cdot J y dy dt \\
&= - \int_0^T \int_{\Omega \times \mathbb{R}^3} \varphi(t) \nabla \xi(X) \cdot J y \gamma_t(dy, dX) dt \\
&= - \int_0^T \int_{\mathbb{R}^3} \varphi(t) \nabla \xi(X) \cdot J \bar{\gamma}_t(X) \alpha_t(dX) dt.
\end{aligned}$$

Combine the above to obtain the equation the equation from (i) for $\zeta(t, x) = \varphi(t) \xi(X)$. Thus, (i) is proved.

Note that, according to (1.4), (3.1) and the triangle inequality, we have

$$\begin{aligned}
\|U_t\|_{L^2(\alpha_t, \mathbb{R}^3)} &\leq W_2(\alpha_t, \chi) \leq W_2(\alpha_0, \chi) + W_2(\alpha_t, \alpha_0) \\
&\leq W_2(\alpha_0, \chi) + \|\nabla P_0 - \nabla P_t\|_{L^2(\Omega; \mathbb{R}^3)},
\end{aligned}$$

where the last inequality is due to the fact that $(\nabla P_0 \times \nabla P_t)_{\#} \chi$ is a transport plan with marginals α_0, α_t . But, according to Definition 2.4, we have $\nabla P \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$, so the displayed inequality above yields $t \mapsto \|U_t\|_{L^2(\alpha_t, \mathbb{R}^3)}$ lies in $L^\infty(0, T)$, which implies $\alpha \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ (since the velocity of minimal norm $\mathbf{v}$ associated to the path $t \mapsto \alpha_t$ will, by definition, satisfy $\|\mathbf{v}_t\|_{L^2(\alpha_t, \mathbb{R}^3)} \leq \|U_t\|_{L^2(\alpha_t, \mathbb{R}^3)}$ for a.e. $t \in [0, T]$ [5]). The continuity of $[0, T] \ni t \mapsto \alpha_t$ in $\mathcal{P}_2(\mathbb{R}^3)$ implies (ii) by means of Proposition 3.4. $\square$

4.2. Weak stability and existence of relaxed Lagrangian solutions. Before our existence result, we need to prove what we call a “weak stability” result, i.e. for appropriate approximations of the initial data and

corresponding relaxed Lagrangian solutions, we obtain convergence (up to a subsequence) to a relaxed Lagrangian solution.

Theorem 4.2. *Let $P_0 \in H^1(\Omega)$, $\{P_0^n\}_n \subset H^1(\Omega)$ be convex and such that $\nabla P_0^n \rightarrow \nabla P_0$ in $L^2(\Omega; \mathbb{R}^3)$. Assume (P^n, σ^n) are relaxed solutions for SG in physical space corresponding to the initial data P_0^n . Then, possibly up to a subsequence, (P^n, σ^n) converges to a relaxed solution (P, σ) corresponding to the initial datum P_0 . The convergence is in the following sense:*

- (i) $\nabla P_t^n \rightarrow \nabla P_t$ in $L^p(\Omega; \mathbb{R}^3)$ for all $t \in [0, T)$ and all $1 \leq p < 2$;
- (ii) σ^n converges weakly as measures to σ .

Furthermore, the corresponding dual space solutions satisfy $W_p(\alpha_t^n, \alpha_t) \rightarrow 0$ for all $t \in [0, T)$ and all $1 \leq p < 2$.

Proof. The measures σ^n have same mass and bounded moments of any order, uniformly in n . By a diagonalization argument, we can extract a subsequence (not relabeled) such that σ^n converges in the Wasserstein distance W_r of any order $1 \leq r < \infty$ to some measure σ on $(0, T) \times \Omega \times \Omega$ with finite moments of any order. Clearly, since σ^n disintegrates like $d\sigma_t^n dt$ and the σ_t^n 's satisfy (2.7) for each integer $n \geq 1$, the limiting σ will too. That is, the measures σ_t given by the t -disintegration (2.6) satisfy (beside the fact that the family $t \mapsto \sigma_t$ is Borel) the property (2.7). Furthermore, by letting $\alpha_0^n := \nabla P_{0\#}^n \chi$, we have, according to Theorem 4.1, that for each $t \in [0, T]$ the measures $\alpha_t^n := \nabla P_{t\#}^n \chi$ are dual space solutions (corresponding to the initial data α_0^n , respectively) in the sense of Theorem 3.2 (i.e., distributional solutions in dual-space which also lie in $AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$). Therefore, according to Theorem 3.2, possibly up to a further subsequence, we obtain a limiting $\alpha \in AC^\infty(0, T; \mathcal{P}_2(\mathbb{R}^3))$ solving SG in dual space. Proposition 3.4 then applies to give that (3.17) and (3.18) from Remark 3.5 are satisfied by the optimal maps P_t pushing χ forward to α_t . Then, for any continuous $\phi : [0, T] \times \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}^3$, we use the marginal property (2.7) to get

$$\left| \int_0^T \int_{\Omega^2} \phi(t, x, y) [\xi(\nabla P^n(t, y)) - \xi(\nabla P(t, y))] \sigma^n(dt, dx, dy) \right| \leq \|\nabla \xi\|_\infty \|\phi\|_\infty \|\nabla P^n - \nabla P\|_{L^1((0, T) \times \Omega; \mathbb{R}^3)}. \quad (4.1)$$

Furthermore, from the uniform bounds on the second moments of α_t (see proof of Theorem 3.2), we get

$$\|\nabla P\|_{L^2((0, T) \times \Omega; \mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} |X|^2 \alpha_t(dX) dt < \infty,$$

which implies (given that ξ is Lipschitz) that $f := \xi \circ \nabla P \in L^2((0, T) \times \Omega)$. Let $\varepsilon > 0$ and approximate f in $L^2((0, T) \times \Omega)$ within $\frac{\varepsilon}{3(T\mathcal{L}^3(\Omega))^{1/2}(\|\phi\|_\infty + 1)}$ distance by $f_\varepsilon \in C_c((0, T) \times \Omega)$. We have

$$\left| \int_0^T \int_{\Omega^2} \phi(t, x, y) [f_\varepsilon(t, y) - f(t, y)] d\mu \right| \leq \frac{\varepsilon}{3}$$

for all Borel measures on $(0, T) \times \Omega^2$ with (t, y) -marginal $\mathcal{L}^1|_{(0, T)} \times \mathcal{L}^3|_{\Omega}$. In particular, this inequality is satisfied by $\mu = \sigma$ and $\mu = \sigma^n$ (for all n). By the narrow convergence of σ^n to σ , we also have

$$\left| \int_0^T \int_{\Omega^2} \phi(t, x, y) f_\varepsilon(t, y) d(\sigma^n - \sigma) \right| \leq \varepsilon/3 \text{ for sufficiently large } n.$$

Consequently, by the triangle inequality, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T \int_{\Omega^2} \phi(t, x, y) \xi(\nabla P(t, y)) \sigma^n(dt, dx, dy) \\ = \int_0^T \int_{\Omega^2} \phi(t, x, y) \xi(\nabla P(t, y)) \sigma(dt, dx, dy), \end{aligned}$$

which, in light of (4.1) and (3.18), yields that

$$[\xi \circ \nabla P^n(t, y)] \sigma^n \text{ converges narrowly to } [\xi \circ \nabla P(t, y)] \sigma. \quad (4.2)$$

Likewise,

$$\begin{aligned} \left| \int_0^T \int_{\Omega^2} Jy \cdot [\nabla \xi(\nabla P^n(t, y)) - \nabla \xi(\nabla P(t, y))] \phi \sigma^n(dt, dx, dy) \right| \\ \leq \|\phi\|_\infty (\sup_{y \in \Omega} |y|) \int_0^T \int_{\Omega} |\nabla \xi(\nabla P^n(t, y)) - \nabla \xi(\nabla P(t, y))| dy dt \end{aligned}$$

and

$$\begin{aligned} \left| \int_0^T \int_{\Omega^2} [J \nabla P^n(t, y) \cdot \nabla \xi(\nabla P^n(t, y)) - J \nabla P(t, y) \cdot \nabla \xi(\nabla P(t, y))] \phi d\sigma^n \right| \\ \leq \|\phi\|_\infty \int_0^T \int_{\Omega} |J \nabla P^n \cdot (\nabla \xi \circ \nabla P^n) - J \nabla P \cdot (\nabla \xi \circ \nabla P)| dy dt. \end{aligned}$$

In both cases we apply (3.18) and Dominated Convergence (recall that $\nabla \xi$ is continuous and bounded) to conclude convergence to zero. Since $(t, y) \mapsto Jy \cdot \nabla \xi(\nabla P(t, y))$ and $(t, y) \mapsto J \nabla P(t, y) \cdot \nabla \xi(\nabla P(t, y))$ are also in $L^2((0, T) \times \Omega)$, we proceed the same way we did for (4.2) to obtain

$$[Jy \cdot \nabla \xi \circ \nabla P^n(t, y)] \sigma^n \text{ converges narrowly to } [Jy \cdot \nabla \xi \circ \nabla P(t, y)] \sigma$$

and

$$[J \nabla P^n(t, y) \cdot \nabla \xi \circ \nabla P^n(t, y)] \sigma^n \rightarrow [J \nabla P(t, y) \cdot \nabla \xi \circ \nabla P(t, y)] \sigma \text{ narrowly.}$$

Finally, we use the convergence of ∇P_0^n to ∇P_0 and the fact that ξ is Lipschitz to infer that $\langle \xi \circ \nabla P_0^n, \zeta(0, \cdot) \rangle_{L^2(\Omega)}$ converges to $\langle \xi \circ \nabla P_0, \zeta(0, \cdot) \rangle_{L^2(\Omega)}$. We now have all the ingredients to pass to the limit in (2.8) as satisfied by (P^n, σ^n) and obtain it for (P, σ) . $\square$

As an immediate consequence, we obtain the following existence result.

Corollary 4.3. *Let $P_0 \in H^1(\Omega)$ be convex. Then there exists a relaxed Lagrangian solution (P, σ) corresponding to the initial datum P_0 .*

Proof. First, we can use Lemma 7.1.10 in [5] to approximate $\nabla P_{0\#}\chi =: \alpha_0 \in \mathcal{P}_2(\mathbb{R}^3)$ with respect to the quadratic Wasserstein distance by smooth, positive probability densities lying in $\mathcal{P}_2(\mathbb{R})$ (just take an everywhere positive convolution kernel in the proof of said lemma). For all integers $k \geq 1$ consider $\xi_k \in C_c^\infty(B(0, k+1))$ such that $0 \leq \xi_k \leq 1$ in $\mathbb{R}^3$ and $\xi_k \equiv 1$ in $B(0, k)$. By Dominated Convergence,

$$\lambda_k := \int_{\mathbb{R}^3} \xi_k(x) \rho(x) dx \rightarrow \int_{\mathbb{R}^3} \rho(x) dx = 1 \text{ as } k \rightarrow \infty.$$

If we place $\rho_k := \xi_k \rho / \lambda_k$, we use $\lambda_k \rightarrow 1$ and Dominated Convergence again to infer that

$$\int_{\mathbb{R}^3} \rho_k(x) \zeta(x) dx \rightarrow \int_{\mathbb{R}^3} \rho(x) \zeta(x) dx \text{ as } k \rightarrow \infty$$

for all $\zeta \in C(\mathbb{R}^3)$ of at most quadratic growth. Thus, ρ_k is a sequence of smooth, compactly supported probability densities converging to ρ in the quadratic Wasserstein distance (once again, we use that fact that narrow convergence and convergence of the p -moments is equivalent to convergence in the p -Wasserstein distance; see, e.g., [21]). In conclusion, we can approximate α_0 in the metric space $(\mathcal{P}_2(\mathbb{R}^3), W_2)$ by smooth, compactly supported densities α_0^n , $n \geq 1$. Denote by ∇P_0^n the optimal maps (with respect to the quadratic cost function) such that $\nabla P_{0\#}\chi = \alpha_0^n$. For each n , let (P^n, F^n) be the weak Lagrangian solutions in physical space constructed in [9, Theorem 2.9] and corresponding to the initial data P_0^n . We now define the measures σ^n on $(0, T) \times \Omega \times \Omega$ by taking $F := F^n$ in (2.5). Since for each $n \geq 1$ the pair (P^n, F^n) satisfies the conditions of Definition 2.1, we have (as seen in the discussion leading to Definition 2.4) that (P^n, σ^n) satisfies Definition 2.4. Since $W_2(\alpha_0^n, \alpha_0) \rightarrow 0$ is equivalent to $\|\nabla P_0^n - \nabla P_0\|_{L^2(\Omega; \mathbb{R}^3)} \rightarrow 0$ (see Theorem 3.4), we may conclude by applying Theorem 4.2. $\square$

4.3. Time continuity of relaxed Lagrangian solutions. As anticipated in Remark 2.5, we shall cover some weaker properties of the relaxed Lagrangian solutions that follow directly from their definition and the time continuity of P .

Remark 4.4. *Define $G_t : \Omega \times \Omega \rightarrow \mathbb{R}^3 \times \mathbb{R}^3$ by $G_t(x, y) = (x, \nabla P_t(y))$, then $t \mapsto G_{t\#}\sigma_t$ is continuous with respect to narrow convergence on $\Omega \times \mathbb{R}^3$: i.e., from the equation, for any $\xi, \zeta \in C_c^1(\mathbb{R}^3)$*

$$\Psi(t) := \int_{\Omega \times \Omega} \zeta(x) \xi(\nabla P_t(y)) \sigma_t(dx, dy) = \int_{\Omega \times \mathbb{R}^3} \zeta(x) \xi(Y) G_{t\#}\sigma_t(dx, dY)$$

is continuous on $[0, T]$.

Finally, we will make use of the time-regularity P enjoys (Theorem 4.1) in order to prove some kind of weak time-regularity for the relaxed Lagrangian solutions. As far (2.11) is concerned, in Remark 2.5 we initiated a discussion why it cannot be expected in general. However, part (ii) of the proposition below makes a weaker claim, which is reminiscent of the initial condition $F(0, \cdot) \equiv \text{Id}$ (yielding $\sigma_0 := (\text{Id} \times \text{Id})_{\#} \chi$) from the regular case $\alpha_t \ll \mathcal{L}^3$ (recall Definition 2.1). More precisely, we have:

Proposition 4.5. (i) *For any $t \in [0, T]$, we have that $s \mapsto G_{t\#} \sigma_s$ is continuous at $s = t$ with respect to narrow convergence on $\Omega \times \mathbb{R}^3$.*

(ii) *We have $G_{0\#} \sigma_0 = G_{0\#} \delta_{\{x=y\}}$, where $\delta_{\{x=y\}} := (\text{Id} \times \text{Id})_{\#} \chi$, and $G_{0\#} \sigma_0$ is understood in the sense of convergence of $G_{t\#} \sigma_t$ as $t \rightarrow 0$ as in Remark 4.4.*

Proof. First, note that all quantities defined below make sense due to the marginal property (2.7) (a proof along the lines of the one for Lemma 2.3 can be readily written). Let $\zeta \in C_c(\Omega)$ and $\xi \in C_c^1(\mathbb{R}^3)$ to write:

$$\Phi(s; t) := \int_{\Omega} \int_{\mathbb{R}^3} \zeta(x) \xi(Y) G_{t\#} \sigma_s(dx, dY) = \int_{\Omega^2} \zeta(x) \xi(\nabla P_t(y)) \sigma_s(dx, dy).$$

So,

$$\begin{aligned} \Phi(s; t) - \Phi(t; t) &= \int_{\Omega^2} \zeta(x) \xi(\nabla P_t(y)) [\sigma_s - \sigma_t](dx, dy) \\ &= \int_{\Omega^2} \zeta(x) [\xi(\nabla P_t(y)) - \xi(\nabla P_s(y))] \sigma_s(dx, dy) \\ &\quad + \Psi(s) - \Psi(t), \end{aligned}$$

where Ψ is the function (depending on ζ and ξ) of t defined in Remark 4.4 above. According to same remark, it therefore suffices to prove that the first term in the right-hand-side vanishes in the limit as $s \rightarrow t$. To this effect, note that we can use (2.7) to write

$$\left| \int_{\Omega^2} \zeta(x) [\xi(\nabla P_t(y)) - \xi(\nabla P_s(y))] \sigma_s(dx, dy) \right| \leq \|\zeta \nabla \xi\|_{\infty} \|\nabla P_t - \nabla P_s\|_{L^1(\Omega; \mathbb{R}^3)}.$$

We conclude the proof of (i) in light of Theorem 4.1 (ii).

To get (ii), take a test function $f(t)\zeta(x)$ instead of a general $\zeta(t, x)$ in (2.8) to see that, in light of (2.6), we have

$$\int_0^T [f'(t)\Psi(t) + f(t)\Lambda(t)] dt + f(0) \int_{\Omega} \xi(\nabla P_0(x)) \zeta(x) dx = 0 \quad (4.3)$$

for all $f \in C_c^1[0, T]$, $\zeta \in C_c(\Omega)$ and $\xi \in C_c^1(\mathbb{R}^3)$, where

$$\Lambda(t) := \int_{\Omega^2} \nabla \xi(\nabla P_t(y)) \cdot J[y - \nabla P_t(y)] \zeta(x) \sigma_t(dx, dy).$$

From (2.7) and Theorem 4.1 (ii)

$$\|\Lambda\|_{L^1(0,T)} \leq \|\zeta \nabla \xi\|_\infty \int_0^T \left(\sup_{y \in \Omega} |y| + \|\nabla P_t\|_{L^1(\Omega; \mathbb{R}^3)} \right) dt < \infty.$$

Thus, from (4.3) get $\Psi \in W^{1,1}(0, T)$ with $\Psi' = \Lambda$ a.e. in $(0, T)$ (thus $\Psi \in C[0, T]$), and

$$\Psi(0) = \int_\Omega \xi(\nabla P_0(x)) \zeta(x) dx,$$

which finishes the proof. $\square$

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