

THE SEMI-GEOSTROPHIC SYSTEM: WEAK-STRONG UNIQUENESS UNDER UNIFORM CONVEXITY

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Abstract. We prove a weak-strong uniqueness result for the Semi-Geostrophic system with constant Coriolis force. The main assumptions on the strong solution are the boundedness of the velocity field as well as the uniform convexity of the Legendre-Fenchel transform of the modified pressure. We give several examples where our results apply, including some classical solutions on the 2-dimensional torus, and the “stationary” solutions for 3DSG (for which the total wind velocity is zero but the pressure may be time-dependent).

Key words. SG system, flows of maps, optimal mass transport, Wasserstein metric, optimal maps, absolutely continuous curves.

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1. Introduction. The Semi-Geostrophic (abbreviated SG in this paper) system is a model of large scale atmosphere/ocean flows, where “large-scale” means that the flow is rotation-dominated. We are interested in solutions to the SG system which satisfy the convexity principle, introduced by Cullen and Purser as a physical stability condition (semi-geostrophic approximation remains accurate for as long as this condition holds).

There is a fair amount of literature on existence of solutions for the SG system: most of it requires some additional conditions/restrictions, which are essentially some form of strict convexity of the initial data. In a recent paper [12], we also prove existence of some appropriately weakened solutions for all convex initial data, i.e. the most general class of physically admissible solutions.

Throughout the entire paper, $\Omega \subset \mathbb{R}^3$ is a given open, bounded set with C^1 boundary, and $T \in (0, \infty)$ is fixed. A version of the 3D Semi-Geostrophic system [4, 7] is

$$\begin{aligned} D_t X &= J(X - x), \\ X &= \nabla P, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u} \cdot \nu &= 0, \quad \text{on } [0, T) \times \partial\Omega, \\ P(0, \cdot) &= P_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where ∇ stands for the spatial gradient, $\nabla \cdot$ the spatial divergence and $D_t := \partial_t + \mathbf{u} \cdot \nabla$. One looks for solutions $(P, \mathbf{u})$ such that $P : [0, T) \times \Omega \rightarrow \mathbb{R}$, $\mathbf{u} : [0, T) \times \Omega \rightarrow \mathbb{R}^3$ which satisfy the Cullen-Purser stability condition (see, e.g., [8]). This amounts to imposing that $P(t, \cdot)$ be convex for all $t \in [0, T)$.

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Here,

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Henceforth, we shall assume without loss of generality that $\mathcal{L}^3(\Omega) = 1$ (otherwise all the measures considered will have total mass equal to the volume of Ω) and denote

$$\chi := \mathcal{L}^3|_{\Omega}.$$

If $\nabla P(t, \cdot)_{\#}\chi =: \rho(t, \cdot) \ll \mathcal{L}^3$, then one can use the Legendre-Fenchel transforms of the maps $P(t, \cdot)$ relative to Ω , namely

$$P^*(t, \cdot) := \sup_{x \in \Omega} \{x \cdot X - P(t, x)\}, \quad (1.2)$$

to formally rewrite (1.1) as the so-called SG in dual variables:

$$\partial_t \rho + \nabla \cdot (U \rho) = 0 \quad \text{in } [0, T) \times \mathbb{R}^3, \quad (1.3)$$

$$\nabla P(t, \cdot)_{\#}\chi = \rho(t, \cdot) \quad \text{for any } t \in [0, T); \quad (1.4)$$

$$U(t, X) = J[X - \nabla P^*(t, X)], \quad (1.5)$$

$$\rho(0, X) = \rho_0(X) \quad \text{for a.e. } X \in \mathbb{R}^3. \quad (1.6)$$

In [11] one can find a comprehensive bibliography of results on SG in dual space. These solutions are not known to be regular enough to be translated into Eulerian solutions of the problem in physical space. Cullen and Feldman [7] pulled the dual space solutions back to physical space in the form of weak Lagrangian solutions in the case $\rho_0 \in L^p(\mathbb{R}^3)$ for some $p > 1$; the question left open in its generalization (to $p \geq 1$) by Faria et al. [10] regards the even more general case, i.e. the case in which ρ_0 may be a singular probability measure. We settled that in [12], where we introduced an appropriate generalization of weak Lagrangian solutions, namely renormalized relaxed Lagrangian solutions.

Recently, existence of Eulerian solutions for a class of initial data, where the conditions include the requirement that the support of $\rho_0 = \nabla P_0_{\#}\chi$ in the dual space is the whole space, was obtained by L. Ambrosio, M. Colombo, G. De Philippis, A. Figalli [2, 3] based on the results of G. De Philippis and A. Figalli [9] on regularity of solutions for the Monge-Ampere equation. Existence of Eulerian solutions in physical space for more general initial data, when the support of ρ_0 in the dual space may have a non-empty boundary, is presently not known.

As far as uniqueness of solutions is concerned, not much is known: in fact, nothing is besides the result by Loper [14], where uniqueness is proved for classical solutions in the class of functions which are Hölder continuous in space uniformly with respect to time, and the very recent one by Cheng, Cullen and Feldman [6], where, besides the short time existence of classical solutions for SG with variable Coriolis force, the authors prove uniqueness of

such solutions in the class of sufficiently regular classical solutions. There is also work by Brenier and Cullen [5] where a formal proof of convergence of solutions for a Navier-Stokes with Boussinesq approximation to a solution of $x - z$ SG is given. Both systems are assumed to have smooth solutions, which is unknown in general even for “nice” initial data. Also, it is assumed that $P(t, \cdot)$ has a smooth convex extension to $\mathbb{R}^3$ such that $\nabla^2 P^*(t, \cdot)$ is bounded away from zero and infinity uniformly for $t \in [0, T]$ (for some $T > 0$).

In this paper we show that the methods of [5] can be extended so as to enable a comparison between two solutions of 3DSG, or 2DSG on the torus. While one of the solutions must be fairly regular (strong solution), the other can be as weak as they get, e.g. a renormalized relaxed Lagrangian solution as in [12].

The most important assumption on the strong solution is the uniform (in time) boundedness away from zero and infinity of $\nabla^2 P^*(t, \cdot)$. This yields the metric equivalence between the L^2 -norm and the relative entropy functional which is instrumental to our proof. It turns out that on $\mathbb{T}^2$ this uniform convexity assumption is equivalent to

$$0 < m \leq \nabla P(t, \cdot)_{\#} \mathcal{L}^2|_{\mathbb{T}^2} \leq M < \infty$$

for some fixed m , M and all $t \in [0, T]$. Rewritten in terms of the dual space solutions this reads $m \leq \rho(t, \cdot) \leq M$ for all $t \in [0, T]$; fortunately, it can be showed that this is satisfied by dual space solutions under the mere assumption that the initial data is as such (see, e.g. [2]).

The paper is organized as follows: in Section 2 we introduce the notion of renormalized relaxed Lagrangian solutions; this is modified slightly from the notion we introduced in [12], as we shall use more general test functions. This modification will help with the proof of the main result, which is done in Section 3. There are three versions of the main theorem: one applies to the case of compactly supported solutions in dual space, one does away with that requirement but still works in the Euclidean setting (3DSG), whereas the last version treats the case of the 2D torus. This last case is the object of Section 4, where we also show that the regular dual-space solutions found by Loeper [14] are unique in the class of renormalized relaxed Lagrangian solutions. The same is shown to be true for the weak (Eulerian) solutions constructed by Ambrosio et al. in [2]. In the last section (Section 5) we discuss a special class of solutions, namely the stationary (zero total wind velocity) ones. This is to be understood in the sense that the total wind velocity $\mathbf{u} \equiv 0$, but as we will see, there are such solutions for which P is time-dependent. By considering restrictions to Ω of certain uniformly convex functions P_0 defined globally in $\mathbb{R}^3$, we construct a large class of classical stationary solutions for SG to which our main uniqueness result applies.

2. Weak Lagrangian solutions; renormalized relaxed Lagrangian solutions. The starting point will be the weak Lagrangian solutions in the

physical space. Such solutions were introduced by Cullen & Feldman [7], and existence of Lagrangian solutions of (1.1) was shown in [7] for any $\rho_0 = \nabla P_0 \# \chi \in L^q(\Omega)$ for $q > 1$; this result was then extended to the case $q = 1$ in [10]. The restriction $\rho_0 \ll \mathcal{L}^3$ amounts to requiring some *strict convexity* property of the potentials $P(t, \cdot)$, which is certainly more than what Cullen and Purser's stability condition imposes (which is simply the convexity of $P(t, \cdot)$).

The following definition (of weak Lagrangian solution) can be found in [11] and generalizes the original definition of weak Lagrangian solution in physical space from [7]; it is inspired by the fact that if the total wind velocity $\mathbf{u}$ is smooth, then there exists a unique Lagrangian flow map associated to it, i.e. there exists $F : [0, T] \times \Omega \rightarrow \Omega$ such that $F_0 = \text{Id}_\Omega$ and $\partial_t F(t, x) = \mathbf{u}(t, F(t, x))$. Moreover, due to $\nabla \cdot \mathbf{u}(t, \cdot) \equiv 0$ and the boundary condition $\mathbf{u}(t, \cdot) \cdot \nu \equiv 0$, we have that $F(t, \cdot)$ are volume preserving in Ω .

In the definition below (and elsewhere) by a convex function on a (not necessarily convex) open set $A \subset \mathbb{R}^d$ we understand the restriction to A of a convex function defined on $\mathbb{R}^d$ with values in $\mathbb{R} \cup \{\infty\}$.

Definition 2.1. *Let $P_0 \in H^1(\Omega)$ be convex. Let $P : [0, T] \times \Omega \rightarrow \mathbb{R}$ satisfy*

$$P \in C([0, T]; H^1(\Omega)), \quad (2.1)$$

$$P(t, \cdot) \text{ is convex in } \Omega \text{ for each } t \in [0, T]. \quad (2.2)$$

Let $F : [0, T] \times \Omega \rightarrow \Omega$ be a Borel map such that

$$F \in C([0, T]; L^2(\Omega; \mathbb{R}^3)). \quad (2.3)$$

Then the pair (P, F) is called a weak Lagrangian solution of (1.1) in $[0, T] \times \Omega$ if

- i. $F(0, x) = x$, $P(0, x) = P_0(x)$ for a.e. $x \in \Omega$,*
- ii. for any $t > 0$ the mapping $F(t, \cdot) : \Omega \rightarrow \Omega$ is Lebesgue measure preserving, in the sense that $F(t, \cdot) \# \chi = \chi$;*
- iii. There exists a Borel map $F^* : [0, T] \times \Omega \rightarrow \Omega$ such that for every $t \in (0, T)$ the map $F^*(t, \cdot) : \Omega \rightarrow \Omega$ is Lebesgue measure preserving: $F^*(t, \cdot) \# \chi = \chi$, and satisfies $F^*(t, F(t, x)) = x$ and $F(t, F^*(t, x)) = x$ for a.e. $x \in \Omega$;*
- iv. The function*

$$Z(t, x) = \nabla P(t, F(t, x)) \quad (2.4)$$

is a distributional solution of

$$\begin{aligned} \partial_t Z(t, x) &= J[Z(t, x) - F(t, x)] && \text{in } [0, T] \times \Omega, \\ Z(0, x) &= \nabla P_0(x) && \text{in } \Omega. \end{aligned} \quad (2.5)$$

Note that the sense in which (2.5) must be satisfied is

$$\int_0^T \int_\Omega [Z \cdot \partial_t \varphi + J(Z - F) \cdot \varphi] dx dt + \int_\Omega \nabla P_0(x) \cdot \varphi(0, x) dx = 0 \quad (2.6)$$

for any $\varphi \in C_c^1([0, T] \times \Omega; \mathbb{R}^3)$. This implies $t \mapsto Z(t, x) \in W^{1,\infty}(0, T; \mathbb{R}^3)$ for $\mathcal{L}^3$ -a.e. $x \in \Omega$, and the differential equation in (2.5) is satisfied in the $\mathcal{L}^4$ -a.e. in $(0, T) \times \Omega$. Indeed, for a.e. $x \in \Omega$ (for which $\nabla P_0(x)$ exists and the first equation in (2.5) is satisfied) we have

$$Z(t, x) = e^{tJ} \nabla P_0(x) - e^{tJ} \int_0^t e^{-sJ} F(s, x) ds, \quad (2.7)$$

which implies

$$\|Z(\cdot, x)\|_{L^\infty(0, T; \mathbb{R}^3)} \leq |\nabla P_0(x)| + tR(\Omega) < \infty \text{ for each } t \in [0, T], \quad (2.8)$$

where $R(\Omega) < \infty$ is chosen such that $\Omega \subset B(0, R(\Omega))$.

Let

$$\begin{aligned} \mathcal{S} := & \{ \xi \in C([0, T] \times \mathbb{R}^3) : \nabla_x \xi \in C([0, T] \times \mathbb{R}^3; \mathbb{R}^3) \text{ and} \\ & \xi(\cdot, X) \text{ is Lipschitz locally uniformly with respect to } X \in \mathbb{R}^3 \}. \end{aligned}$$

Let $\xi \in \mathcal{S}$ and look at

$$\begin{aligned} |\xi(s, Z(s, x)) - \xi(t, Z(t, x))| & \leq |Z(s, x) - Z(t, x)| \int_0^1 |\nabla \xi(s, \zeta(\tau; s, t, x))| d\tau \\ & \quad + |\xi(s, Z(t, x)) - \xi(t, Z(t, x))|, \end{aligned}$$

where $\zeta(\tau; s, t, x) := (1 - \tau)Z(t, x) + \tau Z(s, x)$. So, the map $[0, T] \ni t \mapsto \xi(t, Z(t, x))$ is Lipschitz for χ -a.e. $x \in \Omega$. But

$$\begin{aligned} \frac{\xi(s, Z(s, x)) - \xi(t, Z(t, x))}{s - t} & = \frac{Z(s, x) - Z(t, x)}{s - t} \int_0^1 \nabla \xi(s, \zeta(\tau; s, t, x)) d\tau \\ & \quad + \frac{\xi(s, Z(t, x)) - \xi(t, Z(t, x))}{s - t}, \end{aligned}$$

so if we choose $x \in \Omega$ such that both $Z(\cdot, x)$ and $\xi(\cdot, Z(\cdot, x))$ are Lipschitz, and t such that they are both differentiable at t , we can use the fact that $\zeta(\tau; s, t, x)$ converges as $s \rightarrow t$ uniformly (with respect to τ , by (2.8)) to $Z(t, x)$ and the continuity of $\nabla \xi$ (in both variables) to conclude that the last term in the right hand side has a real limit and, therefore,

$$\frac{d}{dt} \xi(t, Z(t, x)) = \partial_t \xi(t, Z(t, x)) + \nabla \xi(t, Z(t, x)) \cdot J[Z(t, x) - F(t, x)]$$

for a.e. $t \in (0, T)$ (we used (2.5) to write the last term). Consequently, a more general, “renormalized” version of (2.6) is available in the form

$$\begin{aligned} \int_0^T \int_\Omega \left\{ \xi(t, Z(t, x)) \partial_t \zeta(t, x) + \left\{ \partial_t \xi(t, Z(t, x)) + \nabla \xi(t, Z(t, x)) \cdot \right. \right. \\ \left. \left. J[Z(t, x) - F(t, x)] \right\} \zeta(t, x) \right\} dx dt + \int_\Omega \xi(0, \nabla P_0(x)) \zeta(0, x) dx = 0, \quad (2.9) \end{aligned}$$

for any $\xi \in \mathcal{S}$, $\zeta \in C_c^1([0, T] \times \bar{\Omega})$.

In order to prove existence of these weak Lagrangian solutions one needs dual-space solutions such that $\rho(t, \cdot) \ll \mathcal{L}^3$; such dual-space solutions do exist under the assumption $\rho_0 \ll \mathcal{L}^3$ [7], [10]. However, since no uniqueness result is known for weak solutions in dual space, there is no guarantee that all dual-space solutions satisfy $\rho(t, \cdot) \ll \mathcal{L}^3$ if ρ_0 does. In [12] we defined renormalized relaxed Lagrangian solutions in the physical space, replacing $F(t, \cdot) : \Omega \rightarrow \Omega$ by the corresponding transport plans, which are measures σ_t on $\Omega \times \Omega$. Thus we based the definition of relaxed Lagrangian solution on the pair (P, σ) , and tried to capture all the features of (P, F) from Definition 2.1 (for which $\sigma_t = (\text{Id} \times F(t, \cdot))_{\#} \chi$). The main advantage of this new notion of solution is that it exists for any convex $P_0 \in H^1(\Omega)$, there is no requirement that $\rho_0 \ll \mathcal{L}^3$ (and much less that that be satisfied at later times).

More precisely, assuming existence of a weak Lagrangian solution (P, F) in the sense of Definition 2.1, we defined [12] the measure σ on $(0, T) \times \Omega \times \Omega$ by

$$\int_0^T \int_{\Omega} \int_{\Omega} \xi(t, x, y) \sigma(dt, dx, dy) = \int_0^T \int_{\Omega} \xi(t, x, F(t, x)) dx dt, \quad (2.10)$$

for all $\xi \in C(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$. We notice first that the volume-preserving property of $F(t, \cdot)$ shows that σ disintegrates as

$$\int_0^T \int_{\Omega} \int_{\Omega} \xi(t, x, y) \sigma(dt, dx, dy) = \int_0^T \left(\int_{\Omega} \int_{\Omega} \xi(t, x, y) \sigma_t(dx, dy) \right) dt \quad (2.11)$$

for all $\xi \in C(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$, where $[0, T] \ni t \mapsto \sigma_t = (\text{Id} \times F(t, \cdot))_{\#} \chi$ is a Borel family of measures on $\Omega \times \Omega$ such that

$$\text{proj}_x \sigma_t = \text{proj}_y \sigma_t = \chi \quad \text{for all } t \in [0, T]. \quad (2.12)$$

Furthermore, (2.9) becomes, in light of (2.4),

$$\begin{aligned} & \int_0^T \int_{\Omega^2} \left\{ \xi(t, \nabla P(t, y)) \partial_t \zeta(t, x) + \{ (\partial_t \xi)(t, \nabla P(t, y)) + \nabla \xi(t, \nabla P(t, y)) \cdot \right. \\ & \left. J[\nabla P(t, y) - y] \} \zeta(t, x) \right\} \sigma(dt, dx, dy) + \int_{\Omega} \xi(0, \nabla P_0(x)) \zeta(0, x) dx = 0 \end{aligned} \quad (2.13)$$

for all $\xi \in \mathcal{S}$ and all $\zeta \in C_c^1([0, T] \times \bar{\Omega})$. We now have all the ingredients for the definition of *renormalized relaxed Lagrangian solutions*, such as the one in [12]. Note that the only difference is that in [12] we consider test functions $\xi \in C_b^1(\mathbb{R}^3)$ (i.e. time-independent and with bounded gradient). However, a careful inspection of the proofs in [12] reveals that the generalization we employ here does not affect the results in [12].

Definition 2.2. *Let $P_0 \in H^1(\Omega)$ be convex. Consider a family of convex functions $P \in L^\infty(0, T; H^1(\Omega))$ and a Borel family of measures $[0, T] \ni t \mapsto$*

$\sigma_t \in \mathcal{M}(\Omega \times \Omega)$. Let σ be given by $d\sigma = d\sigma_t dt$ as in (2.11). We say that (P, σ) is a relaxed Lagrangian solution for the SG system (1.1) with initial data P_0 if (2.12)–(2.13) are satisfied.

We carry over the assertion of Corollary 4.3 [12] with the one on the conservation of Hamiltonian energy

$$\mathcal{E}(t) = \int_{\Omega} |\nabla P(t, x) - x|^2 dx$$

(as discussed in the preamble to Theorem 4.1 [12]) and state (for further reference):

Theorem 2.3. *Let $P_0 \in H^1(\Omega)$ be convex. Then there exists a renormalized relaxed Lagrangian solution (P, σ) corresponding to the initial datum P_0 . Moreover, the Hamiltonian energy of any such solution is preserved in time.*

Let us now prove a useful proposition; this will enable us to deal with the time-derivatives of some expressions depending on the relaxed solution.

Proposition 2.4. *Let $P_0 \in H^1(\Omega)$ be convex and let (P, σ) be a renormalized relaxed Lagrangian solution corresponding to the initial datum P_0 . Then*

$$\begin{aligned} [0, \infty) \ni t \mapsto \int_{\Omega} x \cdot \nabla P(t, x) dx \text{ is absolutely continuous and} \\ \frac{d}{dt} \int_{\Omega} x \cdot \nabla P(t, x) dx = - \int_{\Omega} Jx \cdot \nabla P(t, x) dx. \end{aligned} \quad (2.14)$$

Proof. Since by (2.12) we have

$$\int_{\Omega} |\nabla P(t, x)|^2 dx = \int_{\Omega} \int_{\Omega} |\nabla P(t, y)|^2 \sigma_t(dx, dy)$$

we can apply (2.13) for $\xi(t, X) := |X|^2$ and any $\zeta(t, x) = \varphi(t) \in C_c^1(0, T)$ to infer that

$$f(t) := \int_{\Omega} |\nabla P(t, y)|^2 dy = \int_{\Omega} \int_{\Omega} |\nabla P(t, y)|^2 \sigma_t(dx, dy)$$

lies in $W^{1,1}(0, T)$ and

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla P(t, x)|^2 dx &= -2 \int_{\Omega} \int_{\Omega} \nabla P(t, y) \cdot J[y - \nabla P(t, y)] \sigma_t(dx, dy) \\ &= -2 \int_{\Omega} Jx \cdot \nabla P(t, x) dx. \end{aligned}$$

Since the Hamiltonian energy is conserved [12], i.e.

$$\int_{\Omega} |\nabla P(t, x) - x|^2 dx = \int_{\Omega} |\nabla P(0, x) - x|^2 dx \text{ for all } t > 0, \quad (2.15)$$

we deduce that

$$t \mapsto \int_{\Omega} x \cdot \nabla P(t, x) dx \text{ lies in } W^{1,1}(0, T)$$

as well, and

$$\frac{d}{dt} \int_{\Omega} |\nabla P(t, x)|^2 dx = 2 \frac{d}{dt} \int_{\Omega} x \cdot \nabla P(t, x) dx.$$

□

3. Main result. We are now in a position to prove our main strong-weak uniqueness result. There are, in fact, three versions of this theorem: the first applies to the case where the supports of the dual-space solutions (which are probability measures on $\mathbb{R}^3$) are (for all times) contained in a bounded set to which the spatial Legendre transform of the modified pressure P admits a uniformly convex extension. The second deals with the case where this uniformly convex extension exists on the whole of $\mathbb{R}^3$; this eliminates the need to impose that dual-space solutions be compactly supported. Finally, in the next section we analyze the case of the 2D torus as the physical space.

Theorem 3.1. *Let (P, σ) be a continuous renormalized relaxed Lagrangian solution for (1.1) in $[0, T] \times \Omega$ such that the following properties are satisfied:*

- (i) *There exists $\mathcal{R} \in (0, \infty)$ and $0 < \delta < \mathcal{R}$ such that $\overline{\nabla P(t, \Omega)} \subset B(0, \mathcal{R} - \delta)$ (open ball centered at the origin) for all $t \in [0, T]$.*
- (ii) *For each $t \in [0, T]$ there exists an extension (still denoted by $P^*(t, \cdot)$) of $P^*(t, \cdot)$ from $\nabla P(t, \Omega)$ to $B(0, \mathcal{R})$ and there exists $\lambda \in (0, \infty)$ such that*

$$\lambda I_3 \leq \nabla^2 P^*(t, X) \text{ for each } t \in [0, T] \text{ and all } X \in B(0, \mathcal{R}). \quad (3.1)$$

(iii) $P^* \in W^{1,\infty}(0, T; W^{2,\infty}(B(0, \mathcal{R}))) \cap L^\infty(0, T; W^{3,\infty}(B(0, \mathcal{R})))$.

Then there exists a real constant C depending only on P such that for any renormalized relaxed Lagrangian solution $(\tilde{P}, \tilde{\sigma})$ for (1.1) which satisfies $\overline{\nabla \tilde{P}(t, \Omega)} \subset B(0, \mathcal{R} - \delta)$ for all $t \in [0, T]$ we have

$$\|\nabla P(t, \cdot) - \nabla \tilde{P}(t, \cdot)\|_{L^2(\Omega; \mathbb{R}^3)} \leq C \|\nabla P_0 - \nabla \tilde{P}_0\|_{L^2(\Omega; \mathbb{R}^3)} \quad (3.2)$$

for all $t \in [0, T]$.

Note that in the above statement we consider renormalized relaxed solutions; however, by Proposition 3.2 below we see that the regularity conditions on P guarantee that any renormalized relaxed Lagrangian solution (P, σ) comes from a weak Lagrangian solution (P, F) in the sense that $\sigma_t = (\text{Id}_\Omega \times F(t, \cdot))_{\#} \chi$ for a.e. $t \in [0, T]$.

Proposition 3.2. *Let (P, σ) as in Theorem 3.1 and denote*

$$\mathbf{u}(t, x) := (\partial_t \nabla P^*)(t, \nabla P(t, x)) + \nabla^2 P^*(t, \nabla P(t, x)) J[\nabla P(t, x) - x]. \quad (3.3)$$

Then the following hold:

(a) $\mathbf{u} \in L^\infty(0, T; W^{1,\infty}(\Omega; \mathbb{R}^3))$ and

$$\nabla \cdot \mathbf{u}(t, \cdot) \equiv 0 \text{ and } \mathbf{u}(t, \cdot) \cdot \nu = 0 \text{ on } \partial\Omega \text{ for a.e. } t \in [0, T]. \quad (3.4)$$

(b) There exists a unique Borel family of maps $F : [0, T] \times \Omega \rightarrow \Omega$ such that $F \in W^{1,\infty}(0, T; L^\infty(\Omega; \mathbb{R}^3))$, $F(t, \cdot)_{\#}\chi = \chi$ for a.e. $t \in [0, T]$ and

$$\partial_t F(t, x) = \mathbf{u}(t, F(t, x)), \quad F(0, x) = x \text{ for a.e. } (t, x) \in (0, T) \times \Omega. \quad (3.5)$$

(c) There exists a unique Borel family of maps $F^* : [0, T] \times \Omega \rightarrow \Omega$ such that for a.e. $t \in [0, T]$ we have $F^*(t, \cdot)_{\#}\chi = \chi$ and $F(t, F^*(t, x)) = F^*(t, F(t, x)) = x$ for a.e. $x \in \Omega$.

(d) For a.e. $t \in [0, T]$ we have $\sigma_t = (\text{Id}_\Omega \times F(t, \cdot))_{\#}\chi$.

(e) (P, F) is a weak Lagrangian solution in the sense of Definition 2.1.

Proof. Note that (2.2) is satisfied due to Definition 2.2. By property (i) of Theorem 3.1 we have that $P \in L^\infty([0, T]; W^{1,\infty}(\Omega))$. From (ii), (iii) of Theorem 3.1 and the obvious identity

$$\nabla P^*(t, \nabla P(t, x)) = x \text{ for all } t \in [0, T] \text{ and all } x \in \Omega \quad (3.6)$$

we deduce there exists $L \in \mathbb{R}$ such that for all $s, t \in [0, T]$ and all $x \in \Omega$

$$\begin{aligned} L|t - s| &\geq |\nabla P^*(t, \nabla P(t, x)) - \nabla P^*(s, \nabla P(t, x))| \\ &= |\nabla P^*(s, \nabla P(t, x)) - \nabla P^*(s, \nabla P(s, x))| \\ &\geq \lambda |\nabla P(t, x) - \nabla P(s, x)|, \end{aligned}$$

so ∇P is Lipschitz in time (uniformly with respect to $x \in \Omega$); we compute the a.e. time derivative to get

$$\partial_t \nabla P(t, x) = -[\nabla^2 P^*(t, \nabla P(t, x))]^{-1} \partial_t \nabla P^*(t, \nabla P(t, x))$$

for a.e. $(t, x) \in (0, T) \times \Omega$. Furthermore, again by (iii) of Theorem 3.1,

$$\begin{aligned} |P^*(s, \nabla P(s, x)) - P^*(t, \nabla P(t, x))| &\leq |P^*(s, \nabla P(t, x)) - P^*(t, \nabla P(t, x))| \\ &\quad + |P^*(s, \nabla P(s, x)) - P^*(s, \nabla P(t, x))| \leq L|t - s| \\ &\quad + \|\nabla P^*\|_\infty |\nabla P(s, x) - \nabla P(t, x)|, \end{aligned}$$

which shows that $P^*(\cdot, \nabla P(\cdot, x))$ is Lipschitz too, uniformly with respect to $x \in \Omega$. Since

$$P^*(t, \nabla P(t, x)) + P(t, x) = x \cdot \nabla P(t, x)$$

we get that P is Lipschitz in time uniformly in $x \in \Omega$; now we employ (3.6) once again to obtain

$$\begin{aligned} \partial_t P(t, x) &= x \cdot \partial_t \nabla P(t, x) - \partial_t P^*(t, \nabla P(t, x)) - \nabla P^*(t, \nabla P(t, x)) \cdot \partial_t \nabla P(t, x) \\ &= -\partial_t P^*(t, \nabla P(t, x)). \end{aligned}$$

In particular, (2.1) is satisfied.

By (iii) we conclude there is a $\Lambda \in (0, \infty)$ such that $\nabla^2 P^*(t, X) \leq \Lambda I_2$, so by (ii) and (3.6) we see that for all $(t, x) \in [0, T] \times \Omega$ we have

$$\nabla^2 P(t, x) = [\nabla^2 P^*(t, \nabla P(t, x))]^{-1} \text{ and } \Lambda^{-1} I_2 \leq \nabla^2 P(t, x) \leq \Lambda^{-1} I_2.$$

We use this and (iii) again to see that $\mathbf{u} \in L^\infty(0, T; W^{1, \infty}(\Omega; \mathbb{R}^3))$ (recall $\mathbf{u}$ is defined in (3.3)).

Since $\nabla P^*(t, \cdot)$ is not defined outside $B(0, \mathcal{R})$, let us consider the cut-off $\eta \in C_c^\infty(B(0, \mathcal{R} - 3\delta/4))$ such that $\eta \equiv 1$ in $B(0, \mathcal{R} - \delta/2)$. Next, let $\tilde{\xi} \in C^1([0, T] \times \mathbb{R}^3)$, $\zeta \in C_c^1([0, T] \times \mathbb{R}^3)$ and define $\xi(t, Y) := \tilde{\xi}(t, S(t, Y))$ for $(t, Y) \in [0, T] \times \mathbb{R}^3$, where $S(t, Y) := \eta(Y) \nabla P^*(t, Y)$ if $Y \in B(0, \mathcal{R})$ and $S(t, Y) := 0$ else. We have that for a.e. $(t, Y) \in (0, T) \times \mathbb{R}^3$

$$\partial_t \xi(t, Y) = \partial_t \tilde{\xi}(t, \eta(Y) \nabla P^*(t, Y)) + \eta(Y) \nabla \tilde{\xi}(t, \nabla P^*(t, Y)) \cdot \partial_t \nabla P^*(t, Y) \quad (3.7)$$

if $Y \in B(0, \mathcal{R} - 3\delta/4)$, and

$$\partial_t \xi(t, Y) = \partial_t \tilde{\xi}(t, 0) \text{ if } Y \in \mathbb{R}^3 \setminus B(0, \mathcal{R} - 3\delta/4).$$

From (iii) we have a uniform in time-space bound on ∇P^* and $\partial_t \nabla P^*$; it follows that $\partial_t \xi$ is bounded in $[0, T] \times \mathbb{R}^3$. Next we look at (by taking into account the symmetry of ∇P^*)

$$\begin{aligned} \nabla \xi(t, Y) &= \eta(Y) \nabla^2 P^*(t, Y) \nabla \tilde{\xi}(t, \nabla P^*(t, Y)) \\ &\quad + \nabla \eta(Y) \otimes \nabla \tilde{\xi}(t, \eta(Y) \nabla P^*(t, Y)) \nabla P^*(t, Y) \end{aligned}$$

if $Y \in B(0, \mathcal{R} - 3\delta/4)$ (and $\nabla \xi = 0$ else) and use (iii) again to see that $\nabla \xi$ is bounded and jointly continuous in $[0, T] \times \mathbb{R}^3$ as well. Therefore, the pair of test functions (ξ, ζ) is admissible in (2.13). Note that, since $\eta(\nabla P(t, y)) = 1$ for all $(t, y) \in [0, T] \times \Omega$, (3.7) implies

$$(\partial_t \xi)(t, \nabla P(t, y)) = \partial_t \tilde{\xi}(t, y) + \nabla \tilde{\xi}(t, y) \cdot (\partial_t \nabla P^*)(t, \nabla P(t, y))$$

and

$$\nabla \xi(t, \nabla P(t, y)) = \nabla^2 P^*(t, \nabla P(t, y)) \nabla \tilde{\xi}(t, y).$$

If we apply (2.13) to the pair of test functions (ξ, ζ) and use the symmetry of $\nabla^2 P^*(t, \nabla P(t, y))$ and (3.3), we get

$$\begin{aligned} \int_0^T \int_\Omega \int_\Omega \{ \partial_t [\tilde{\xi}(t, y) \zeta(t, x)] + \zeta(t, x) \nabla \tilde{\xi}(t, y) \cdot \mathbf{u}(t, y) \} \sigma(dt, dx, dy) \\ = - \int_\Omega \tilde{\xi}(0, x) \zeta(0, x) dx. \end{aligned} \quad (3.8)$$

Now specialize to $\zeta(t, x) := \varphi(t) \psi(x)$ with $\varphi \in C_c^1(0, T)$ and $\tilde{\xi}(t, y) := \vartheta(y)$; since (3.8) holds for any such φ , we deduce that

$$t \mapsto \int_\Omega \int_\Omega \psi(x) \vartheta(y) \sigma_t(dx, dy) \text{ is absolutely continuous on } [0, T]$$

and

$$\frac{d}{dt} \int_{\Omega} \int_{\Omega} \psi(x) \vartheta(y) \sigma_t(dx, dy) = \int_{\Omega} \int_{\Omega} \psi(x) \nabla \vartheta(y) \cdot \mathbf{u}(t, y) \sigma_t(dx, dy). \quad (3.9)$$

In particular, for $\psi \equiv 1$ we obtain (in light of (2.12))

$$\int_{\Omega} \nabla \vartheta(y) \cdot \mathbf{u}(t, y) dy = 0 \text{ for all } \vartheta \in C^1(\mathbb{R}^3). \quad (3.10)$$

Thus, (a) is proved. Then (b), (c) follow from the regularity of $\mathbf{u}$ and the classical theory of divergence-free vector fields which have traces parallel to the boundary. Moreover, if we take $\zeta(t, x) := \varphi(t) \psi(x)$ for any $\varphi \in C_c^1([0, T])$ in (3.8) and ψ and ϑ as before, we use (3.9) to see that

$$\lim_{t \rightarrow 0^+} \int_{\Omega} \int_{\Omega} \psi(x) \vartheta(y) \sigma_t(dx, dy) = \int_{\Omega} \psi(x) \vartheta(x) dx$$

and this holds for arbitrary ψ, ϑ ; we infer that there exists a time-continuous (with respect to narrow convergence) family $\{\bar{\sigma}_t\}_{t \in [0, T]}$ of Borel probability measures on $\Omega \times \Omega$ such that $\bar{\sigma}_t = \sigma_t$ for a.e. $t \in [0, T]$ and

$$\bar{\sigma}_0 = (\text{Id} \times \text{Id})_{\#} \chi. \quad (3.11)$$

So, we may assume σ has these properties. Due to (2.12), we can disintegrate σ_t as

$$\int_{\Omega} \int_{\Omega} \psi(x) \vartheta(y) \sigma_t(dx, dy) = \int_{\Omega} \vartheta(y) \left(\int_{\Omega} \psi(x) \nu_{t,y}(dx) \right) dy.$$

Denote

$$f(t, y) := \int_{\Omega} \psi(x) \nu_{t,y}(dx),$$

which, from the disintegration, is a Borel map in time-space. By (3.9) and (3.11), we deduce

$$\partial_t f + \nabla \cdot (\mathbf{u} f) = 0 \text{ in the sense of distributions and } f(0, \cdot) = \psi.$$

Due to the regularity of $\mathbf{u}$, we infer that the equation above has a unique solution, given by

$$f(t, y) = f(0, F^*(t, y)) = \psi(F^*(t, y)).$$

It follows that

$$\int_{\Omega} \int_{\Omega} \psi(x) \vartheta(y) \sigma_t(dx, dy) = \int_{\Omega} \psi(F^*(t, y)) \vartheta(y) dy = \int_{\Omega} \psi(x) \vartheta(F(t, x)) dx,$$

where we used the volume preserving property of $F(t, \cdot)$ and the fact that $F^*(t, F(t, x)) = x$ for a.e. $x \in \Omega$. Since this holds for arbitrary $\psi, \vartheta \in C^1(\mathbb{R}^3)$, we conclude the proof of (d). Finally, (e) follows easily from (d). $\square$

We now have all the necessary ingredients to prove Theorem 3.1:

Proof of Theorem 3.1: Consider, as in [5], the so-called Bregman (or relative entropy) functional

$$\eta[P^*](t, x) := P^*(t, \tilde{\mathbf{y}}(t, x)) - P^*(t, \mathbf{y}(t, x)) - \nabla P^*(t, \mathbf{y}(t, x)) \cdot [\tilde{\mathbf{y}}(t, x) - \mathbf{y}(t, x)],$$

where

$$\mathbf{y}(t, x) := \nabla P(t, x), \quad \tilde{\mathbf{y}} := \nabla \tilde{P}(t, x).$$

Let us, for notational relief, denote by $\|\cdot\|$ the $\|\cdot\|_{L^2(\Omega; \mathbb{R}^3)}$ -norm. From (ii) and (iii) we deduce

$$\lambda_0 \|\nabla P(t, \cdot) - \nabla \tilde{P}(t, \cdot)\|^2 \leq \int_{\Omega} \eta[P^*](t, x) dx \leq \lambda_0^{-1} \|\nabla P(t, \cdot) - \nabla \tilde{P}(t, \cdot)\|^2 \quad (3.12)$$

for some $\lambda_0 \in (0, 1)$ and each $t \in [0, T)$. Therefore, if we denote

$$E(t) := \int_{\Omega} \eta[P^*](t, x) dx,$$

it suffices to prove that

$$\dot{E}(t) \leq C \|\nabla P(t, \cdot) - \nabla \tilde{P}(t, \cdot)\|^2 \text{ for all } t \in [0, T]$$

for a real constant C (depending only on P) to get the desired inequality. By contrast to [5], here the solutions $\mathbf{y}$ and $\tilde{\mathbf{y}}$ are not necessarily classical (so they may not even be differentiable with respect to t), so we need to proceed more carefully. First off, let us use (3.6) and (2.4), (2.12) to see that

$$\begin{aligned} E(t) &= \int_{\Omega} \int_{\Omega} P^*(t, \tilde{\mathbf{y}}(t, z)) \tilde{\sigma}_t(dx, dz) \\ &\quad - \int_{\Omega} P^*(t, \mathbf{y}(t, z)) \sigma_t(dx, dz) - \int_{\Omega} x \cdot (\tilde{\mathbf{y}} - \mathbf{y}) dx. \end{aligned}$$

Note that if we consider the test functions $\xi \equiv P^*$ and $\zeta(t, x) = \varphi(t) \in C_c^1(0, T)$ in (2.13) we infer that the first integral in the right hand side of the above display is absolutely continuous as a function of t and

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} \int_{\Omega} P^*(t, \tilde{\mathbf{y}}(t, z)) \tilde{\sigma}_t(dx, dz) \\ &= \int_{\Omega} \int_{\Omega} \{(\partial_t P^*)(t, \tilde{\mathbf{y}}(t, z)) + \nabla P^*(t, \tilde{\mathbf{y}}(t, z)) \cdot J[\tilde{\mathbf{y}}(t, z) - z]\} \sigma_t(dx, dz) \\ &= \int_{\Omega} \{(\partial_t P^*)(t, \tilde{\mathbf{y}}(t, z)) + \nabla P^*(t, \tilde{\mathbf{y}}(t, z)) \cdot J[\tilde{\mathbf{y}}(t, z) - z]\} dz, \end{aligned}$$

where we have used the marginal property (2.12) for the last equality. Obviously, the same holds for the second integral. We may now differentiate $E(t)$

with respect to time and use Proposition 2.4 to get (after a cancellation due to (3.6))

$$\begin{aligned}\dot{E}(t) &= \int_{\Omega} [(\partial_t P^*)(t, \tilde{\mathbf{y}}(t, x)) - (\partial_t P^*)(t, \mathbf{y}(t, x))] dx \\ &\quad + \int_{\Omega} \nabla P^*(t, \tilde{\mathbf{y}}(t, x)) \cdot G(x, \tilde{\mathbf{y}}(t, x)) dx + \int_{\Omega} Jx \cdot \tilde{\mathbf{y}}(t, x) dx \\ &\quad - \int_{\Omega} \nabla P^*(t, \mathbf{y}(t, x)) \cdot G(x, \mathbf{y}(t, x)) dx - \int_{\Omega} Jx \cdot \mathbf{y}(t, x) dx,\end{aligned}$$

where we have denoted $G(x, y) := J(y - x)$. After some obvious cancellations (due to (3.6)), we get

$$\begin{aligned}\dot{E}(t) &= \int_{\Omega} [(\partial_t P^*)(t, \tilde{\mathbf{y}}(t, x)) - (\partial_t P^*)(t, \mathbf{y}(t, x))] dx \\ &\quad + \int_{\Omega} \nabla P^*(t, \tilde{\mathbf{y}}(t, x)) \cdot G(x, \tilde{\mathbf{y}}(t, x)) dx + \int_{\Omega} Jx \cdot \tilde{\mathbf{y}}(t, x) dx.\end{aligned}\quad (3.13)$$

Let us now investigate the second term in the right hand side. We begin by noticing that the boundedness of $\nabla^2 P^*$ in space-time (by (iii)) and (3.12) imply that

$$\int_{\Omega} [\nabla P^*(t, \tilde{\mathbf{y}}) - \nabla P^*(t, \mathbf{y})] \cdot [G(x, \tilde{\mathbf{y}}) - G(x, \mathbf{y})] dx$$

is bounded from above by $CE(t)$. By (3.6) and the fact that $J^T = -J$, we have

$$\int_{\Omega} \nabla P^*(t, \mathbf{y}) \cdot [G(x, \tilde{\mathbf{y}}) - G(x, \mathbf{y})] dx = - \int_{\Omega} Jx \cdot (\tilde{\mathbf{y}} - \mathbf{y}) dx.$$

We consequently infer

$$\begin{aligned}\int_{\Omega} \nabla P^*(t, \tilde{\mathbf{y}}) \cdot G(x, \tilde{\mathbf{y}}) dx + \int_{\Omega} Jx \cdot \tilde{\mathbf{y}} dx \\ \leq CE(t) + \int_{\Omega} \nabla P^*(t, \tilde{\mathbf{y}}) \cdot G(x, \mathbf{y}) dx + \int_{\Omega} Jx \cdot \mathbf{y} dx.\end{aligned}\quad (3.14)$$

Now let us deal with the first term in the right hand side of (3.13). By (iii) we have that $\partial_t \nabla^2 P^*$ is bounded in space-time, so

$$\int_{\Omega} \eta [\partial_t P^*](t, x) dx \leq \tilde{C} \|\tilde{\mathbf{y}} - \mathbf{y}\|^2 \leq CE(t),$$

which yields

$$\int_{\Omega} [(\partial_t P^*)(t, \tilde{\mathbf{y}}) - (\partial_t P^*)(t, \mathbf{y})] dx \leq CE(t) + \int_{\Omega} (\partial_t \nabla P^*)(t, \mathbf{y}) \cdot (\tilde{\mathbf{y}} - \mathbf{y}) dx.\quad (3.15)$$

In view of (3.13), (3.14) and (3.15) we see that it suffices to estimate

$$S(t) := \int_{\Omega} \nabla P^*(t, \tilde{\mathbf{y}}) \cdot G(x, \mathbf{y}) dx + \int_{\Omega} (\partial_t \nabla P^*)(t, \mathbf{y}) \cdot (\tilde{\mathbf{y}} - \mathbf{y}) dx + \int_{\Omega} Jx \cdot \mathbf{y} dx$$

in terms of $E(t)$ in order to finish the proof. Due to (i), (iii) and (3.12) again, we have that

$$\int_{\Omega} [\nabla P^*(t, \tilde{\mathbf{y}}) - \nabla P^*(t, \mathbf{y}) - \nabla^2 P^*(t, \mathbf{y})(\tilde{\mathbf{y}} - \mathbf{y})] \cdot G(x, \mathbf{y}) \leq CE(t).$$

Since we have

$$\int_{\Omega} \nabla P^*(t, \mathbf{y}) \cdot G(x, \mathbf{y}) dx = \int_{\Omega} x \cdot J(\mathbf{y} - x) dx = - \int_{\Omega} Jx \cdot \mathbf{y} dx,$$

we get that the previously displayed inequality is equivalent to

$$\begin{aligned} \int_{\Omega} \nabla P^*(t, \tilde{\mathbf{y}}) \cdot G(x, \mathbf{y}) dx \\ \leq CE(t) + \int_{\Omega} \nabla^2 P^*(t, \mathbf{y})(\tilde{\mathbf{y}} - \mathbf{y}) \cdot G(x, \mathbf{y}) dx - \int_{\Omega} Jx \cdot \mathbf{y} dx. \end{aligned}$$

Thus,

$$S(t) \leq CE(t) + \int_{\Omega} [(\partial_t \nabla P^*)(t, \mathbf{y}) + \nabla^2 P^*(t, \mathbf{y})G(x, \mathbf{y})] \cdot (\tilde{\mathbf{y}} - \mathbf{y}) dx. \quad (3.16)$$

But, by (3.3), we have

$$(\partial_t \nabla P^*)(t, \mathbf{y}(t, x)) + \nabla^2 P^*(t, \mathbf{y}(t, x))G(x, \mathbf{y}(t, x)) = \mathbf{u}(t, x).$$

Since for all $t \in [0, T)$ we have that $\tilde{\mathbf{y}}(t, \cdot) - \mathbf{y}(t, \cdot)$ is the gradient of a $W^{1,\infty}(\Omega)$ function, we use Proposition 3.2 (a) to infer that the integral in the right-hand-side of (3.16) vanishes. $\square$

Clearly, one immediate consequence is uniqueness of sufficiently regular solutions among weak solutions.

Corollary 3.3. *Let $P_0 \in W^{1,\infty}(\Omega)$ and assume (P, F) is a weak Lagrangian solution on $[0, T)$ for some $T \in (0, \infty)$. Assume that for each $t \in [0, T)$ the function $P^*(t, \cdot)$ admits a uniformly convex extension to the ball $B(0, \mathcal{R}(T))$, where $\mathcal{R}(T)$ is any positive number such that $\mathcal{R}(T) > \|\nabla P_0\|_{\infty} + TR(\Omega)$. If P^* also satisfies condition (iii) from Theorem 3.1, then (P, F) is unique among weak Lagrangian solutions with the same initial data.*

Proof. Note that (2.7) guarantees that any weak Lagrangian solution originating at P_0 satisfies condition (i) from Theorem 3.1 for any $\mathcal{R} > \|\nabla P_0\|_{\infty} + TR(\Omega)$ and sufficiently small $\delta > 0$. Take $\mathcal{R} := \mathcal{R}(T)$. Since $\nabla P_0(\Omega) \subset B(0, \mathcal{R} - \delta)$ and $\tilde{P}_0 = P_0$, we deduce that there exists $\mathcal{R} > \delta' > \delta$ such $\nabla \tilde{P}_0(\Omega) \subset B(0, \mathcal{R} - \delta')$. But $(\tilde{P}, \tilde{F})$ satisfies (2.5) with $\tilde{P}_0 = P_0$, so

$$\nabla \tilde{P}(t, \tilde{F}(t, x)) = e^{tJ} \nabla P_0(x) - e^{tJ} \int_0^t e^{-sJ} \tilde{F}(s, x) ds.$$

It follows that there exists a time $0 < \bar{t} < T$ which depends only on δ and δ' such that $\overline{\nabla \tilde{P}(t, \Omega)} \subset B(0, \mathcal{R} - \delta)$ for all $t \in [0, \bar{t}]$. By Theorem 3.1 we deduce $\nabla \tilde{P}(t, \cdot) \equiv \nabla P(t, \cdot)$ for all $t \in [0, \bar{t}]$ and, by Proposition 3.2, $\tilde{F}(t, \cdot) \equiv F(t, \cdot)$ for all $t \in [0, \bar{t}]$. From (2.1) and (2.3) we deduce that the subset of times $\mathcal{T} \subset [0, T)$ such that $\nabla \tilde{P}(t, \cdot) \equiv \nabla P(t, \cdot)$ and $\tilde{F}(t, \cdot) \equiv F(t, \cdot)$ for all $t \in \mathcal{T}$ is relatively closed in $[0, T)$. By a similar argument to the one above (where we showed the existence of $\bar{t}$) we can also show that $\mathcal{T}$ is relatively open in $[0, T)$. But $0 \in \mathcal{T}$; this means $\mathcal{T} = [0, T)$. $\square$

Remark 3.4. *The corollary above can be formulated somewhat differently, so that it refers to renormalized relaxed solutions satisfying the conditions in Theorem 3.1. However, we can produce a much “cleaner” statement in the case of weak Lagrangian solutions originating at some $P_0 \in W^{1,\infty}(\Omega)$. Indeed, notice (from (2.7)) that such initial modified pressure function guarantees that condition (i) from Theorem 3.1 is automatically satisfied for some $\mathcal{R}$ and δ . Furthermore, as seen in the proof below, the restriction that the weak solution $\tilde{P}$ satisfy the same condition Theorem 3.1 (i) (for the same $\mathcal{R}$ and δ) can be eliminated as well.*

The requirement that $\overline{\nabla P(t, \Omega)}$ and $\overline{\nabla \tilde{P}(t, \Omega)}$ lie in a compact set for all $t \in [0, T)$ can be eliminated if global conditions are imposed on P^* . For example, in principle it is enough to have condition (ii) globally in space, and $\partial_t \nabla^2 P^*$ and $\nabla^3 P^*$ globally bounded in time-space (i.e. in $[0, T) \times \mathbb{R}^3$) for the conclusion of Theorem 3.1 to hold. To be precise, we state:

Theorem 3.5. *Let (P, σ) be a renormalized relaxed Lagrangian solution for (1.1) in $[0, T) \times \Omega$ such that the following properties are satisfied:*

- (i) *For each $t \in [0, T)$ there exists an extension (still denoted by $P^*(t, \cdot)$) of $P^*(t, \cdot)$ from $\nabla P(t, \Omega)$ to the whole $\mathbb{R}^3$ such that $P^* \in C^1([0, T]; C^2(\mathbb{R}^3))$;*
- (ii) *There exists $\lambda \in (0, \infty)$ such that*

$$\lambda I_3 \leq \nabla^2 P^*(t, X) \text{ for each } t \in [0, T) \text{ and all } X \in \mathbb{R}^3. \quad (3.17)$$

- (iii) $\nabla^2 P^* \in L^\infty(0, T; W^{1,\infty}(\mathbb{R}^3)) \cap W^{1,\infty}(0, T; L^\infty(\mathbb{R}^3))$.

Then there exists a real constant C depending only on P such that for any renormalized relaxed Lagrangian solution $(\tilde{P}, \tilde{\sigma})$ for (1.1) the inequality (3.2) holds for all $t \in [0, T)$.

Proof. From (i) and (iii) we get that $\nabla^2 P^*$ and $\partial_t \nabla^2 P^* = \nabla^2 \partial_t P^*$ are bounded in time-space. So for each $t \in [0, T)$ we have that $\nabla P^*(t, \cdot)$ and $\partial_t \nabla P^*(t, \cdot) = \nabla \partial_t P^*(t, \cdot)$ have at most linear growth, which implies $P^*(t, \cdot)$ and $\partial_t P^*(t, \cdot)$ have at most quadratic growth. Since $\mathbf{y}(t, \cdot), \tilde{\mathbf{y}}(t, \cdot) \in L^2(\Omega; \mathbb{R}^3)$ for any $t \in [0, T)$ we infer that

$$P^*(t, \mathbf{y}(t, \cdot)), P^*(t, \tilde{\mathbf{y}}(t, \cdot)), \partial_t P^*(t, \mathbf{y}(t, \cdot)), \partial_t P^*(t, \tilde{\mathbf{y}}(t, \cdot)) \in L^1(\Omega; \mathbb{R}^3).$$

The proof continues as in the proofs of Proposition 3.2 and Theorem 3.1, except now we only have $\nabla P(t, \cdot) \in L^2(\Omega; \mathbb{R}^3)$ (instead of L^∞), which leads

to $\mathbf{u} \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^3))$ (as opposed to $\mathbf{u} \in L^\infty(0, T; W^{1,\infty}(\Omega; \mathbb{R}^3))$ as before). Also, there is no need for the cut-off argument anymore. $\square$

Since we now do not have to keep track of the compactness of the sets $\nabla \tilde{P}(t, \Omega)$, we can formulate a uniqueness result for renormalized relaxed solutions:

Corollary 3.6. *Renormalized relaxed Lagrangian solutions satisfying the assumptions (i)-(ii) from Theorem 3.5 are unique among renormalized relaxed Lagrangian solutions with the same initial data.*

Example 3.7. Let $\Omega := B(0, 1)$, $\mathbf{a} \in \mathbb{R}^3$ and $P_0(x) = |x|^2/2 + \mathbf{a} \cdot x$. It is easy to see that

$$P(t, x) = \frac{1}{2}|x|^2 + e^{tJ}\mathbf{a} \cdot x$$

is a time-dependent (unless $\mathbf{a} = 0$) stationary (in the sense that the total wind velocity $\mathbf{u} = 0$) solution (starting at P_0). Clearly, $\nabla P(t, \Omega) = B(e^{tJ}\mathbf{a}, 1)$ for all $t \geq 0$. Then we use (1.2) to compute $P^*(t, X) = |X - e^{tJ}\mathbf{a}|^2/2$ for all $X \in B(e^{tJ}\mathbf{a}, 1)$; naturally, $P^*(t, \cdot)$ admits an extension (same formula) to the whole $\mathbb{R}^3$. It is trivial to check that (i)-(iii) are satisfied for any $\mathcal{R} > 1 + |\mathbf{a}|$, $\delta > 0$ sufficiently small, and any $T > 0$. Thus, according to Theorem 3.1, we have that for any renormalized relaxed Lagrangian solution $(\tilde{P}, \tilde{\sigma})$ with $\cup_{t \in [0, T)} \nabla \tilde{P}(t, \Omega) \subset B(0, \mathcal{R})$ the inequality (3.2) is satisfied.

Interestingly enough, it is easy to see that Theorem 3.5 applies as well. Thus, the restriction $\cup_{t \in [0, T)} \nabla \tilde{P}(t, \Omega) \subset B(0, \mathcal{R})$ is unnecessary for concluding (3.2). In particular, that means that the solution P is unique among all renormalized relaxed Lagrangian solutions (not necessarily compactly supported in dual space).

Remark 3.8. It is not difficult to see that the boundedness of $\nabla^3 P^*$ is, in light of the other assumptions (of Theorem 3.1 or Theorem 3.5), equivalent to $\nabla \mathbf{u} \in L^\infty((0, T) \times \Omega; \mathbb{R}^3 \times \mathbb{R}^3)$.

4. SG on the 2-d torus. In [2] the authors prove existence of weak Eulerian solutions for SG on the 2-d flat torus $\mathbb{T}^2$ under the requirement that the dual-space densities ρ stay bounded away from zero and infinity. In [2] Section 5 they show that their weak Eulerian solutions yield weak Lagrangian solutions in the spirit of [7].

Definition 4.1. Let $P_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be convex and such that $P_0 - |\text{Id}|^2/2$ is periodic. Let $r \in [1, \infty)$ and $P : [0, T) \times \Omega \rightarrow \mathbb{R}$ satisfy

$$P \in L^\infty([0, T); W_{loc}^{1,\infty}(\mathbb{R}^2)) \cap C([0, T); W_{loc}^{1,r}(\mathbb{R}^2)), \quad (4.1)$$

$$P(t, \cdot) \text{ is convex and } P(t, \cdot) - |\text{Id}|^2/2 \text{ is periodic for each } t \in [0, T). \quad (4.2)$$

Let $F : [0, T) \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a Borel map such that

$$F \in C([0, T); L_{loc}^r(\mathbb{R}^2; \mathbb{R}^2)). \quad (4.3)$$

Then the pair (P, F) is called a weak Lagrangian solution of (1.1) in $[0, T) \times \mathbb{T}^2$ if

- i. $F(0, x) = x$, $P(0, x) = P_0(x)$ for a.e. $x \in \mathbb{T}^2$,
- ii. for any $t > 0$ the mapping $F(t, \cdot) : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is Lebesgue measure preserving, in the sense that $F_{t\#}\chi = \chi$;
- iii. There exists a Borel map $F^* : [0, T) \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that for every $t \in (0, T)$ the map $F^*(t, \cdot) : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is Lebesgue measure preserving: $F_{t\#}^*\chi = \chi$, and satisfies $F_t^* \circ F_t(x) = x$ and $F_t \circ F_t^*(x) = x$ for a.e. $x \in \mathbb{T}^2$;
- iv. The function

$$Z(t, x) = \nabla P(t, F(t, x)) \quad (4.4)$$

is a distributional solution of

$$\begin{aligned} \partial_t Z(t, x) &= J[Z(t, x) - F(t, x)] && \text{in } [0, T) \times \mathbb{T}^2, \\ Z(0, x) &= \nabla P_0(x) && \text{in } \mathbb{T}^2. \end{aligned} \quad (4.5)$$

A version of Theorem 3.1 can be similarly proved in the periodic case, namely:

Theorem 4.2. *Let (P, F) be a weak Lagrangian solution for (1.1) in $[0, T) \times \mathbb{T}^2$. Assume further:*

- (i) *There exists $\lambda \in (0, \infty)$ such that*

$$\lambda I_2 \leq \nabla^2 P^*(t, x) \text{ for each } t \in [0, T) \text{ and all } x \in \mathbb{T}^2. \quad (4.6)$$

- (ii) $P^* \in W^{1,\infty}(0, T; W^{2,\infty}(\mathbb{T}^2)) \cap L^\infty(0, T; W^{3,\infty}(\mathbb{T}^2))$.

Then there exists a real constant C depending only on P such that for any weak Lagrangian solution $(\tilde{P}, \tilde{F})$ for (1.1) we have

$$\|\nabla P(t, \cdot) - \nabla \tilde{P}(t, \cdot)\|_{L^2(\mathbb{T}^2; \mathbb{T}^2)} \leq C \|\nabla P_0 - \nabla \tilde{P}_0\|_{L^2(\mathbb{T}^2; \mathbb{T}^2)} \quad (4.7)$$

for all $t \in [0, T)$.

Proof. The proof of Theorem 3.1 carries over, once we check that the integral

$$\int_{\mathbb{T}^2} [P^*(t, \tilde{\mathbf{y}}(t, x)) - P^*(t, \mathbf{y}(t, x)) - x \cdot [\tilde{\mathbf{y}}(t, x) - \mathbf{y}(t, x)] dx$$

is well-defined (on the torus). For that it suffices to add and subtract $|\mathbf{y}|^2/2$ and $|\tilde{\mathbf{y}}|^2/2$ in the integrand, then use the parallelogram identity together with the fact that $P^*(t, x) - |x|^2/2$, $\tilde{P}^*(t, x) - |x|^2/2$, $\mathbf{y}(t, x) - x$ and $\tilde{\mathbf{y}}(t, x) - x$ are periodic as functions of x . $\square$

Remark 4.3. *In the periodic case we can have that the support of the measure $\rho(t, \cdot)$ is the whole torus, so $\nabla P(t, \mathbb{T}^2) = \mathbb{T}^2$ if $\rho(t, \cdot) > 0$ a.e. on $\mathbb{T}^2$. Since (i) from Theorem 4.6 guarantees $\rho(t, \cdot) \geq \lambda$ a.e. on $\mathbb{T}^2$, we have that $\nabla P(t, \mathbb{T}^2) = \mathbb{T}^2$ for all $t \in [0, T)$. This makes the list of assumptions much shorter in Theorem 4.6 by comparison with Theorem 3.1, since in the latter $\nabla P(t, \Omega)$ will, in general, vary with t .*

Of course, in the case of $\mathbb{T}^2$, we also have a uniqueness result which reads:

Corollary 4.4. *Weak Lagrangian solutions satisfying the assumptions (i)-(ii) from Theorem 4.2 are unique among weak Lagrangian solutions with the same initial data.*

The proof is even simpler than the proof of Corollary 3.3, as there is no compactness of support issue for dual space solutions in this case.

Remark 4.5. *Finally, the velocity field $\mathbf{u}$ for the weak (Eulerian) solutions constructed by Ambrosio et al. [2] is shown in Section 5 of [2] to admit a regular Lagrangian flow F , and, consequently, (P, F) is a weak Lagrangian solution. Therefore, Theorem 4.4 applies to give uniqueness of those solutions as well.*

4.1. Local (in time) uniqueness for solutions with $C^{1,\alpha}(\mathbb{T}^2)$ initial densities. Fix $0 < \alpha \leq 1$ and let $\rho \in C^{1,\alpha}(\mathbb{T}^2)$ be a probability density on the 2D flat torus $\mathbb{T}^2$. It is known (e.g., [2]) that there exists a convex function $P \in C^{3,\alpha}(\mathbb{T}^2)$ such that $\nabla P_{\#}\chi = \rho$, where $\chi := \mathcal{L}^2|_{\mathbb{T}^2}$. Let us assume without loss of generality that the average of P (and of all the optimal map potentials in the sequel) over $\mathbb{T}^2$ is zero. According to Theorem 2.2 (iii) [2] and the many references therein, there exists a real-valued function $\mathcal{C} = \mathcal{C}(m, M, a)$ nondecreasing in each of its three real arguments such that if $0 < m \leq \rho \leq M < \infty$ we have

$$\|P\|_{C^{3,\alpha}(\mathbb{T}^2)} + \|P^*\|_{C^{3,\alpha}(\mathbb{T}^2)} \leq \mathcal{C}(m, M, \|\rho\|_{C^{1,\alpha}(\mathbb{T}^2)}) \quad (4.8)$$

and

$$[\mathcal{C}(m, M, \|\rho\|_{C^{0,\alpha}(\mathbb{T}^2)})]^{-1} I_2 \leq \nabla^2 P + \nabla^2 P^* \leq \mathcal{C}(m, M, \|\rho\|_{C^{0,\alpha}(\mathbb{T}^2)}) I_2. \quad (4.9)$$

Let us now state a result whose first part (about existence, bounds and regularity of the solution ρ) was proved by Loeper in [14] (Corollary 3.4.3) in slightly modified wording (and in the particular case $k = 1$). The second part, about the regularity of P and P^* , follows from the observations above.

Theorem 4.6. *Fix $0 < \alpha \leq 1$ and let $\rho_0 \in C^{1,\alpha}(\mathbb{T}^2)$ be a positive probability density on the 2D flat torus $\mathbb{T}^2$. Then there exists $T_{\rho_0} =: T > 0$ such that the dual-space problem (1.3)–(1.6) admits a solution $\rho \in L^\infty(0, T; C^{1,\alpha}(\mathbb{T}^2))$ such that*

$$\min_{\mathbb{T}^2} \rho_0 \leq \rho(t, \cdot) \leq \max_{\mathbb{T}^2} \rho_0 \text{ for each } t \in [0, T].$$

Moreover, $P, P^* \in L^\infty(0, T; C^{3,\alpha}(\mathbb{T}^2))$ and they satisfy and (4.8) and (4.9) with $m := \min \rho_0$ and $M := \max \rho_0$.

We can now use Theorem 4.2 to prove that this solution is unique in the class of renormalized relaxed Lagrangian solutions. Since the solution considered in [14] lies in dual space, what we meant in the previous sentence refers to the weak Lagrangian solution corresponding to it.

Corollary 4.7. *Under the assumptions in Theorem 4.6, the solution (P, F) is unique among weak Lagrangian solutions on the time interval $[0, T_{\rho_0}]$, where $\rho_0 =: \nabla P_0_{\#}\chi$.*

Proof. Clearly, (4.8) and (4.9) yield (i) and the boundedness of $\nabla^3 P^*$ from (ii) (we refer to the list of assumptions of Theorem 4.2). What is left is to prove the remainder of the time-regularity conditions spelled out in (ii). This basically follows from the argument used in the proof of [2] Lemma 3.5 (i). For any matrix $A = [a_{ij}]$ we denote by $M(A)$ the cofactor matrix of A . Let us start from the observation that $\det \nabla^2 P^*(t, x) = \rho(t, x)$, so, as in said proof, we have

$$\frac{\rho(s, x) - \rho(t, x)}{s - t} = \sum_{i,j=1}^2 m_{i,j}(s, t; x) \partial_{ij} \left(\frac{P^*(s, x) - P^*(t, x)}{s - t} \right), \quad (4.10)$$

where

$$m_{ij}(s, t; x) := \int_0^1 M_{i,j}(\tau \nabla^2 P^*(s, x) + (1 - \tau) \nabla^2 P^*(t, x)) d\tau. \quad (4.11)$$

We regard (4.10) as an equation in the unknown $[P^*(s, x) - P^*(t, x)]/(s - t)$ (as a function of x for $s \neq t$ fixed). From (4.9) we deduce there exists $\lambda \in (0, 1)$ such that

$$\lambda I_2 \leq M(\tau \nabla^2 P^*(s, x) + (1 - \tau) \nabla^2 P^*(t, x)) \leq \lambda^{-1} I_2 \quad (4.12)$$

for all i, j, s, t, x . By Theorem 4.6, $\nabla^2 P^*(t, \cdot)$ is $C^{1,\alpha}$ uniformly with respect to time; it follows that the coefficients of the elliptic PDE (4.10) are $C^{1,\alpha}$ uniformly with respect to time. But (4.11), (4.12) also show that the PDE is uniformly elliptic on $\mathbb{T}^2$, with ellipticity constants independent of t, s . Thus, there exists a real constant C depending on

$$\left\| \frac{\rho(s, \cdot) - \rho(t, \cdot)}{s - t} \right\|_{L^\infty(0, T; C^{0,\alpha}(\mathbb{T}^2))}$$

such that

$$\left\| \frac{P^*(s, \cdot) - P^*(t, \cdot)}{s - t} \right\|_{L^\infty(0, T; C^{2,\alpha}(\mathbb{T}^2))} \leq C. \quad (4.13)$$

Now, ρ satisfies

$$\partial_t \rho(t, X) + J[X - \nabla P^*(t, X)] \cdot \nabla \rho(t, X) = 0 \text{ as distributions}, \quad (4.14)$$

which, in view of $\rho \in L^\infty(0, T; C^{1,\alpha}(\mathbb{T}^2))$ and $P^* \in L^\infty(0, T; C^{3,\alpha}(\mathbb{T}^2))$, implies

$$\partial_t \rho \in L^\infty(0, T; C^{0,\alpha}(\mathbb{T}^2)).$$

This shows that the real constant C from (4.13) may be chosen independently of $s, t \in [0, T]$, so the time-regularity conditions from (ii) of Theorem 4.2 are satisfied. $\square$

In fact, this is a classical solution for SG on $\mathbb{T}^2$, as the following proposition shows:

Proposition 4.8. *Any solution with the properties listed in Theorem 4.6 is a classical solution for SG on the time interval $[0, T_{\rho_0})$.*

Proof. Fix $\beta \in (0, \alpha)$, $t_0 \in (0, T_{\rho_0})$ and consider a sequence $\{h_n\}_n$ which converges to zero and such that $\{t_0 + h_n\}_n \subset (0, T_{\rho_0})$. From the uniform bound

$$\left\| \frac{\rho(t_0 + h_n, \cdot) - \rho(t_0, \cdot)}{h_n} \right\|_{L^\infty(0, T; C^{0, \alpha}(\mathbb{T}^2))} \leq C < \infty$$

we deduced in the proof of Theorem 4.7 we conclude that there is a subsequence $\{n_k\}_k$ such that

$$\frac{\rho(t_0 + h_{n_k}, \cdot) - \rho(t_0, \cdot)}{h_{n_k}} \rightarrow \partial_t \rho(t_0, \cdot) \text{ in } C^{0, \beta}(\mathbb{T}^2). \quad (4.15)$$

Just as in the proof of Theorem 4.7, we also have

$$\left\| \frac{P^*(t + h_{n_k}, \cdot) - P^*(t, \cdot)}{h_{n_k}} \right\|_{L^\infty(0, T; C^{2, \alpha}(\mathbb{T}^2))} \leq C, \quad (4.16)$$

which shows that possibly up to a further subsequence (not relabeled) we get

$$\frac{P^*(t_0 + h_{n_k}, \cdot) - P^*(t_0, \cdot)}{h_{n_k}} \rightarrow \theta(t_0, \cdot) \text{ in } C^{2, \beta}(\mathbb{T}^2). \quad (4.17)$$

From (4.10) and the regularity of P^* we see that $\theta(t_0, \cdot)$ solves

$$\partial_t \rho(t_0, y) = \sum_{i, j=1}^2 M_{i, j}(\nabla^2 P^*(t_0, y)) \partial_{ij} \theta(t_0, y),$$

which is also uniformly elliptic. Since (4.17) implies $\int_{\mathbb{T}^2} \theta(t_0, x) dx = 0$, and the last displayed equation admits a unique solution with prescribed average, we deduce that the limit in (4.17) is independent of the extracted subsequence.

Thus,

$$\frac{P^*(t_0 + h_n, \cdot) - P^*(t_0, \cdot)}{h_n} \rightarrow \theta(t_0, \cdot) \text{ in } C^{2, \beta}(\mathbb{T}^2) \quad (4.18)$$

and since $\{h_n\}_n$ is arbitrary, we conclude that P^* is differentiable at t_0 and $\theta(t_0, y) = \partial_t P^*(t_0, y)$ for all $y \in \mathbb{T}^2$; (4.18) also implies that $\nabla P^*(t_0, y)$ is differentiable at t_0 . Since t_0 is arbitrary in $[0, T_{\rho_0})$, we have that ∇P^* is differentiable in time everywhere in $(0, T_{\rho_0}) \times \mathbb{T}^2$. We next claim that it follows from $\nabla P(t, \nabla P^*(t, y)) = y$ (for all $t \in [0, T_{\rho_0})$) that ∇P is differentiable in time; indeed, we have that

$$\begin{aligned} \nabla P(s, \nabla P^*(s, y)) - \nabla P(s, \nabla P^*(t, y)) &= y - \nabla P(s, \nabla P^*(t, y)) \\ &= \nabla P(t, \nabla P^*(t, y)) - \nabla P(s, \nabla P^*(t, y)). \end{aligned}$$

It follows

$$\begin{aligned} & \nabla P(t, \nabla P^*(t, y)) - \nabla P(s, \nabla P^*(t, y)) \\ &= \int_0^1 (\nabla^2 P)(s, (1 - \tau)\nabla P^*(t, y) + \tau\nabla P^*(s, y)) d\tau \cdot [\nabla P^*(s, y) - \nabla P^*(t, y)]. \end{aligned} \quad (4.19)$$

We now use (4.8) and $\nabla P^*(t, \nabla P(t, x)) = x$ for all $x \in \mathbb{T}^2$ and $t \in [0, T_{\rho_0})$ to conclude

$$\nabla^2 P^*(t, \nabla P(t, x)) \nabla^2 P(t, x) = I_2 = \nabla^2 P(t, x) \nabla^2 P^*(t, \nabla P(t, x)). \quad (4.20)$$

As in the beginning of the proof of Proposition 3.2, we can show here as well that ∇P is continuous in time-space, so (4.20) (along with the regularity of P^*) guarantees that $\nabla^2 P$ is jointly continuous in time-space. This shows that the integral in the right hand side of (4.19) converges to $\nabla^2 P(t, \nabla P^*(t, y))$ as $s \rightarrow t$; since ∇P^* is differentiable in time (as proved above), (4.19) implies that $\nabla P(\cdot, \nabla P^*(t, y))$ is differentiable at t for any $y \in \mathbb{T}^2$. In particular, we can choose $y = \nabla P(t, x)$ to finish proving the claim about the time-differentiability of ∇P . So now, for all $y \in \mathbb{T}^2$, we can differentiate $\nabla P(t, \nabla P^*(t, y)) = y$ with respect to t to obtain

$$\partial_t \nabla P(t, \nabla P^*(t, y)) + \nabla^2 P(t, \nabla P^*(t, y)) \partial_t \nabla P^*(t, y) = 0,$$

i.e. (upon setting $y = \nabla P(t, x)$)

$$\partial_t \nabla P(t, x) + \nabla^2 P(t, x) (\partial_t \nabla P^*)(t, \nabla P(t, x)) = 0 \text{ pointwise in } (0, T_{\rho_0}) \times \mathbb{T}^2.$$

Finally, we use (3.3) to replace

$$(\partial_t \nabla P^*)(t, \nabla P(t, x)) = \mathbf{u}(t, x) - \nabla^2 P^*(t, \nabla P(t, x)) J[\nabla P(t, x) - x]$$

and use (4.20) to conclude, upon setting $X(t, x) := \nabla P(t, x)$, that (1.1) is satisfied in the classical sense. $\square$

Remark 4.9. *As in the previous sections, similar results can be proved for renormalized relaxed Lagrangian solutions. One could, of course, define them on the torus by mimicking their definition from the 3D case.*

5. Stability of stationary solutions for 3DSG. Some of the results in this paper apply to a class of solutions whose velocity trivially satisfies the boundedness of the Jacobian assumption (see Remark 3.8). These are solutions that can explicitly be shown to satisfy the regularity required of the strong solution in order for the inequality (3.2) to hold. “Stationary” in this context only means $\mathbf{u} \equiv 0$, which does not exclude the possibility that P be time-dependent. An obvious time-independent solution is given by

$$P(t, x) = \frac{1}{2}(x_1^2 + x_2^2) + g(x_3) \text{ for all } t \geq 0$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function (so that the Cullen-Purser stability condition is satisfied). Note that this is a classical solution even if g is not differentiable everywhere.

Assume Ω is simply connected. Next we shall endeavor to find all the stationary solutions satisfying the Cullen-Purser stability condition. Setting $\mathbf{u} = 0$ in the first equation of (1.1) yields the following equation for $X(t, x) := \nabla P(t, x)$:

$$\partial_t X(t, x) = J[X(t, x) - x], \quad X(0, \cdot) = \nabla P_0.$$

The solution is

$$X_1(t, x) = (\cos t)\partial_1 P_0(x) - (\sin t)\partial_2 P_0(x) + (1 - \cos t)x_1 + (\sin t)x_2, \quad (5.1)$$

$$X_2(t, x) = (\sin t)\partial_1 P_0(x) + (\cos t)\partial_2 P_0(x) - (\sin t)x_1 + (1 - \cos t)x_2, \quad (5.2)$$

$$X_3(t, x) = \partial_3 P_0(x).$$

Assume for now that P_0 is at least C^2 and let us check first that X is a gradient (with respect to x). We start by comparing

$$\partial_2 X_1 = \cos t \partial_{12} P_0 - \sin t \partial_{22} P_0 + \sin t, \quad \partial_1 X_2 = \sin t \partial_{11} P_0 + \cos t \partial_{12} P_0 - \sin t$$

and see that equality holds for all $x \in \Omega$ and all $t \in [0, T]$ (for some $T > 0$) if and only if

$$\partial_{11} P_0 + \partial_{22} P_0 \equiv 2 \text{ in } \Omega. \quad (5.3)$$

Likewise, we compare

$$\partial_3 X_1 = \cos t \partial_{13} P_0 - \sin t \partial_{23} P_0 \text{ to } \partial_1 X_3 = \partial_{13} P_0$$

and

$$\partial_3 X_2 = \sin t \partial_{13} P_0 + \cos t \partial_{23} P_0 \text{ to } \partial_2 X_3 = \partial_{23} P_0$$

to see that for equalities to hold inside a nondegenerate time interval one must necessarily have $\partial_{13} P_0 = \partial_{23} P_0 = 0$, so $\partial_3 P_0(x) = f(x_3)$ (independent of x_1, x_2). So,

$$P_0(x) = p_0(x_1, x_2) + g_0(x_3), \quad (5.4)$$

and for P_0 to be convex we necessarily must have that p_0 and g_0 are both convex on Ω (if we regard them as functions of three variables, even though they are independent of some). Note further that $\partial_i P_0 = \partial_i p_0$ for $i = 1, 2$ and (5.3) becomes

$$\partial_{11} p_0 + \partial_{22} p_0 \equiv 2 \text{ in } \Omega. \quad (5.5)$$

Thus, we conclude that the decomposition (5.4) and the equation (5.5) ensure that $X(t, \cdot)$ is the gradient of a function $P(t, \cdot)$ which is analytic in time and in space. Further conditions may need to be imposed to guarantee that P is convex on Ω . The fact that $\partial_3 P(t, x) = \partial_3 P_0(x) = g'_0(x_3)$ shows that

$$P(t, x) = p(t, x_1, x_2) + g_0(x_3),$$

where $p(0, x_1, x_2) = p_0(x_1, x_2)$. Let us, for convenience, denote $x := (x_1, x_2)$ in what follows, since p depends only on these coordinates. Likewise, ∇ or ∇^2 will be used to denote operators on $\mathbb{R}^2$. We venture to identify necessary and sufficient conditions for $p(t, \cdot)$ to be convex. We use the expressions for X_1 , X_2 and replace $\partial_{11}p_0 = 2 - \partial_{22}p_0$ (due to (5.5)) to find

$$\partial_{11}p = -\cos t \partial_{22}p_0 - \sin t \partial_{12}p_0 + 1 + \cos t, \quad (5.6)$$

$$\partial_{22}p = \sin t \partial_{12}p_0 + \cos t \partial_{22}p_0 + 1 - \cos t, \quad (5.7)$$

and

$$\partial_{12}p = \cos t \partial_{12}p_0 - \sin t \partial_{22}p_0 + \sin t. \quad (5.8)$$

Since the convexity of $p(t, \cdot, \cdot)$ for $t \in [0, T]$ is equivalent to $\partial_{ii}p \geq 0$ ($i = 1, 2$), and $\partial_{11}p \partial_{22}p - (\partial_{12}p)^2 \geq 0$ in $[0, T] \times \Omega$, we next look for necessary and sufficient conditions on p_0 such that these inequalities are satisfied for all $t \in [0, T]$. Denote $\Delta_0 := (1 - \partial_{22}p_0)^2 + (\partial_{12}p_0)^2$ and use (5.5) to see that $\Delta_0 = 1 - \det \nabla^2 p_0$. Since p_0 is convex, we deduce that

$$0 \leq \Delta_0(x) = 1 - \det \nabla^2 p_0(x) \leq 1 \text{ for all } x \in \Omega. \quad (5.9)$$

If $\partial_{22}p_0 = 1$ and $\partial_{12}p_0 = 0$ in $A \subset \Omega$, then $\partial_{11}p(t, \cdot) = \partial_{22}p(t, \cdot) = 1$ and $\partial_{12}p(t, \cdot) = 0$ in A for all $t \geq 0$, so the desired inequalities are trivially satisfied in A . In $\Omega \setminus A$ we have $\Delta_0 > 0$ and $1 > \det \nabla^2 p_0 \geq 0$. For all $x \in \Omega \setminus A$ we let $\omega(x) \in \mathbb{R}$ be such that

$$\sin \omega(x) = \frac{\partial_{22}p_0(x) - 1}{\sqrt{\Delta_0(x)}} \text{ and } \cos \omega(x) = \frac{\partial_{12}p_0(x)}{\sqrt{\Delta_0(x)}}.$$

So, from (5.6) and (5.7) we see that

$$\partial_{11}p(t, x) = 1 - \sqrt{\Delta_0(x)} \sin(t + \omega(x)), \quad \partial_{22}p(t, x) = 1 + \sqrt{\Delta_0(x)} \sin(t + \omega(x)), \quad (5.10)$$

while (5.8) becomes

$$\partial_{12}p = \sqrt{\Delta_0(x)} \cos(t + \omega(x)).$$

Since $0 \leq \Delta_0 \leq 1$, it follows that $\partial_{ii}p(t, \cdot) \geq 0$ in Ω for $i = 1, 2$ and

$$\det \nabla^2 p(t, x) = 1 - \Delta_0(x) = \det \nabla^2 p_0(x). \quad (5.11)$$

Thus, $p(t, \cdot)$ is convex for all $t \geq 0$ if and only if p_0 is convex. Let us also record that (5.10) also implies that the trace of $\nabla^2 p$ is conserved in time, i.e.

$$\partial_{11}p(t, x) + \partial_{22}p(t, x) = 2 \text{ for all } x \in \Omega \text{ and all } t \geq 0. \quad (5.12)$$

In fact, since we wish to prove a stability result for these stationary solutions by employing Theorem (3.1), let us assume p_0 is uniformly convex, i.e. (in light of (5.5)) there exists $d_0 \in (0, 1]$ such that

$$\det \nabla^2 p_0 \geq d_0 \text{ in } \Omega. \quad (5.13)$$

In light of (5.12) and (5.11) we deduce that $p(t, \cdot)$ is uniformly convex, uniformly with respect to $t \geq 0$. Note that the computations above can be performed in any open ball $B(0, R)$ which contains Ω , including the whole $\mathbb{R}^3$. Indeed, if p_0 is a convex function defined on $\mathbb{R}^2$ and such that (5.5) and (5.13) are satisfied, it suffices to complement that with a function $g_0 : \mathbb{R} \rightarrow \mathbb{R}$ such that g'' is bounded away from zero and infinity to see by (5.9), (5.11) and (5.12) that condition (i) from Theorem 3.1 is satisfied by $P(t, x) = p(t, x_1, x_2) + g_0(x_3)$ in $\mathbb{R}^3$.

Let us summarize what we have just proved.

Proposition 5.1. *Let Ω' be convex such that $\Omega \subset \Omega'$ and let $P_0 : \Omega' \rightarrow \mathbb{R}$ be convex. Then:*

- (i) *A stationary Cullen-Purser solution for SG starting at P_0 exists on some time interval $[0, T]$ if and only if $P_0(x) = p_0(x_1, x_2) + g_0(x_3)$ for some convex function $p_0 = p_0(x_1, x_2)$ such that $\text{tr} \nabla^2 p_0(x_1, x_2) = 2$ for any $(x_1, x_2) \in \text{proj}_{(x_1, x_2)} \Omega$, and some convex function $g_0 : \text{proj}_{x_3} \Omega' \rightarrow \mathbb{R}$. In this case, T can be taken $T = \infty$.*
- (ii) *The solution from (i) is unique (among stationary Cullen-Purser solutions) and has the form $P(t, x) = p(t, x_1, x_2) + g_0(x_3)$ for some convex function p which is analytic in space-time.*
- (iii) *If, besides the conditions on p_0 from (i), we also have that p_0 is uniformly convex, then $p(t, \cdot)$ is uniformly convex, uniformly with respect to $t \in [0, \infty)$.*
- (iv) *If ∇P_0 is bounded, then so is ∇P , uniformly in time and space.*

We next prove the following global result.

Theorem 5.2. *Let $p_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be uniformly convex and such that $\text{tr} \nabla^2 p_0 \equiv 2$ in $\mathbb{R}^2$. Let $g_0 \in C^3(\mathbb{R})$ be such that $c \leq g_0'' \leq c^{-1}$ for some $c \in (0, 1)$. Then the stationary Cullen-Purser solution for SG starting at $P_0(x) := p_0(x_1, x_2) + g_0(x_3)$ satisfies the conditions (i) (uniformly with respect to $t \in [0, \infty)$) and (ii) from Theorem 3.1 (with bounds independent of $t \in [0, \infty)$). Moreover, this solution is compactly supported.*

Proof. Condition (ii) (global in space) of Theorem 3.1 is given by Proposition 5.1 (iii). Since $p_0(x_1, x_2) - (x_1^2 + x_2^2)/2$ is harmonic in $\mathbb{R}^2$ we deduce that it is analytic. By (5.1) and (5.2) we deduce ∇P is analytic in time-space, periodic (of period 2π) in time, and there exists $\mathcal{R} \in (0, \infty)$ such that $\nabla P(t, \Omega) \subset B(0, \mathcal{R})$ for all $t \in [0, \infty)$. In particular, $\nabla P \in C^\infty([0, \infty) \times \mathbb{R}^3; \mathbb{R}^3)$ and so its inverse (with respect to x) ∇P^* enjoys the same regularity, along with time-periodicity. Thus, (i)-(iii) are satisfied for any $\mathcal{R} \in (0, \infty)$ sufficiently large. $\square$

Naturally, we can apply a “uniform” (with respect to time) version of Theorem 3.1 to infer:

Corollary 5.3. *Let p_0 and g_0 be and satisfy the same conditions as in Theorem 5.2. Then the stationary Cullen-Purser solution P originating at $P_0(x) := p_0(x_1, x_2) + g_0(x_3)$ is stable in the class of compactly supported renormalized relaxed Lagrangian solutions for (1.1). More precisely, for any time horizon $T \in (0, \infty]$ and any renormalized relaxed Lagrangian solution $(\tilde{P}, \tilde{\sigma})$*

such that $\nabla \tilde{P}(t, \Omega) \subset B(0, \tilde{\mathcal{R}}) \Subset \mathbb{R}^3$ for all $t \in [0, T)$, there exists a constant C depending only on P and $\tilde{\mathcal{R}}$ such that (3.2) holds for all $t \in [0, T)$.

We conclude by noticing that if $T \in (0, \infty)$, then weak Lagrangian solutions in the sense of Cullen and Feldman [7] do, in fact, satisfy that the sets $\nabla P(t, \Omega)$ stay inside a time-independent compact set for all $t \in [0, T)$ if $\nabla P_0(\Omega)$ is bounded; this can be easily seen from (2.5). So, we can also state:

Corollary 5.4. *Let p_0 and g_0 be and satisfy the same conditions as in Theorem 5.2. Then the stationary Cullen-Purser solution P originating at $P_0(x) := p_0(x_1, x_2) + g_0(x_3)$ is stable on $[0, T)$ (for all $T \in (0, \infty)$) in the class of all weak Lagrangian solutions for (1.1) with $\tilde{P}_0 \in W^{1,\infty}(\Omega)$. More precisely, for any time horizon $T \in (0, \infty)$, any $\mathcal{R} \in (0, \infty)$ and any weak Lagrangian solution $(\tilde{P}, \tilde{F})$ originating at some $\tilde{P}_0 \in W^{1,\infty}(\Omega)$ such that $\nabla \tilde{P}_0(\Omega) \subset B(0, \mathcal{R})$ there exists a constant C depending only on P , T and $\mathcal{R}$ such that (3.2) holds for all $t \in [0, T)$. In particular, the solution P is unique on $[0, \infty)$ in the class of weak Lagrangian solutions originating at P_0 .*

6. Uniqueness without the Cullen-Purser stability condition. We would like to emphasize here that for all the above results the convexity of $\tilde{P}$ is unnecessary. Roughly speaking, renormalized relaxed Lagrangian solutions enjoying the regularity prescribed for our strong solution P are unique even in the class of renormalized relaxed Lagrangian solutions *without* the Cullen-Purser stability condition.

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