Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons

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1 Introduction

In this paper, we construct several new families of Kähler-Ricci solitons and give examples of flowing through a singularity. Our solitons live on complex line bundles over $\mathbb{CP}^{n-1}$, $n \geq 2$, and complement the examples of Cao [Cao96, Cao97] and Koiso [Koi90]. Certain new solitons have initial condition equal to the metric cone $\mathbb{C}^n/\mathbb{Z}_k$, $k > n$, endowed with the quotient flat metric. Another new soliton can be combined with one of Cao's solitons to construct a noncompact Ricci flow that is smooth and self-similarly shrinking for $t < 0$, becomes a metric cone at at $t = 0$, and then continues by smooth, self-similar expansion for $t > 0$, with a change of topology. It has a unique singular point in space and time. We also present shrinking solitons on certain spaces with orbifold-type point singularities.

A solution $(M, g(t))$ of the Ricci flow

$$\frac{\partial}{\partial t} g(x,t) = -2Rc(g)(x,t) \qquad x \in M, \quad 0 \leq t < T, \tag{1}$$

is called a *Ricci soliton* if there exist scalars $\sigma(t)$ and diffeomorphisms ψ_t such that

$$g(t) = \sigma(t)\, \psi_t^*(g_0), \qquad 0 \leq t < T, \tag{2}$$

where $g_0 = g(0)$. For (2) to hold, it is necessary and sufficient that the identity

$$-2Rc(g_0) = \mathcal{L}_X g_0 + 4\lambda g_0 \tag{3}$$

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hold for some constant λ and some vector field X on M.[1] In fact X is the initial velocity field of ψ_t. By rescaling, one may assume that $\lambda \in \{-1, 0, 1\}$. These three cases correspond to solitons of shrinking, translating, and expanding type, respectively (see §2.1).

If X vanishes, a Ricci soliton is an Einstein metric. More generally, equation (3) is an elliptic system for the pair (g_0, X). Ricci solitons frequently arise as limits of parabolic dilations about singularities; see for instance [Ha93b, Ha95, Cao97]. Their study yields useful information about singularity formation and other things: for example, Li–Yau–Hamilton differential Harnack inequalities are exact on appropriate solitons; see [Ha88, Cao92, Ha93a].

Now suppose (M^{2n}, J, g_0) is a Kähler manifold. The basis of the Kähler-Ricci flow is the remarkable fact that the Ricci flow $g(t)$ remains Kähler with respect to the complex structure J. For Ricci solitons, where $g(t) = \sigma(t)\phi_t^*(g_0)$, $g(t)$ will also be holomorphic with respect to the complex structure $\phi_t^*(J)$. It is then natural to impose the constraint that $\psi_t^*(J) = J$, or equivalently, that the $(1,0)$ part of X is a holomorphic vector field:

$$\partial X^\beta / \partial \bar{z}^{\bar{\alpha}} = 0. \tag{4}$$

We shall see that (3) and (4) together are equivalent to the complex analogue of (3), namely

$$-2R_{\alpha\bar{\beta}} = \mathcal{L}_X h + 4\lambda h_{\alpha\bar{\beta}} \tag{5}$$

for a real vector field X. We call a metric satisfying (5) a *Kähler-Ricci solition.*

Of particular interest is the case where

$$X = \operatorname{grad} Q$$

for some real-valued function Q. A Ricci soliton with this property is called a *gradient Ricci soliton.* In the Kähler case, this implies that X is automatically holomorphic; in fact, (3) and (5) become equivalent. We call such a metric a *gradient Kähler-Ricci soliton.* (See §2.2 for details.)

Let us briefly survey some general results concerning the existence and uniqueness of Kähler-Ricci solitons in the compact case. Good background is the monograph [Tia00] or the introduction to [Tia97].

A Kähler-Ricci soliton (including a Kähler-Einstein metric) can only exist in the so-called *canonical case* — that is, if the first Chern class c_1 is positive, negative, or zero, and the Kähler class $[\omega]$ is parallel to c_1 (see §2.3). In this case, $[\omega(t)]$ evolves parallel to itself, and the metric (suitably renormalized) may be expected to approach an ideal representative of its Kähler class.

If c_1 is zero or negative, the Ricci flow exists forever and converges (after rescaling) to the unique Kähler-Einstein metric in the given Kähler class [Cao85] (see also [Yau77, Yau78, Aub76, Aub78]).

When $c_1 > 0$, the Ricci flow remains smooth until the volume goes to zero [Cao85]. What does the rescaled metric converge to? Because some positive manifolds do not carry a Kähler-Einstein metric, one is led to conjecture that

[1] The factor 4 corresponds to a time lapse of $1/4$.

after pullback by a diffeomorphism, the metric converges to a (non-Einstein) Kähler-Ricci soliton.

This conjecture is true for complex surfaces. The existence of positive Kähler-Einstein metrics on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ for $3 \le k \le 8$ was solved in [Tia87, TY87, Tia90], whereas the existence of Kähler-Ricci solitons on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ and $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ was proven in [Cao96, Koi90] and [WZ02] respectively. This covers all compact postive surfaces in the case $n = 2$. But for $n \ge 3$, the conjecture is false: there are positive Kähler manifolds of dimension $n = 3$ with no Kähler-Einstein metric and no holomorphic vector field, hence no Kähler-Ricci soliton [Tia97]. In such manifolds, singularity formation must embody a more subtle mechanism. More recently, a holomorphic obstruction to the existence of Kähler-Ricci solitons has been announced [TZ99]. For an analytic criterion relating the existence of Kähler-Ricci solitons to certain Moser-Trudinger inequalities, see [CTZ02].

A soliton on a compact Kähler manifold, if it exists, is unique modulo the identity component of the holomorphic automorphism group [TZ99, TZ00, TZ01], generalizing the classic result for Kähler-Einstein metrics [BM87].

The results of the present paper are also related to the construction of Kähler-Einstein metrics on open manifolds; see for example [Cal79, TY90, TY91, TY86, Hit79] and others.

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1.1 Ansatz and Method

In this paper we pursue an ansatz due to Cao [Cao96] to produce new solitons on line bundles over $\mathbb{CP}^{n-1}$.

First some nomenclature. Let L^ℓ denote the total space of the holomorphic line bundle over $\mathbb{CP}^{n-1}$ with twisting number $\ell \in \mathbb{Z}$, characterized by the equation $\langle c_1, [\Sigma]\rangle = \ell$, where c_1 is the first Chern class of the line bundle and $\Sigma \approx \mathbb{CP}^1$ is a positively oriented generator of $H_2(\mathbb{CP}^{n-1}; \mathbb{Z})$.

When $\ell \neq 0$, we can construct L^ℓ as follows. Let $\mathbb{Z}_k$ act on $\mathbb{C}^n\backslash\{0\}$ by $z \mapsto e^{2\pi i/k} z$. Then the complement of the zero-section of L^ℓ is biholomorphic to $(\mathbb{C}^n\backslash\{0\})/\mathbb{Z}_k$ with $k = |\ell|$. Thus a negative line bundle L^{-k} is obtained by gluing a $\mathbb{CP}^{n-1}$ into $(\mathbb{C}^n\backslash\{0\})/\mathbb{Z}_k$ at $z = 0$, whereas the positive line bundle L^k is is obtained by gluing in a $\mathbb{CP}^{n-1}$ at infinity. In particular L^{-1} (the tautological line bundle) is the blow-up of $\mathbb{C}^n$ at $z = 0$. In the negative case, the zero section is an isolated (non-perturbable) complex hypersurface of L^{-k}.

We impose a $U(n)$ symmetry according to the following ansatz. We call a soliton *trivial* if it is Einstein. Assume $n \ge 2$.

1.1 Ansatz g shall be a nontrivial $U(n)$-invariant gradient Kähler-Ricci soliton metric defined on $(\mathbb{C}^n\backslash\{0\})/\mathbb{Z}_k$.

In addition, we require the following boundary conditions:

(A) At $z = 0$, we shall see that $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ is always incomplete. We require that the metric completion is obtained either by adding a $\mathbb{CP}^{n-1}$ smoothly, adding a point smoothly (when $k = 1$), or adding an orbifold point (when $k \geq 2$).

(B) At $|z| = \infty$, we require either that $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ is complete, or that the metric completion is obtained by adding a $\mathbb{CP}^{n-1}$ smoothly.

These conditions yield complete Kähler-Ricci solitons, compact or noncompact with at most a point singularity.

Method Here is how we solve it. The rotational symmetry and the gradient condition allow us to reduce equation (3) to a single fourth-order ODE for the Kähler potential, whose solution can be expressed in terms of explicit integrals. Two of the constants of integration have no geometric significance and can be ignored. We are left with a two-parameter family of $U(n)$-invariant Kähler-Ricci gradient solitons defined locally in $\mathbb{C}^n$ (and hence in the quotient $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$). These computations, which are due to Cao, are outlined in §2, §3, and §4.

In sections §5, §6, and §8, we vary the parameters so that g is defined on all of $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ and matches boundary conditions (A) and (B). Several combinations have already been explored [Koi90, Cao96, Cao97]. See the table at the end of the section for a comprehensive list.

We are most interested in the case of L^ℓ, which contains a $\mathbb{CP}^{n-1}$ as zero-section. It is well known that under the Kähler-Ricci flow, the volume of a complex subvariety of M evolves in a fixed way depending only on the first Chern class. We apply this principle in §2.3 to establish the following restriction on the possibilities.

1.2 Lemma *Suppose $g(t)$ is a Kähler-Ricci soliton on $L = L^\ell$, $\ell \in \mathbb{Z}$. Then:*
(i) if $\ell > -n$, then $g(t)$ must be a shrinking soliton;
(ii) if $\ell = -n$, then $g(t)$ must be a translating soliton;
(iii) if $\ell < -n$, then $g(t)$ must be an expanding soliton.

Kähler Cones To interpret our results, let $\mathcal{C}^{n,p}$ denote $\mathbb{C}^n \backslash \{0\}$ endowed with the Kähler metric

$$h(z) := |z|^{2p-2} \left(\delta_{\alpha\bar{\beta}} + (p-1)|z|^{-2} \bar{z}^{\bar{\alpha}} z^{\beta}\right) dz^\alpha d\bar{z}^{\bar{\beta}}, \qquad z \neq 0, \tag{6}$$

where $p > 0$; this comes from the Kähler potential $P(z) = |z|^{2p}/p$. Let $\mathcal{C}^{n,k,p}$ denote the quotient by $\mathbb{Z}_k$. The reader may verify that that the metric completion $\overline{\mathcal{C}^{n,k,p}}$ is a metric cone with respect to the homotheties $z \mapsto az$, and that all $U(n)$-invariant Kähler cones on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ have this form up to a constant multiple. Note that $p = 1$ gives a flat metric, with an orbifold-type singularity at the origin if $k \geq 2$. More generally, the cone angle around the origin in each complex line is $2\pi p/k$, and the density ratio $\mathrm{Vol}_{\mathcal{C}^{n,k,p}}(B_1)/\mathrm{Vol}_{\mathbb{C}^n}(B_1)$ is p^n/k.

It will turn out that soliton metrics satisfying the completeness condition (B) are automatically *conelike at infinity*: that is, smoothly asymptotic to $\mathcal{C}^{n,k,p}$ at infinity for some p.

1.2 Results

We now explore the various cases in turn, including an overview of the already known cases.

Expanding solitons ($\mathbb{C}^n$ and $\ell < -n$) We begin with the expanding solitons found by Cao on $\mathbb{C}^n$. They provide a forward evolution of the initial cone $C^{n,p}$ for any $p > 0$. We discuss their construction briefly in §7.

1.3 Theorem (Cao [Cao97]) *Let $n \geq 2$. For each $p > 0$, there exists a complete homothetically expanding $U(n)$-invariant gradient Kähler-Ricci soliton*

$$\mathcal{Y}_t^{n,p} = (\mathbb{C}^n, g_p(t)), \qquad 0 < t < \infty, \qquad p > 0,$$

with conelike end modelled on $\mathcal{C} = \mathcal{C}^{n,p}$. It is unique up to isometry.

As $t \searrow 0$, the metric $g(t)$ converges smoothly to that of $\mathcal{C}$ on the complement of the zero section. The metric space $\mathcal{Y}_t^{n,p}$ converges to $\overline{\mathcal{C}}$ in the pointed Gromov–Hausdorff sense.

Our first result is an expanding soliton L^{-k} for any $k > n$, $p > 0$. It provides a forward evolution for $\mathcal{C}^{n,k,p}$. The proof is in §5.

1.4 Theorem (Expanding solitons) *Let $n \geq 2$. For each integer $k > n$ and real number $p > 0$, there exists a complete homothetically expanding $U(n)$-invariant gradient Kähler-Ricci soliton*

$$\mathcal{Z}_t^{n,k,p} = (L^{-k}, g(t)), \qquad 0 < t < \infty,$$

with conelike end modelled on $\mathcal{C} = \mathcal{C}^{n,k,p}$. It is unique up to isometry.

As $t \searrow 0$, the metric $g(t)$ converges smoothly to that of $\mathcal{C}$ on the complement of the zero section. The metric space $\mathcal{Z}_t^{n,k,p}$ converges to $\overline{\mathcal{C}}$ in the pointed Gromov–Hausdorff sense.

Besides the above flow, one can always take the quotient of $\mathcal{Y}_t^{n,p}$ by $\mathbb{Z}_k$ and consider it to be a forward evolution of $\mathcal{C}^{n,k,p}$. Let us call this quotient $\mathcal{Y}_t^{n,k,p}$. It has a persistent point singularity at $z = 0$ modelled on the orbifold $\mathbb{C}^n/\mathbb{Z}_k$.

Are these singular evolutions really Ricci flows? The singular set has (parabolic) codimension at least 4 for $\mathcal{Y}_t^{n,k,p}$ and at least 6 for $\mathcal{Z}_t^{n,k,p}$. Heuristically, one expects that a singular set of codimension greater than 2 cannot carry a concentration of curvature, and therefore does not disqualify the flow from being a weak solution. Further evidence comes from the static case, where the Gromov–Hausdorff limit of smooth Kähler-Einstein manifolds has singularities of codimension at least 4 [CCT97]. So pending a conclusive notion of weak solution, we regard both $\mathcal{Y}_t^{n,k,p}$ and $\mathcal{Z}_t^{n,k,p}$ as valid Ricci flows.

Let us discuss uniqueness of the initial value problem. For $0 < k \leq n$, the flow $\mathcal{Y}_t^{n,k,p}$ is the only known forward evolution of $\mathcal{C}^{n,k,p}$ (at least according to our Ansatz: see Proposition 9.3). In particular there is no known smooth forward evolution of $\mathcal{C}^{n,k,p}$ when $2 \leq k < n$. For $k > n$, there is nonuniqueness: $\mathcal{C}^{n,k,p}$ can flow either by $\mathcal{Y}_t^{n,k,p}$ or by $\mathcal{Z}_t^{n,k,p}$.

An interesting special case is when $p = 1$, so the initial condition is the flat cone $\mathbb{C}^n/\mathbb{Z}_k$. For $k > n$, besides remaining still, it can begin to move under the Ricci flow. This may seem surprising, since one might expect a flat metric to remain static under the Ricci flow. But this phenomenon is already familiar from other geometric heat flows (such Yang-Mills flow and harmonic map heat flow [Ilm95a, Ilm95b, Fan99, Gas00]). For example: consider 3 possible mean curvature evolutions starting at a cross in $\mathbb{R}^2$.

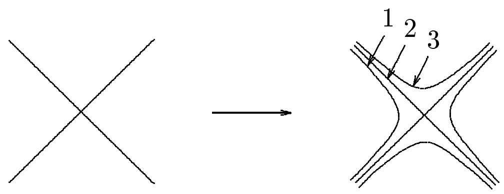

Figure 1. Three mean curvature flows

In fact, a stationary cone in $\mathbb{R}^n$ has a nonstatic evolution if and only if it is not area minimizing [Ilm95b]. So for Ricci-flat metric cones, we may regard the absence of a nonstatic Ricci flow as a dynamical criterion that substitutes for a minimizing condition.

Note that the cross actually has infinitely many mean curvature evolutions, since it may begin moving at an arbitrary time $t_{break} \in \mathbb{R}$. Analogously, we expect the singular flow $\mathcal{Y}_t^{n,k,p}$ to be *fragile* for $k > n$ in the sense that at any time the orbifold point can spontaneously desingularize to a tiny copy of the expanding soliton $\mathcal{Z}_t^{n,k,1}$, which then grows approximately self-similarly for awhile, and the flow has moved away from $\mathcal{Y}_t^{n,k,p}$.

Several problems concerning the Ricci evolution of Kähler cones appear in §10.

Translating solitons ($\ell = -n$) Translating solitons were found by Cao on L^{-n} and on $\mathbb{C}^n$ [Cao96] by compactifying $\mathbb{C}^n \backslash \{0\}$ at $z = 0$. In both cases, in the sphere S^{2n-1} at metric distance $d \gg 0$ from $z = 0$, the Hopf fibers $U(1) \cdot z \cong S^1$ have diameter $O(1)$, whereas the $\mathbb{CP}^{n-1}$-direction has diameter $O(\sqrt{d})$. As the latter behavior resembles the cigar soliton on $\mathbb{C}$, this asymptotic behavior is called *cigar-paraboloid.*

Shrinking solitons ($-n < \ell < 0$) By contrast with the expanding case, shrinking solitons are relatively hard to produce. For example, there is no $U(n)$-invariant shrinking soliton metric on $\mathbb{C}^n$ (Proposition 9.2). The following theorem yields a unique $U(n)$-invariant shrinking soliton on L^{-k} for each $0 < k < n$; the aperture p of the limiting cone at $t = 0$ is determined by n and k. It is proven in §6.

1.5 Theorem (Shrinking solitons) *Let $n \geq 2$. For each integer $0 < k < n$, there exists a complete homothetically shrinking $U(n)$-invariant gradient Kähler-Ricci soliton*

$$\mathcal{X}_t^{n,k} = (L^{-k}, g(t)), \qquad -\infty < t < 0,$$

with a conelike end. It is unique up to isometry.

The cone at spatial infinity has the form $\mathcal{C} = \mathcal{C}^{n,k,p}$ where $p = p(n,k)$. As $t \nearrow 0$, the metric $g(t)$ converges smoothly to that of $\mathcal{C}$ on the complement of the zero section. The metric space $\mathcal{X}_t^{n,k}$ converges to $\overline{\mathcal{C}}$ in the pointed Gromov–Hausdorff sense.

Flowing through the singularity By matching the cone angle at $t = 0$, we may regard $\mathcal{Y}_t^{n,k,q}$ as a continuation of $\mathcal{X}_t^{n,k}$ past the singular time. We concentrate on the case $k = 1$ since it has the most smoothness.

Set $q := p(n,1)$ and define

$$\mathcal{N}_t := \begin{cases} \mathcal{X}_t^{n,1} & t < 0 \\ \mathcal{C}^{n,q} & t = 0 \\ \mathcal{Y}_t^{n,q} & t > 0. \end{cases}$$

For $t < 0$, $\mathcal{N}_t$ is biholomorphic to L^{-1}, has a conelike end, and is homothetically shrinking. At $t = 0$ the singular fiber (the zero section) pinches off and $\mathcal{N}_t$ becomes a Kähler cone with an isolated singularity. Then the singularity smooths out: for $t > 0$, $\mathcal{N}_t$ is biholomorphic to $\mathbb{C}^n$, has a conelike end, and is homothetically expanding. A blowdown has occured.

Regularity across $t = 0$ is expressed by the following theorem, proven in §7.

1.6 Theorem (Flow-through) *The flow $\mathcal{N}_t$ is smooth in space-time except for an isolated point singularity at $t = 0$, and solves the Ricci flow where it is smooth.*

The smoothness should be understood as follows: for $t \leq 0$ the space-time manifold is

$$W_- := (L^{-1} \times (-\infty, 0]) \setminus (L_0 \times \{0\})$$

where L_0 is the zero-section, whereas for $t \geq 0$ it is

$$W_+ := (\mathbb{C}^n \times [0, \infty)) \setminus \{(0,0)\}.$$

These glue together along their common boundary $\mathbb{C}^n \setminus \{0\}$ at $t = 0$ to form a smooth space-time manifold W of real dimension $2n+1$ foliated by the spatial hypersurfaces $\mathcal{N}_t$. The assertion of the theorem is that $g(x,t)$ is smooth on W and solves the Ricci flow, and that the metric completion of the spacetime metric $g(x,t) + dt^2$ is obtained by adding just one point. The flow is continuous in the Gromov–Hausdorff topology.

This is the first known example for the Kähler-Ricci flow of what we call a *local singularity*: the singularity at $t = 0$ involves only part of the metric space $\mathcal{N}_0$. In the much-studied canonical case, the manifold shrinks off all at once, presumably to a point (at least, the volume of each analytic cycle goes simultaneously to zero), so a local singularity in the sense expressed here cannot occur. By contrast, in the non-canonical case, various parts shrink or grow at different rates, like jangling springs, so local singularities must occur in some cases. See Example 2.2.

Similar phenomena — shrinking and expanding solitons, flow-through with a point singularity at $t = 0$, and the relative difficulty of finding shrinking solitons as against expanding ones — occur for other geometric heat equations, including mean curvature flow [ACI95, AIV, Ilm95b], harmonic map heat flow [Ilm95a, Ilm95b, Fan99, Gas00], Yang-Mills heat flow [Gas00], and the equation $u_t = \Delta u + u^p$ [Tro87, BQ89, Lep90]. Related examples exist for nonlinear wave equations [STZ94] and nonlinear Schrödinger equations [Mer98].

Compact shrinking solitons There is no complete soliton on L^k, $k > 0$, (Proposition 9.1). Indeed, the soliton metric is always incomplete at $z = 0$, and completing it leads to compact solitons in two ways.

Let F_k denote the k-twisted projective line bundle $\mathbb{CP}^1 \hookrightarrow F_k \to \mathbb{CP}^{n-1}$ of Calabi [Cal82]. It is obtained from L^k by adding a $\mathbb{CP}^{n-1}$ at $z = 0$ (or equivalently from L^{-k} by adding a $\mathbb{CP}^{n-1}$ at infinity). Shrinking solitons are known to exist on F_k, $0 < k < n$ [Koi90, Cao96]. There can be no soliton on F_k for $k \geq n$ because then the $\mathbb{CP}^{n-1}$ at infinity shrinks, whereas the $\mathbb{CP}^{n-1}$ at $z = 0$ does not. See Lemma 1.2 and Example 2.2.

Let G_k denote the compact space $L^k \cup \{0\}$. Note that $G_k = \mathbb{CP}^n/\mathbb{Z}_k$ branched over $z = 0$ and over the $\mathbb{CP}^{n-1}$ at infinity (when $k \geq 2$). We have the following theorem.

1.7 Theorem (Shrinking Orbifolds) *For each $k > 0$, there is a $U(n)$-invariant shrinking gradient Kähler-Ricci soliton metric on G_k, $k > 0$.*

In the $k = 1$ case this is just the Fubini-Study metric on $\mathbb{CP}^n$. For $k \geq 2$ it has an orbifold-type singularity at $z = 0$ modelled on $\mathbb{C}^n/\mathbb{Z}_k$. For $k > n$ we expect that the flow is fragile: it can spontaneously desingularize to F_k via a small, but growing copy of (nearly) $\mathcal{Z}^{n,k,1}$ at the orbifold point.

1.3 Table of solitons

We list all known gradient Kähler-Ricci solitons arising as the metric completion of a $U(n)$-invariant ansatz on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$, $n \geq 2$, $k > 0$ under various closing conditions.

We have excluded familiar Einstein manifolds and examples with singularities of real codimension 2. We allow point singularities. (‡) indicates that solitons with point singularities can be obtained as a metric quotient, so they are not separately listed. (1) indicated a 1-parameter family of solitons parametrized by the cone aperture p. In the remaining cases there is a single soliton for each k and n.

Topology and type	**k-fold symmetry**	**Behavior as $z \to 0$**	**Behavior as $\|z\| \to \infty$**	**Source**
L^{-k} noncompact expanding	$n+1, \dots$	smooth $\mathbb{CP}^{n-1}$	conelike (1)	present paper
$\mathbb{C}^n$ (‡) noncompact expanding	1	smooth point	conelike (1)	[Cao97]
L^{-n} noncompact translating	n	smooth $\mathbb{CP}^{n-1}$	cigar-paraboloid	[Cao96]
$\mathbb{C}^n$ (‡) noncompact translating	1	smooth point	cigar-paraboloid	[Cao96]
L^{-k} noncompact shrinking	$1, \dots, n-1$	smooth $\mathbb{CP}^{n-1}$	conelike	present paper
F_k compact shrinking	$1, \dots, n-1$	smooth $\mathbb{CP}^{n-1}$	smooth $\mathbb{CP}^{n-1}$	[Koi90] [Cao96]
G_k compact shrinking	$2, \dots, n-1$	orbifold point	smooth $\mathbb{CP}^{n-1}$	present paper

2 Behavior of solitons

In this section we derive the equations for a gradient Kähler-Ricci soliton, establish the conservation laws that rule the growth of volume, and apply it to the bundle L^ℓ.

2.1 Ricci solitons

Let us show that (1) and (2) are equivalent to (3).

First suppose that $(M^n, g(t))$ is a solution of the Ricci flow (1) having the special form (2). We may assume $\psi_0 = \mathrm{id}$ and $\sigma(0) = 1$. Then

$$-2Rc(g_0) = \left.\frac{\partial g}{\partial t}\right|_{t=0} = \sigma'(0)\, g_0 + \mathcal{L}_Y g_0,$$

where Y_t is the family of vector fields generating the diffeomorphisms ψ_t. So g_0 satisfies (3) with $X = Y_0$.

Conversely, suppose that g_0 satisfies (3). Set

$$\sigma(t) := 1 + 4\lambda t,$$

and let ψ_t denote the diffeomorphisms generated by the time-varying vector fields

$$Y_t(x) := \frac{1}{\sigma(t)} X(x), \qquad \psi_0 = \mathrm{id}\,.$$

(We assume that X is a complete vector field. This will be always hold in our examples: see (22), (23).) Set

$$g(t) := \sigma(t)\psi_t^* g_0.$$

Then $g(t)$ has the form (2). To see that $g(t)$ is a solution of the Ricci flow (1), calculate

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \sigma'(t)\psi_t^*(g_0) + \sigma(t)\psi_t^*(\mathcal{L}_{Y_t} g_0) \\
&= \psi_t^*\,(4\lambda g_0 + \mathcal{L}_X g_0) \\
&= \psi_t^*(-2Rc(g_0)) \qquad\qquad \text{by (3)} \\
&= -2Rc(g).
\end{aligned}$$

This proves (1), and the desired equivalence.

2.2 Ricci solitons on Kähler manifolds

Let (M, g, J) be Kähler. Note that g and Rc are real and J-invariant, and so their $\alpha\beta$ and $\bar\alpha\bar\beta$ parts vanish. The Hermitian metric $h = h_{\alpha\bar\beta}\, dz^\alpha d\bar z^{\bar\beta}$ is given by

$$h := g - ig(J\cdot, \cdot) =: g - i\omega,$$

and the Hermitian Ricci tensor $R = R_{\alpha\bar\beta}\, dz^\alpha d\bar z^{\bar\beta}$ by

$$R := Rc - iRc(J\cdot, \cdot) =: Rc - i\rho.$$

Note that R is double the $\alpha\bar\beta$ component of Rc.[2]

Now consider the complex soliton equation, namely

$$-2R_{\alpha\bar\beta} = \mathcal{L}_X h + 4\lambda h_{\alpha\bar\beta}, \tag{5}$$

[2] Note that $\omega = (i/2)\, h_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^{\bar\beta}$ and $\rho = (i/2)\, R_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^{\bar\beta}$. For us $\sigma \wedge \nu := \sigma \otimes \nu - \nu \otimes \sigma$.

where X is a real vector field. By taking the real and complex parts, this is equivalent to the conjunction of

$$-2Rc = \mathcal{L}_X g + 4\lambda g \tag{3}$$

and

$$-2\rho = \mathcal{L}_X \omega + 4\lambda\omega. \tag{3i}$$

Now (3) and (3i) differ only by a term involving $\mathcal{L}_X J$, and by taking the difference, we quickly deduce that X is holomorphic. Conversely, if we assume (3) together with $\mathcal{L}_X J = 0$, we deduce (3i). So (5) is equivalent to (3) together with the statement that X is holomorphic, as claimed in the introduction. This is what we call a Kähler-Ricci soliton.

On the other hand, from (3) alone we can only deduce

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0,$$

which does not quite say that X is holomorphic.

In the case of a gradient soliton, $\nabla_\alpha X_\beta$ is symmetric and it follows that X is automatically holomorphic for any Ricci soliton. The execution: assume (3) and let $X^i = g^{ij} D_j Q$ with Q real. Then $\mathcal{L}_X g = 2D^2 Q$, and (3) becomes

$$-2Rc = 2D^2 Q + 4\lambda g. \tag{7}$$

The Hessian decomposes as

$$D^2 Q = \nabla^2 Q + \nabla\bar{\nabla} Q + \bar{\nabla}\nabla Q + \bar{\nabla}^2 Q,$$

where $D = \nabla + \bar{\nabla}$ is the decomposition of the connection into complex and anti-complex parts. Since (7) is real, it suffices to consider only the $\alpha\beta$ and $\alpha\bar{\beta}$ components. Thus (7) is equivalent to

$$-R_{\alpha\bar{\beta}} = 2\nabla_\alpha \bar{\nabla}_{\bar{\beta}} Q + 2\lambda h_{\alpha\bar{\beta}}, \qquad 0 = \nabla_\alpha \nabla_\beta Q. \tag{8}$$

The second equation yields by conjugation

$$\bar{\nabla}_{\bar{\alpha}} X^\beta = \bar{\nabla}_{\bar{\alpha}} \left(h^{\beta\bar{\gamma}} \nabla_{\bar{\gamma}} Q \right) = 0 \tag{9}$$

where X^β is the complex part $(X + iJX)/2$ of X. This says that the flow vector field of a gradient Ricci soliton on a Kähler manifold is holomorphic, as claimed.

In local coordinates $z = (z^1, \ldots, z^n)$ the metric can be represented by a Kähler potential $P : U \subseteq M \to \mathbb{R}$ via

$$h_{\alpha\bar{\beta}} = \frac{\partial^2 P}{\partial z^\alpha \, \partial \bar{z}^{\bar{\beta}}}. \tag{10}$$

The Hermitian Ricci curvature equals[3]

$$R_{\alpha\bar{\beta}} = -2 \frac{\partial^2}{\partial z^\alpha \, \partial \bar{z}^{\bar{\beta}}} \log\det h.$$

[3] This corresponds to $\rho = -i\partial\bar{\partial} \log\det h$.

The operator $\nabla\bar{\nabla}$ depends only on the complex structure and is given in coordinates by

$$\nabla_\alpha \bar{\nabla}_{\bar{\beta}} Q = \frac{\partial^2 Q}{\partial z^\alpha \, \partial \bar{z}^{\bar{\beta}}}.$$

So we can write the first equation of (8) as

$$\frac{\partial^2}{\partial z^\alpha \, \partial \bar{z}^{\bar{\beta}}} (\log \det h - Q - \lambda P) = 0.$$

We are interested in the case of expanding or shrinking solitons, so we take $\lambda = \pm 1$. Modifying P by an element in the kernel of $\nabla\bar{\nabla}$ if necessary, we may assume

$$Q = \log \det h - \lambda P. \tag{11}$$

The operator ∇ is independent of the connection on vector fields of type $(1,0)$, and (9) takes the form

$$\frac{\partial}{\partial \bar{z}^{\bar{\alpha}}} \left(h^{\beta\bar{\gamma}} \frac{\partial}{\partial \bar{z}^{\bar{\gamma}}} Q \right) = 0.$$

Inserting Q, we obtain

$$\frac{\partial}{\partial \bar{z}^{\bar{\alpha}}} \left[h^{\beta\bar{\gamma}} \frac{\partial}{\partial \bar{z}^{\bar{\gamma}}} (\log \det h - \lambda P) \right] = 0. \tag{12}$$

In turn, this equation implies (9) and (8). So h will be a gradient Kähler-Ricci soliton if and only if h possesses a Kähler potential P that satisfies (12). Note that this is a fourth-order equation in P.

2.3 Conservation laws

We recall the well-known principle that the Kähler class evolves in a fixed way, which yields an exact prediction of the volume of complex varieties at any time t. We give two examples, and prove Lemma 1.2.

On a Kähler manifold, the Ricci flow becomes

$$\frac{\partial \omega}{\partial t} = -2\rho.$$

Thus the Kähler class $[\omega]$ in $H^2(M, \mathbb{R})$ evolves by

$$[\omega(t)] = [\omega_0] - 2t[\rho] = [\omega_0] - 4\pi t c_1, \tag{13}$$

where $c_1 = c_1(TM)$ is the first Chern class of M and $[\rho] = 2\pi c_1$.

Let Σ be a complex curve in M. By integrating $\omega(t)$ over Σ, we find the following exact prediction for area,

$$|\Sigma|_t = |\Sigma|_0 - 4\pi t \langle c_1, [\Sigma] \rangle.$$

If c_1 is positive on Σ, then in principle, Σ will disappear at the *formal vanishing time*

$$T(\Sigma) := \frac{|\Sigma|_0}{4\pi\langle c_1, [\Sigma]\rangle}.$$

We call this phenomenon the *robot behavior of vanishing cycles.* Clearly $g(t)$ must develop a singularity at or before time $T(\Sigma)$.

Suppose that $g(t)$ is a Ricci soliton. Then

$$[\omega(t)] = \sigma(t)[\phi_t^*(\omega_0)] = \sigma(t)[\omega_0], \qquad \sigma(t) = 1 + 4\lambda t,$$

since a cohomology class is unchanged under a flow by diffeomorphisms. Combining with (13) (or just taking the homology class of (3i)) yields

$$\lambda[\omega_0] = -\pi c_1. \tag{14}$$

If M is compact, this falls under the canonical case, as mentioned in the introduction.

We give two examples.

2.1 Example (Proof of Lemma 1.2) Let L denote L^ℓ and let $L_0 \cong \mathbb{CP}^{n-1} \subseteq L$ be the zero-section. We compute the first Chern class of the complex manifold L. We have

$$TL|L_0 = TL_0 \oplus NL_0 \cong TL_0 \oplus L$$

where NL_0 is the normal bundle of L_0 in L. By the Whitney product formula, we have in $H^2(L_0, \mathbb{Z})$,

$$i^*(c_1(TL)) = c_1(TL|L) = c_1(TL_0) + c_1(L),$$

where $i : L_0 \hookrightarrow L$ is the embedding. Let $\Sigma \subseteq L_0$ be a $\mathbb{CP}^1$ that generates $H_2(L_0, \mathbb{Z}) \cong \mathbb{Z}$. Then

$$\begin{aligned}\langle c_1(TL), [\Sigma]\rangle &= \langle c_1(TL_0), [\Sigma]\rangle + \langle c_1(L), [\Sigma]\rangle \\ &= n + \ell.\end{aligned} \tag{15}$$

(A suitable reference is [GH78], Chapter 3, §3.) Integrating (14) over Σ, we obtain

$$\lambda|\Sigma|_0 = -\pi(n + \ell). \tag{16}$$

Thus the sign of λ must be opposite to the sign of $n + \ell$. This proves Lemma 1.2.

2.2 Example Let F_k be the k-twisted $\mathbb{CP}^1$ bundle over $\mathbb{CP}^{n-1}$ described above. How does it evolve with an arbitrary starting metric?

Let Σ_0 be a $\mathbb{CP}^1$ in the zero-section (at $z = 0$), let Σ_∞ be a $\mathbb{CP}^1$ in the ∞-section, and let T be a fiber. Then $H_2(F_k, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[\Sigma_0]$ and $[T]$, and one can compute

$$[\Sigma_0] + k[T] = [\Sigma_\infty].$$

From (15) we have for $c_1 = c_1(TF_k)$

$$\langle c_1, [\Sigma_0]\rangle = n - k, \qquad \langle c_1, [\Sigma_\infty]\rangle = n + k, \qquad \langle c_1, [T]\rangle = 2.$$

First consider the case $0 < k < n$. Then the areas of all three curves shrink at fixed rates; but which one shrinks off first? This depends on the initial metric only through the ratio of the areas $|\Sigma_0|$ and $|T|$, and there are three possibilities.

(1) If $|\Sigma_0|/|T| = (n-k)/2$ (canonical case), we believe that the entire manifold remains smooth until it shrinks to a point (a global singularity) and is asymptotic to a shrinking soliton on F_k as presented in [Koi90, Cao96].

(2) If $|\Sigma_0|/|T| > (n-k)/2$, then T shrinks before Σ_0. In fact, all the fibers have the same area, and it is reasonable to conjecture that the manifold remains smooth until the final instant, when it converges as a metric space to $\mathbb{CP}^{n-1}$.

(3) The evolution is most interesting in the case that $|\Sigma_0|/|T| < (n-k)/2$, so that Σ_0 shrinks first. We conjecture the following evolution: as the zero-section shrinks, there forms around it a local singularity modelled on the shrinking soliton $\mathcal{X}_t^{n,k}$ of Theorem 1.5. At the singular time, the zero-section shrinks to a point and the manifold develops a cone singularity. If $k = 1$, the singularity heals smoothly as in Theorem 1.6, the manifold becomes $\mathbb{CP}^n$, and eventually shrinks to a point with the Fubini-Study metric. If $2 \le k < n$, the space becomes G_k with a persistent orbifold singularity, and eventually shrinks off self-similarly to a point, with a singularity modelled on the soliton of Theorem 1.7. See §10 for further questions.

In the case $k > n$, Σ_0 expands in area, so the only possibility is that the fiber shrinks first as in case (2) above.

3 The ODE for rotationally invariant solitons

In this section we reduce the Kähler-Ricci soliton equations in the $U(n)$-invariant case to an ODE which we can solve nearly explicitly. Our calculations follow those of Cao [Cao96]. In the final part we give a lemma of Calabi that addresses the closing condition at $z = 0$.

3.1 $U(n)$-invariant metrics on $\mathbb{C}^n\backslash\{0\}$

Let h be a $U(n)$-invariant Kähler metric on $\mathbb{C}^n\backslash\{0\}$. Because $H^2(\mathbb{C}^n\backslash\{0\}) = 0$, by the ∂-Poincaré lemma we may assume that the representation (10) holds on all of $\mathbb{C}^n\backslash\{0\}$. Averaging the potential P with respect to action of $U(n)$, we may assume that P is also $U(n)$ invariant and thus $SO(2n)$ invariant. So we may write $P = P(r)$, where

$$r := \log|z|^2.$$

Define

$$\phi := P_r, \tag{17}$$

and compute from (10)

$$h = [e^{-r}\phi\,\delta_{\alpha\bar\beta} + e^{-2r}(\phi_r - \phi)\bar z^{\bar\alpha} z^{\beta}]\, dz^\alpha \otimes d\bar z^{\bar\beta}. \tag{18}$$

The associated Riemannian metric is

$$
\begin{aligned}
g &= \phi\,(g_{S^{2n-1}} - \eta\otimes\eta) + \phi_r(dr\otimes dr/4 + \eta\otimes\eta) \\
&= \phi\, g_{\mathrm{FS}} + \phi_r g_{\mathrm{cyl}},
\end{aligned} \tag{19}
$$

where η restricts to the 1-form $d\theta$ on each complex line, g_{FS} is the pullback of the standard Fubini-Study metric of $\mathbb{CP}^{n-1}$ to $\mathbb{C}^n\backslash\{0\}$, and g_{cyl} restricts to a round cylinder metric on each punctured complex line. Note that the size of g_{FS} is such that the area of a $\mathbb{CP}^1$ in $\mathbb{CP}^{n-1}$ is π.

From this formula it is clear that h is positive definite, hence a Kähler metric, if and only if

$$
\phi > 0 \qquad \text{and} \qquad \phi_r > 0. \tag{20}
$$

3.2 $U(n)$-invariant gradient Kähler-Ricci solitons

We now specialize (12) to the case of $U(n)$-invariant solitons on $\mathbb{C}^n\backslash\{0\}$. In this section we will solve the equation locally, and in the next three sections impose particular boundary conditions at $z=0$ and at infinity.

Substituting equations (17) and (18) into (12), we obtain after some computation the fourth-order ODE

$$
P_{rrrr} - 2\frac{P_{rrr}^2}{P_{rr}} + nP_{rrr} - (n-1)\frac{P_{rr}^3}{P_r^2} + \lambda(P_{rrr}P_r - P_{rr}^2) = 0. \tag{21}
$$

Thus one should in principle expect four arbitrary constants. We shall see that only two of these are geometrically significant.

Rather than work with (21) directly, it is easier to perform the following reduction: taking $Q = \log\det h - \lambda P$ as in (11), one observes that

$$
X^\beta = h^{\beta\bar\gamma}\frac{\partial Q}{\partial \bar z^{\bar\gamma}} = \frac{Q_r}{P_{rr}} z^\beta, \tag{22}
$$

so X is holomorphic if and only if there is $\mu\in\mathbb{C}$ such that

$$
Q_r = \mu P_{rr}. \tag{23}
$$

Since the iz^β direction is a Killing field, we may assume that $\mu\in\mathbb{R}$. This is a third-order equation for P, and μ is the first arbitrary constant. Taking $\phi = P_r$ as in (17), we write this as a second-order equation for ϕ:

$$
(\log\phi_r)_r + (n-1)(\log\phi)_r - \mu\phi_r - \lambda\phi - n = 0. \tag{24}
$$

(The reduction can also be accomplished by the standard method of rewriting (21) as an exact differential; indeed, one recovers (21) by solving (24) for μ and differentiating.)

We shall assume in what follows that $\mu\neq 0$, since otherwise $\operatorname{grad} Q = 0$ and (8) forces h to be an Einstein metric, which we have excluded.

Since $\phi_r > 0$, we can regard r as a function of ϕ and write $\phi_r = F(\phi)$. A computation yields that F satisfies the linear equation

$$F' + \left(\frac{n-1}{\phi} - \mu\right) F - (n + \lambda\phi) = 0. \tag{25}$$

We solve this to find

$$\phi_r = \phi^{1-n} e^{\mu\phi}(\nu + \lambda I_n + n I_{n-1}), \tag{26}$$

where ν is the second arbitrary constant and

$$I_m := \int \phi^m e^{-\mu\phi}\, d\phi.$$

After integrating by parts and doing a wee bit of algebra to find I_m, we write (26) as the separable first-order equation

$$\phi_r = F(\phi) := \frac{\nu e^{\mu\phi}}{\phi^{n-1}} - \frac{\lambda}{\mu}\phi - \frac{\lambda+\mu}{\mu^{1+n}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j \phi^{j+1-n}. \tag{27}$$

We may integrate this to get an implicit solution $r = r(\phi)$. (In low dimensions, the integration can even be carried out.) The third arbitrary constant arises during this integration and represents translation in r, hence dilation in z. The fourth arises when one integrates ϕ to get P. The third and fourth constants are not geometrically significant. The geometric degrees of freedom reside in μ and ν.

4 Closing Conditions

In this section we show how to add a $\mathbb{CP}^{n-1}$ to $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ at zero or at infinity, or add a point at zero. We postpone the conelike condition at infinity to sections §5 and §6.

The following lemma is an immediate consequence of (25). It gives an overview of the zeroes of F.

4.1 Lemma (Zeroes of F) *If $\phi \neq 0$ is a zero of F, then*

$$F'(\phi) = n + \lambda\phi. \tag{28}$$

Consequently,

(1) If $\lambda = 1$, then F has at most one positive zero a.

(2) If $\lambda = -1$, then F has at most two positive zeroes. It has at most one zero $0 < a \leq n$ and at most one zero $b \geq n$.

4.1 Adding a $\mathbb{CP}^{n-1}$ at zero or infinity

We derive the condition for adding a $\mathbb{CP}^{n-1}$ smoothly to $(\mathbb{C}^n\backslash\{0\})/\mathbb{Z}_k$. As we shall see, this corresponds to a positive, simple zero of F. The sign of F' determines whether it is added at $z=0$ or $|z|=\infty$.

Let us start with $z=0$. Assume that g is a $U(n)$-invariant expanding or shrinking gradient soliton defined on a neighborhood of the zero section of L^{-k}, $k>0$. Then ϕ exists for $-\infty<r<r_0$, and $\phi>0$, $\phi_r>0$ by (20). Let

$$a := \lim_{r\to-\infty}\phi(r).$$

By (19), any $\mathbb{CP}^1$ in the zero-section has area $|\Sigma|_g=\pi a>0$. Thus the Robot Principle (16) yields

$$\begin{array}{lll} a=k-n, & k>n & \text{expanding case,} \\ a=n-k, & 0<k<n & \text{shrinking case.} \end{array} \tag{29}$$

Since $\phi\searrow a$ as $r\to-\infty$, we deduce that

$$F(a)=0,$$

and F is positive on $(a,a+\varepsilon]$. Note from (27) that $F(a)=0$ is equivalent to the relation

$$\nu=\nu_a^\lambda(\mu):=e^{-\mu a}\left[\frac{\lambda}{\mu}a^n+\frac{\lambda+\mu}{\mu^{1+n}}\sum_{j=0}^{n-1}\frac{n!}{j!}(\mu a)^j\right]. \tag{30}$$

Conversely, let us examine the geometric effect of a zero of F satisfying

$$F(a)=0,\qquad a>0,\qquad \beta:=F'(a)>0.$$

Let ϕ be a solution of $\phi_r=F(\phi)$ with

$$\phi\searrow a\qquad\text{as}\qquad r\to-\infty \tag{31}$$

The ODE is conjugate by a smooth diffeomorphism $\psi=\Phi(\phi)$ to the equation $\psi_r=\beta\psi$, so ϕ has the form

$$\phi(r)=a+e^{\beta r}A(e^{\beta r}) \tag{32}$$

as $\phi\to a$, where A is a smooth function defined on $(-\varepsilon,\varepsilon)$ with $A(0)>0$. In addition

$$|\phi_r(r)|\le Ce^{\beta r}$$

as $r\to-\infty$, so by (19), the metric distance to $z=0$ is

$$\int_{-\infty}^0 C\sqrt{\phi_r}\,dr\le\int_{\infty}^0 Ce^{\beta r/2}\,dr<\infty, \tag{33}$$

and the metric is incomplete near $z = 0$.

Integrating (32) in r, the Kähler potential P has the form

$$P = ar + f(e^{\beta r}) = 2a \log |z| + f(|z|^{2\beta}), \tag{34}$$

where f is smooth at zero, $f'(0) = A(0)/\beta > 0$. The metric completion of $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ at $z = 0$ is obtained by adding a $\mathbb{CP}^{n-1}$. Each complex line through the origin acquires a cone singularity with central angle $2\pi\beta/k$. The following lemma tells that the induced metric is indeed smooth when $\beta = k$.

4.2 Lemma (Calabi [Cal82]) *When $\beta = k$, the Kähler potential (34) induces a smooth Kähler metric on a neighborhood of the zero-section in L^{-k}.*

Proof. We will use the fact that f is smooth at zero and $f'(0) > 0$. Recall from (19) that

$$h = P_r\, h_{FS} + P_{rr} h_{cyl} =: h_1 + h_2$$

on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$. We aim to show that h extends smoothly to $L := L^{-k}$. Now h_1 is smooth, since

$$h_1 = a\, h_{FS} = a\, \pi^*(h_0),$$

where h_0 is the Fubini-Study metric on $\mathbb{CP}^{n-1}$ and $\pi : L \to \mathbb{CP}^{n-1}$ is the projection. We write h_2 as

$$h_2 = \nabla \bar{\nabla} P_2,$$

where $P_2(z) := f(e^{kr}) = f(|z|^{2k})$ is considered as a function on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$. It suffices to show that P_2 extends to a smooth function on L. It is possible to choose a smooth (nonholomorphic!) local coordinate

$$w : \pi^{-1}(U) \subseteq L \to \mathbb{C}$$

over $U \subseteq \mathbb{CP}^{n-1}$ which is complex linear on fibers, with the property that $|w| = |z|^k$, that is, $|w(p(z))| = |z|^k$ for $z \in \mathbb{C}^n \backslash \{0\}$, where $p : \mathbb{C}^n \backslash \{0\} \to L \backslash L_0$ is the canonical k-fold cover. In particular, $|w|^2$ is a smooth function on L (in fact, it is a bundle metric) and

$$e^{kr} = |z|^{2k} = |w|^2,$$

so $P_2 = f(|w|^2)$ defines a smooth function on L. Thus h extends smoothly to L. Along the zero section, $h = ah_0 + kf'(0)\, dw\, d\bar{w}$, so h is positive and therefore Kähler. ∎

Now set $a := \lambda(k - n) > 0$ as required in (29), and let ϕ be a solution of $\phi_r = F(\phi)$ with $\phi \searrow a$ as $r \to -\infty$ as in (31). Does the corresponding metric g actually compactify smoothly at $z = 0$? From (29) and (28) we deduce

$$\beta = F'(a) = n + \lambda a = n + k - n = k,$$

so by Lemma 4.2, g does indeed induce a smooth metric on a neighborhood of the zero-section of L^{-k}. (This is not a coincidence: a dirac of curvature along the $\mathbb{CP}^{n-1}$ would be a source term in the conservation law (13).)

A development similar to the above can be given for the case $|z| = \infty$, $F(b) = 0$, $F'(b) < 0$, producing a metric on L^k. We summarize both results as follows.

4.3 Lemma *Let $k > 0$.*

(1) A solution of $\phi_r = F(\phi)$ with $\phi \searrow a$ as $r \to -\infty$ induces a smooth metric on a neighborhood of the zero section of L^{-k} if and only if $a = \lambda(k - n) > 0$, and $F(a) = 0$ or equivalently $\nu = \nu_a^\lambda(\mu)$. It is an expanding soliton for $k > n$, shrinking for $0 < k < n$.

(2) A solution ϕ of $\phi_r = F(\phi)$ with $\phi \nearrow b$ as $r \to \infty$ induces a smooth metric on a neighborhood of the zero section of L^k if and only if $b = k + n > 0$ and $F(b) = 0$, or equivalently $\nu = \nu_b^{-1}(\mu)$. It is a shrinking soliton.

4.2 Adding a point at $z = 0$

We derive the condition for adding a point smoothly to $\mathbb{C}^n \backslash \{0\}$ at $z = 0$. Heuristically speaking, this corresponds to passing the cross-sectional area $a \to 0$ in the above construction.

So suppose ϕ represents a smooth soliton defined on a neighborhood of $z = 0$ in $\mathbb{C}^n$. Then $\phi \to 0$ as $r \to -\infty$. We may write F as

$$F(\phi) = \frac{\nu e^{\mu\phi}}{\phi^{n-1}} - \frac{\nu_0(\mu)}{\phi^{n-1}} \sum_{j=0}^{n-1} \frac{\mu^j \phi^j}{j!} - \frac{\lambda\phi}{\mu},$$

where in agreement with (30),

$$\nu_0(\mu) := \frac{n!(\mu + \lambda)}{\mu^{n+1}}.$$

Then

$$F(\phi) \approx \frac{\nu - \nu_0(\mu)}{\phi^{n-1}}$$

near $\phi = 0$.

If $\nu < \nu_0(\mu)$, then $F < 0$ on $0 < \phi \leq \varepsilon$, contradicting $\phi_r > 0$. If $\nu > \nu_0(\mu)$, then ϕ reaches 0 at some finite point $r_0 > -\infty$, contradicting existence of ϕ as $r \to -\infty$. We deduce that

$$\nu = \nu_0(\mu).$$

Conversely, let us set $\nu = \nu_0(\mu)$. Then F simplifies to

$$F(\phi) = \phi + \nu_0(\mu) \sum_{j=2}^{\infty} \frac{\mu^{j+n-1}\phi^j}{(j+n-1)!},$$

which is smooth at $\phi = 0$ with $F(0) = 0$, $F'(0) = 1$. So there exists a solution ϕ of $\phi_r = F(\phi)$ defined on some interval $(-\infty, r_1]$ with $\phi > 0$, $\phi_r > 0$, and $\phi \searrow 0$ as $r \to -\infty$. Since $F'(0) = 1$, the same argument as for (32) shows that

$$\phi(r) = e^r K(e^r)$$

where K is smooth at 0, $K(0) > 0$. Then the Kähler potential P satisfies

$$P = e^r L(e^r) = |z|^2 L(|z|^2),$$

where L is smooth at 0, $L(0) > 0$. Thus P induces a smooth Kähler metric on a neighborhood of 0 in $\mathbb{C}^n$. We have proven the following Lemma.

4.4 Lemma *Let $k = 1$. A solution of $\phi_r = F(\phi)$ induces a smooth metric on a neighborhood of 0 in $\mathbb{C}^n$ if and only if*

$$F(0) = 0,$$

that is, if and only if $\nu = \nu_0(\mu)$.

5 Expanding solitons on line bundles

In this section we construct a 1-parameter family of expanding solitons on L^{-k} for each $k > n$, thereby proving Theorem 1.4. The closing condition at $z = 0$ reduces the parameter space (μ, ν) by 1 dimension. Completeness as $|z| \to \infty$ comes for free. The solution is asymptotic to a cone at spatial infinity, and therefore this cone is also the initial condition of the expanding flow defined for $t > 0$. Any cone aperture $p > 0$ can be realized.

Proof of Theorem 1.4. 1. We first show existence. Set $\lambda = 1$. By Lemma 4.3, the closing condition at $z = 0$ can be satisfied if and only if we set

$$\nu = \nu_a^{+1}(\mu), \qquad a = k - n > 0.$$

For each $\mu \neq 0$, we obtain a solution of $\phi_r = F(\phi)$ defined on a maximal interval $-\infty < r < r_1$, such that $F(a) = 0$ and $\phi \searrow a$ as $\phi \to -\infty$, and a corresponding expanding soliton defined on a neighborhood of the zero section of L^{-k}. For each μ, ϕ is unique up to translation and g is unique up to isometry.

For $\lambda = 1$, (27) becomes

$$\phi_r = F(\phi) := \frac{\nu e^{\mu\phi}}{\phi^{n-1}} - \frac{\phi}{\mu} - \frac{1+\mu}{\mu^{1+n}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j \phi^{j+1-n}.$$

By Ansatz 1.1, we wish ϕ to be defined for all $-\infty < r < \infty$.

Now recall from Lemma 4.1 with $\lambda = 1$ that a is the only positive zero of F, and $F > 0$ on (a, ∞). Thus ϕ increases without bound as r converges to its maximal value $r_1 \leq \infty$.

Now if $\mu > 0$, then examining (30) with $\lambda = 1$, we find that $\nu > 0$, so $\phi_r = F(\phi)$ is dominated by the superlinear term $\nu\phi^{1-n}e^{\mu\phi} > 0$ for large ϕ. This implies that $r_1 < \infty$ which contradicts the Ansatz. We previously excluded $\mu = 0$. Thus we have proven the following additional necessary condition.

5.1 Lemma *An expanding soliton on L^{-k} must have $\mu < 0$.*

Conversely, assume that $\mu < 0$. Then the exponential term in $F(\phi)$ becomes tame, and the dominant term for large ϕ is the linear term $-\phi/\mu > 0$. This implies that ϕ exists for all $-\infty < r < \infty$. Note that the conditions $\phi > 0$, $\phi_r > 0$ of (20) are satisfied automatically and we get a Kähler metric. It is an expanding soliton on L^{-k}.

2. We next verify that L^{-k} is conelike at infinity and in particular complete. We may write $\phi_r = F(\phi)$ in the form

$$\phi_r = -\frac{\phi}{\mu} + G\left(\frac{1}{\phi}\right),$$

where G is smooth at zero. Rewriting the equation in terms of $\psi := 1/\phi$ and invoking the principle used in (32), we find that

$$\phi(r) = e^{-r/\mu} B(e^{r/\mu}), \tag{35}$$

for large r, where B is smooth and $B(0) > 0$. Comparing with the Kähler cone potential $|z|^{2p}/p$ following (6), we see that g is spatially asymptotic to the Kähler cone $C^{n,k,p}$ with

$$p = -1/\mu.$$

Evidently any cone aperture $p > 0$ can be realized.

Let us investigate the degree of closeness to the cone. By (18) the metric has the form (recall $\mu < 0$)

$$\begin{aligned} h(z) &= \left[\phi|z|^{-2}\,\delta_{\alpha\bar\beta} + (\phi_r - \phi)|z|^{-4}\bar z^{\bar\alpha} z^{\beta}\right]\, dz^{\alpha}\, d\bar z^{\bar\beta} \\ &= \Bigg\{ |z|^{-2-2/\mu} B\left(|z|^{2/\mu}\right)\delta_{\alpha\bar\beta} \\ &+ \left[\left(-\frac{1}{\mu} - 1\right)|z|^{-2/\mu} B\left(|z|^{2/\mu}\right) + \frac{1}{\mu} B'\left(|z|^{2/\mu}\right)\right] |z|^{-4}\bar z^{\bar\alpha} z^{\beta}\Bigg\}\, dz^{\alpha}\, d\bar z^{\bar\beta} \end{aligned}$$

If $d(z)$ is the intrinsic distance between z and the vertex, the metric h has the form

$$h = h_{cone} + O(d^{-2})$$

in a neighborhood of infinity, and the i^{th} covariant derivative of $O(d^{-2})$ goes to zero like d^{-2-i} as $d \to \infty$.

In summary, we have proven that for each $\mu < 0$, there exists up to isometry a unique expanding soliton defined on L^{-k}, $k > n$, which is conelike at infinity.

3. To complete the proof of Theorem 1.4, we derive the time evolution of the soliton and show smooth convergence to the cone (away from the zero-section) as t approaches the singular time. Referring to §2.1, let us change σ by a shift in time so that $\sigma(t) = 4t$. Then the singular time is $t = 0$, and the solution metric h derived above coincides with $h(\cdot, 1/4)$. By (22) and (23), the flow vector is

$$Y^{\beta}(z,t) = \frac{\mu}{4t} z^{\beta},$$

which induces the diffeomorphisms

$$\psi_t(z) = (4t)^{\mu/2} z.$$

The corresponding time-dependent metric is

$$h(z,t) = 4t\,(\psi_t^* h)(z), \qquad z \in L^{-k}, \quad t > 0.$$

We calculate the pullback to find

$$\begin{aligned} h(z,t) &= (4t)^{1+\mu} h_{\alpha\bar{\beta}}\left((4t)^{\mu/2} z\right)\, dz^\alpha\, d\bar{z}^{\bar{\beta}} \\ &= \Bigg\{ |z|^{-2-2/\mu} B\left(4t|z|^{2/\mu}\right) \delta_{\alpha\bar{\beta}} \\ + \Bigg[\Big(-\frac{1}{\mu} - 1\Big)&|z|^{-2/\mu} B\left(4t|z|^{2/\mu}\right) + \frac{4t}{\mu} B'\left(4t|z|^{2/\mu}\right)\Bigg] |z|^{-4} \bar{z}^{\bar{\alpha}} z^\beta \Bigg\}\, dz^\alpha\, d\bar{z}^{\bar{\beta}}. \end{aligned}$$

for $|z| > 0$ and $t > 0$.

Since B is smooth at 0, we see that this expression is actually smooth up to $t = 0$ for $z \neq 0$, that is, h is smooth on the chart $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k \times [0, \infty)$. In fact, because of the smooth closing condition at $z = 0$, the metric extends to a smooth metric on the space-time manifold

$$(L^{-k} \times [0, \infty)) \setminus (L_0 \times \{0\})$$

where L_0 is the zero section. At $t = 0$, the metric is

$$h(z,0) = B(0) \left\{ |z|^{-2-2/\mu} \delta_{\alpha\bar{\beta}} + \left(-\frac{1}{\mu} - 1\right) |z|^{-2/\mu} |z|^{-4} \bar{z}^{\bar{\alpha}} z^\beta \right\} dz^\alpha\, d\bar{z}^{\bar{\beta}},$$

which is the cone metric $h_{C^{n,k,p}}(\cdot)$ on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$. So $g(t)$ emerges locally smoothly from the cone metric $g(0)$ except on the zero-section, which erupts out of the vertex.

Fix a point x_0 in the zero-section. The diameter of a fixed tubular neighborhood of L_0 of the form $|z| \leq \varepsilon$ in L^{-k} has $g(t)$-diameter bounded by $C\varepsilon$ as $t \searrow 0$. This together with the above smooth convergence shows that $(L^{-k}, g(t), x_0)$ converges to $(\overline{C^{n,k,-1/\mu}}, g(0), 0)$ in the pointed Gromov–Hausdorff topology. This completes the proof of Theorem 1.4. ■

6 Shrinking solitons on line bundles

In this section we construct a shrinking soliton on L^{-k} for each $k < n$, thereby proving Theorem 1.5. In the case of shrinking solitons, the two boundary conditions determine a unique solution. Again, the solution converges to a cone as $t \nearrow 0$, but in the shrinking case, only one such cone can be realized for each k. As mentioned in the introduction, it is typical for geometric heat flows that shrinking solutions are rarer than expanding ones.

Proof of Theorem 1.5. 1. We first show existence. Set $\lambda = -1$. By Lemma 4.3, the closing condition at $z = 0$ can be satisfied if and only if we set

$$\nu = \nu_a^{-1}(\mu), \qquad a = n - k > 0,$$

or equivalently $F(a) = 0$. For each $\mu \neq 0$, we obtain a solution of $\phi_r = F(\phi)$ defined on a maximal interval $-\infty < r < r_1$, such that $\phi \searrow a$ as $\phi \to -\infty$, and a corresponding shrinking soliton g (unique up to isometry) defined on a neighborhood of the zero section of L^{-k}.

For $\lambda = -1$, (27) becomes

$$\phi_r = F(\phi) := \frac{\nu e^{\mu\phi}}{\phi^{n-1}} + \frac{\phi}{\mu} + \frac{1-\mu}{\mu^{1+n}} \sum_{j=0}^{n-1} \frac{n!}{j!} \mu^j \phi^{j+1-n}.$$

As in the previous proof, we wish to ensure that ϕ is defined for all $-\infty < r < \infty$ and is complete as $|z| \to \infty$.

Recall by Lemma 4.1 with $\lambda = -1$ that F has at most one zero b in the interval (a, ∞). As ν increases, b passes to infinity and beyond. In the expanding case, in which b was always "beyond" infinity, there was a continuum of solutions with a tame exponential term for large ϕ. In the shrinking case, however, the exponential term is always wild when b is beyond infinity, so the only acceptable solution comes when b just barely passes to infinity. This reduces the dimension of the solution space to zero via the following additional necessary condition.

6.1 Lemma *A complete shrinking soliton on L^{-k} must have $\mu > 0$ and $\nu = 0$.*

Proof. If $\mu < 0$, then F is dominated for large ϕ by $\phi/\mu < 0$, so F has a second zero $b > a$. Then ϕ exists for all r and $\phi \to b$ as $r \to \infty$, but arguing as for (32) and (33), we see that the metric is incomplete as $|z| \to \infty$. This excludes $\mu < 0$. We previously excluded $\mu = 0$. Thus $\mu > 0$.

If $\nu < 0$, then F is dominated for large ϕ by $\nu\phi^{1-n}e^{\mu\phi} < 0$, and again F has a second zero $b > a$, which contradicts completeness as before. If $\nu > 0$, then F is dominated for large ϕ by the superlinear term $\nu\phi^{1-n}e^{\mu\phi} > 0$. Then there is no second zero b, so $F > 0$ on (a, ∞) and $\phi \to \infty$ as r converges to a maximal value $r_1 < \infty$, which contradicts existence for all r. Thus $\nu = 0$. ■

In view of $\nu = 0$ we rewrite $F(a) = 0$ as

$$f(a, \mu) := \frac{n!}{a^{n-1}\mu^{n+1}} \left(\frac{a^n \mu^n}{n!} + (1-\mu) \sum_{j=0}^{n-1} \frac{a^j \mu^j}{j!} \right) = 0.$$

We next show that there is a unique solution μ of this equation for the given a.

6.2 Lemma *For each $0 < \phi < n$ there exists a unique positive root μ of $f(\phi, \mu) = 0$. In fact $\mu > 1$.*

Proof. Write f in the alternate form

$$f(\phi,\mu) = \frac{n!}{\phi^{n-1}\mu^{n+1}} \left(\sum_{j=0}^{n} \frac{(\phi-j)\phi^{j-1}}{j!}\mu^j \right). \tag{36}$$

Then there is exactly one sign change in the sequence of coefficients

$$1, \ldots, \frac{(\phi-j)\phi^{j-1}}{j!}, \ldots, \frac{(\phi-n)\phi^{n-1}}{n!},$$

of the inner polynomial, so there is at most one positive root μ of $f(\phi,\mu) = 0$. But $f(\phi,1) = \phi > 0$ for $\mu = 1$, whereas $f(\phi,\mu) \sim (\phi - n)/\mu < 0$ as $\mu \to \infty$, so there is exactly one positive root μ, and $\mu > 1$. ■

6.3 Remark By refining the technique of the Lemma, it is possible to show that $1 < \mu < 2$ for $\phi = 1, \cdots, n-1$.

We now turn the tables and show that these data yield a shrinking soliton on L^{-k}. Set $\nu = 0$ and let $\mu = \mu(n,k) > 0$ be the unique solution of $f(a,\mu) = 0$ as given by Lemma 6.2. The equation is now $\phi_r = F(\phi) = f(\phi,\mu)$. By our choice of μ and ν (or by direct inspection of (36) for $\phi \geq n$ together with Lemma 4.1), we have $F(a) = 0$ and $F > 0$ on (a,∞), so ϕ increases without bound as r converges to its maximal value r_1. The exponential term in $F(\phi)$ is missing, and the dominant term for large ϕ is the linear term $\phi/\mu > 0$. Thus ϕ exists for all $-\infty < r < \infty$. The conditions $\phi > 0$, $\phi_r > 0$ are automatic and the metric is Kähler. This defines a shrinking soliton on L^{-k}.

2. We now show that g is conelike at infinity and hence complete. We may write $\phi_r = F(\phi)$ in the form

$$\phi_r = \frac{\phi}{\mu} + H\left(\frac{1}{\phi}\right),$$

where H is smooth at zero, so ϕ exists for all $r \in \mathbb{R}$, and by the same argument as for (35), we get

$$\phi(r) = e^{r/\mu} D(e^{-r/\mu})$$

for large r, where D is smooth at zero and $D(0) > 0$. Recalling (6), we see that h is spatially asymptotic to the Kähler cone $C^{n,k,p(n,k)}$, where

$$p(n,k) := 1/\mu.$$

In summary, we have proven that up to isometry there exists a unique complete shrinking soliton defined on L^{-k}, $0 < k < n$, and it is conelike at infinity.

3. To complete the proof, we derive the time evolution of the soliton. Analogously to §5, we compute h via (18), and with $\sigma(t) = -4t$, $Y(z,t) = -(\mu/4t)z$

and $\psi_t(z) = (-4t)^{-\mu/2}z$, $t<0$, we compute the evolving metric to be

$$\begin{aligned}
h_-(z,t) &= -4t\,(\psi_t^* h)(z) \qquad\qquad z\in L^{-k},\quad t<0,\\
&= \left\{ |z|^{-2+2/\mu} D\left(-4t|z|^{-2/\mu}\right)\delta_{\alpha\bar\beta}\right.\\
&\left.+\left[\left(\frac{1}{\mu}-1\right)|z|^{2/\mu}D\left(-4t|z|^{-2/\mu}\right)+\frac{4t}{\mu}D'\left(-4t|z|^{-2/\mu}\right)\right]|z|^{-4}\bar z^{\bar\alpha}z^{\beta}\right\} dz^\alpha\, d\bar z^{\bar\beta},
\end{aligned}$$

for $z\neq 0$, $t<0$. This is a smooth, complete, shrinking soliton metric metric on L^{-k}, and the metric h computed above occurs as $h_-(\cdot,-1/4)$.

Since D is smooth at 0 and by the closing condition at $z=0$, the above expression actually defines a smooth metric on the space-time manifold

$$W_- := (L^{-1}\times(-\infty,0])\setminus(L_0\times\{0\})$$

where L_0 is the zero section. At $t=0$, the metric is

$$h_-(z,0) = D(0)\left\{|z|^{-2+2/\mu}\delta_{\alpha\bar\beta}+\left(\frac{1}{\mu}-1\right)|z|^{2/\mu}|z|^{-4}\bar z^{\bar\alpha}z^{\beta}\right\} dz^\alpha\, d\bar z^{\bar\beta}, \tag{37}$$

which is the cone metric $h_{\mathcal{C}^{n,k,p(n,k)}}(\cdot)$ on $(\mathbb{C}^n\setminus\{0\})/\mathbb{Z}_k$. So $g(t)$ converges locally smoothly to the cone metric $g(0)$ except on the zero-section, which falls into the vertex. The Gromov–Hausdorff convergence goes as before. This completes the proof of Theorem 1.5. ∎

7 Flowing through the singularity

In this section we prove Theorem 1.6. We begin by outlining Cao's construction of the expanding soliton $\mathcal{Y}_t^{n,q}$ on $\mathbb{C}^n$, and then show how to paste it across the cone $\mathcal{C}^{n,q}$ at $t=0$ to the shrinking soliton $\mathcal{X}_t^{n,1}$ on L^{-1}.

Fix $q>0$ and let $\phi:(-\infty,r_1)\to\mathbb{R}$ be the solution of $\phi_r = F(\phi)$ constructed in Lemma 4.4 corresponding to $\lambda := 1$, $\mu := -1/q < 0$, $\nu := \nu_0(\mu)$. By the Lemma, ϕ induces a smooth metric on a neighborhood of $z=0$ in $\mathbb{C}^n$. Note that F is positive for $0<\phi\le\varepsilon$, whereas for large ϕ,

$$\phi_r = -\frac{\phi}{\mu}+G\left(\frac{1}{\phi}\right),$$

where G is smooth at zero. Recall $\mu<0$. Then by Lemma 4.1, $F>0$ for all $\phi>0$. As in the argument for (35), ϕ exists for all $r\in\mathbb{R}$ and has the form

$$\phi(r) = e^{-r/\mu}E(e^{r/\mu})$$

for large r, where E is a smooth at zero and $E(0)>0$. Therefore g is smoothly asymptotic to $C^{n,q}$ at infinity.

We derive the time evolution of the soliton. Analogously to §5 and §6, we compute h via (18), and with $\sigma(t) = 4t$, $Y(z,t) = (\mu/4t)z$ and $\psi_t(z) = (4t)^{\mu/2}z$, $t > 0$, we compute the evolving metric to be

$$\begin{aligned} h_+(z,t) &= 4t(\psi_t^* h)(z) \qquad\qquad z \in \mathbb{C}^n, \quad t>0, \\ &= \Big\{ |z|^{-2-2/\mu} E\left(4t|z|^{2/\mu}\right) \delta_{\alpha\bar\beta} \\ &+ \Big[\Big(-\frac{1}{\mu} - 1 \Big) |z|^{-2/\mu} E\left(4t|z|^{2/\mu}\right) + \frac{4t}{\mu} E'\left(4t|z|^{2/\mu}\right) \Big] |z|^{-4} \bar z^{\bar\alpha} z^{\beta} \Big\}\, dz^\alpha \, d\bar z^{\bar\beta}, \end{aligned}$$

for $z \neq 0$, $t > 0$. This is a smooth, complete, expanding soliton metric metric on $\mathbb{C}^n$, and the metric h computed above occurs as $h_+(\cdot, 1/4)$.

Since E is smooth at zero and by the closing condition at $z = 0$, the expression actually defines a smooth metric on the space-time manifold

$$W_+ := \mathbb{C}^n \times [0,\infty) \setminus \{(0,0)\}.$$

At $t = 0$, the metric is

$$h_+(z,0) = E(0) \left\{ |z|^{-2-2/\mu} \delta_{\alpha\bar\beta} + \left(-\frac{1}{\mu} - 1 \right) |z|^{-2/\mu} |z|^{-4} \bar z^{\bar\alpha} z^{\beta} \right\} dz^\alpha \, d\bar z^{\bar\beta}, \quad (38)$$

which is the cone metric $h_{\mathcal{C}^{n,k,p}}(\cdot)$ on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_k$, $p = -1/\mu$. So $g(t)$ converges locally smoothly to the cone metric $g(0)$ except at $z = 0$. The Gromov–Hausdorff convergence goes as in §5 and §6. This completes the construction.

Proof of Theorem 1.6. We now paste the shrinking soliton for $t \le 0$ to the correct expanding soliton for $t \ge 0$.

By the proof of Theorem 6, $\mathcal{X}_t^{n,1}$ is given by the smooth metric $h_-(z,t)$ defined on

$$W_- := (L^{-1} \times (-\infty, 0]) \setminus (L_0 \times \{0\}). \quad (39)$$

By the above discussion, the expanding soliton $\mathcal{Y}_t^{n,q}$ is given by the smooth metric $h_+(z,t)$ defined on

$$W_+ := \mathbb{C}^n \times [0,\infty) \setminus \{(0,0)\}$$

Note that the boundaries of W_- and W_+ as smooth manifolds are each equal to $\mathbb{C}^n \setminus \{0\}$. So define the space-time manifold

$$W := W_- \cup W_-,$$

where ∂W_- is identified with ∂W_+ by the identity map.

Set $q := p(n,1)$ and pull back h_+ by a homothety of $\mathbb{C}^n \setminus \{0\}$ so that $E(0) = D(0)$. Define h on W by

$$h(z,t) := \begin{cases} h_-(z,t) & \text{on } W_- \\ h_+(z,t) & \text{on } W_+. \end{cases}$$

By (37) and (38), we have

$$h_-(z,0) = h_+(z,0) \qquad \text{on } (\mathbb{C}^n \backslash \{0\}) \times \{0\} = \partial W_- = \partial W_+,$$

so h is well-defined and continuous on W.

Let us show that h is smooth on W. The spatial derivates of h_- and of h_+ agree at $t = 0$, and h_- and h_+ solve the same parabolic equation, so

$$\frac{\partial h_-}{\partial t}(z,0) = \frac{\partial h_+}{\partial t}(z,0), \qquad z \neq 0.$$

It follows from this that $\partial h/\partial t$ exists and is continuous on W and that h is C^1 on W. Similar considerations apply to all higher time and mixed derivatives. Thus h is C^∞ on W, as claimed, and solves the Ricci flow even across $t = 0$. The metric completion of the flow has a unique singular point in spacetime. The Gromov–Hausdorff continuity follows from Theorems 1.3 and 1.5. ■

8 Shrinking solitons on compact orbifolds

Proof of Theorem 1.7. We wish to construct a soliton on G_k. It must be a shrinking soliton, as we can see by applying Lemma 1.2 to $G_k \setminus \{0\} \cong L^k$. So set $\lambda := -1$.

By Lemma 4.4 the orbifold-singularity condition at $z = 0$ is equivalent to

$$F(0) = 0,$$

or equivalently

$$\nu = \nu_0(\mu) := \frac{(\mu - 1)n!}{\mu^{n+1}}. \tag{40}$$

By Lemma 4.3, adding a $\mathbb{CP}^{n-1}$ smoothly at $|z| = \infty$ is equivalent to

$$F(n+k) = 0. \tag{41}$$

Conversely, if both of these conditions hold, then $F'(0) > 0$ and $F'(n+k) < 0$. So by Lemma 4.1, $F > 0$ on $(0, n+k)$, and the solution ϕ exists for all r with

$$\phi \to 0 \quad \text{as } r \to -\infty, \qquad \phi \to n+k \quad \text{as } r \to \infty.$$

Then by Lemma 4.4 and Lemma 4.3, ϕ yields a shrinking soliton metric on G_k with an orbifold singularity at $z = 0$, as required.

So it suffices to satisfy (40) and (41) simultaneously. Using (40), we rewrite (41) as

$$f_0(b, \mu) := \frac{n!}{\mu^{n+1} b^n} \sum_{j=n+1}^{\infty} \frac{\mu^j b^j}{j!}(j - b) = 0, \tag{42}$$

where $b := n + k$.

We consider only the case $\mu > 0$.

If $b = n+1$ then $\mu = 0$, which yields the Fubini-Study metric on $\mathbb{CP}^n$. (This was excluded earlier in any case.)

If $b = n + k > n + 1$, then there is exactly one sign change in the coefficient sequence

$$\frac{b^{n+1}}{(n+1)!}(n+1-b), \ldots, \frac{b^j}{j!}(j-b), \ldots,$$

so there is at most one positive root μ of (42). We have

$$f_0(b,0) = b\left(1 - \frac{b}{n+1}\right) < 0, \qquad f_0(b,1) = b > 0 \quad (\text{flat } \mathbb{C}^n).$$

So there exists exactly one positive root $\mu = \mu(k)$ of (42), and it satisfies $0 < \mu < 1$. ∎

9 Impossibilities

Suppose ϕ is a solution of $\phi_r = F(\phi)$ defined for $-\infty < r < \infty$, and let g be the induced metric on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$. By the proofs of Lemma 4.1, Lemma 4.3, Lemma 4.4, Theorem 1.4, and Theorem 1.5, the following picture has emerged.

At $z = 0$, g is necessarily incomplete (see (32), (33), and the proof of Lemma 4.4). The metric completion at $z = 0$ is obtained either by adding a point smoothly (when $k = 1$), an orbifold point (when $k \geq 2$), or a $\mathbb{CP}^{n-1}$. In the latter case the metric can be smooth or singular along the added $\mathbb{CP}^{n-1}$.

At $|z| = \infty$, the induced metric on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ is either complete, in which case it is conelike, or incomplete, in which case the metric completion is obtained by adding a $\mathbb{CP}^{n-1}$. In the latter case the metric can be smooth or singular along the added $\mathbb{CP}^{n-1}$.

Using this information, we rule out a few possibilities.

9.1 Proposition *There is no complete, $U(n)$-invariant gradient Kähler-Ricci soliton metric on L^k, $k > 0$.*

Proof. By the above discussion, a metric on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ satisfying Ansatz 1.1 is necessarily incomplete at $z = 0$. ∎

9.2 Proposition *There is no complete, $U(n)$-invariant gradient Kähler-Ricci soliton metric on $\mathbb{C}^n$ of shrinking type, except for the flat metric.*

Proof. As in the proof of Lemma 6.1, the existence of $\phi(r)$ as $r \to \infty$ and completeness of the metric implies that

$$\nu = 0.$$

By Lemma 4.4, the closing condition of adding a point at $z = 0$ implies that

$$\nu = \frac{n!(\mu + \lambda)}{\mu^{n+1}}.$$

Since $\lambda = -1$, we get $\mu = 1$, which yields the flat metric. ■

A more general proposition is the following.

9.3 Proposition (Initial and terminal cones)

(1) For $0 < k < n$, the only expanding soliton satisfying the $U(n)$-invariant Ansatz 1.1 with initial condition the cone $\mathcal{C}^{n,k,p}$ is the expanding orbifold $\mathcal{Y}_t^{n,k,p}$.

(2) For $k > n$, the only shrinking soliton satisfying the $U(n)$-invariant Ansatz 1.1 with terminal condition the cone $\mathcal{C}^{n,k,p}$ is the flat cone $\mathbb{C}^n/\mathbb{Z}_k$.

Proof. Let g be the metric induced on $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$. At $|z| = \infty$ the metric must be complete in order to satisfy the initial (or terminal) condition. At $z = 0$, the completion process adds either a $\mathbb{CP}^{n-1}$ (smoothly) or an orbifold point. For $0 < k \leq n$, the former is excluded by Lemma 1.2 and the latter necessarily leads to $\mathcal{Y}_t^{n,k,p}$. For $k > n$, the former is excluded by Lemma 1.2 and the latter leads to flat $\mathbb{C}^n$ by Proposition 9.2. ■

10 Directions for further research

We mention some problems concerning the evolution of a Kähler metric past singularities. Several of these are familiar to the experts.

1. Does the Kähler-Ricci flow of M remain smooth until it is forced to become singular because the volume of some analytic cycle goes to zero?

The volume of a complex subvariety V^k in M at time t is given by

$$P_V(t) := \int_V (\omega(0) - 4\pi c_1 t)^k,$$

a polynomial of degree k in t. Define $T(V)$ to be the first positive zero of $P_V(t)$, if it exists. Then the question asks whether $g(t)$ remains smooth until

$$T_{\text{robot}} := \inf_V T(V).$$

As mentioned in the introduction, this is known to be true in the canonical case [Cao85]. On the other hand, its analogue for the closely related Lagrangian mean curvature flow is false, as shown by the cusp singularity of a plane curve.

2. Does a singularity of the (unnormalized) Kähler-Ricci flow necessarily form along an analytic subvariety?

Within a given Kähler class, the singular set of a sequence of Kähler potentials is an analytic subvariety [Tia87]; see also [Nad90].

3. Does the Ricci flow of a Kähler manifold remain Kähler after a singularity?

After all, certain Kähler manifolds have no Kähler-Einstein metric, but do carry a non-Kähler Einstein metric (for example $\mathbb{CP}^1 \# \overline{\mathbb{CP}^1}$ [Pag78]).

4. Does the Ricci flow evolve uniquely after a singularity? If not, is there a selection principle for the best flow?

Ideally a nonuniqueness example would start smooth, develop a point singularity, then become smooth again in more than one way. Such a phenomenon is known for the mean curvature flow, the harmonic map heat flow, and the equation $u_t = \Delta u + u^p$ [Ilm95b]. No known selection principle is adequate for these equations.

5. At a local singularity, does the Ricci flow always simplify the complex structure though a blowdown?

For example, let M be a Kähler manifold given by blowing up $\mathbb{CP}^n$ at j points. If the exceptional divisors are small in the initial metric, show that they all blow down and disappear. What else can happen? (See Example 2.2.)

What about trying to use the Ricci flow to undo the algebraic-geometrical resolution of more complicated singularities?

What about the analytic space $\mathbb{C}^n/\mathbb{Z}_k$, $k > n$, which can evolve to L^{-k}? Can this space (presumably not endowed with the flat metric) be reached from a smooth initial manifold, and how complicated must the initial manifold be? (See Proposition 9.3.)

6. When does a Ricci-flat Kähler cone possess a nontrivial Ricci evolution?

An answer to this question would be an analogue for intrinsic geometry of well-known criteria for a stationary cone in $\mathbb{R}^N$ to be area-minimizing [Law89, Law91]. For example, it seems reasonable to conjecture that the flat cone $\mathbb{C}^n/\mathbb{Z}_k$ possesses no Ricci evolution other than itself for $0 < k < n$.

7. Construct a self-similarly expanding Kähler-Ricci flow starting from any given Kähler cone.

The solution will sometimes be singular.

8. Define and construct weak solutions of the Kähler-Ricci flow.

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