Nonlocal geometric expansion of convex planar curves*

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Abstract

We consider a class of nonlocal geometric equations for expanding curves in the plane, arising in the study of evolutions governed by Monge-Kantorovich mass transfer. We construct convex solutions, given convex initial data. In order to obtain such solutions, we develop a new version of Perron's method. We give applications to the problem of characterizing fast/slow diffusion limits.

In this paper we study the following equation describing the expansion of a bounded convex set $\Omega_t \subset \mathbf{R}^2$, $t \geq 0$:

$$V = F(\kappa, \gamma, t) \quad \text{on } \Gamma_t, \tag{1}$$

where

$$\left\{ \begin{array}{lllll} \Gamma_t & = & \partial\Omega_t, & & \\ V & = & V_{\Gamma_t}(y) & = & \text{outer normal velocity of } \Gamma_t \text{ at } y \in \Gamma_t, \\ \gamma & = & \gamma_{\Omega_t}(y) & = & \text{radius of the largest disk in } \overline{\Omega}_t \text{ which touches } \Gamma_t \text{ at } y, \\ \kappa & = & \kappa_{\Gamma_t}(y) & = & \text{curvature of } \Gamma_t \text{ at } y. \end{array} \right. \tag{2}$$

Note that $\kappa\gamma \leq 1$, since if the curvature of Γ_t at $y \in \Gamma_t$ is positive, then the radius of a disk within Ω_t touching Γ_t at y cannot exceed the radius of curvature at y. The function F in the right-hand side of (1) is thus defined on the set $\mathcal{A} \times [0, \infty)$, where

$$\mathcal{A} = \{\ (\kappa, \gamma) \in \mathbf{R}^1 \times [0, \infty)\ \mid\ \kappa\gamma \leq 1\ \}. \tag{3}$$

We assume that F is nonnegative on $\mathcal{A} \times [0, \infty)$ and satisfies some additional conditions stated below.

Equation (1) represents a nonlocal geometric curvature motion, nonlocality coming from the quantity $\gamma_{\Omega_t}(y)$. Such equations arise in the study of certain variational evolution problems related to the Monge-Kantorovich mass transfer. One example is a variational evolution problem which describes the limits as $p \to \infty$ of solutions of p-Laplacian evolution equations, considered by L.C. Evans, M. Feldman and R.F. Gariepy [EFG97]. This problem can be regarded as a model of

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collapsing sandpiles. A form of Equation (1) related to this model was derived heuristically in [EFG97], and the right-hand side in this case is

$$F(\kappa,\gamma,t) = \frac{\gamma}{3(t+a)} \cdot \frac{3-2\kappa\gamma}{2-\kappa\gamma}, \quad a>0. \tag{4}$$

Another example is a model of compression molding introduced by G. Aronsson and L.C. Evans [AE], based on a Hele-Shaw model for highly non-Newtonian fluids. This model leads to Equation (1) with

$$F(\kappa,\gamma,t) = \gamma\left(1-\frac{\kappa\gamma}{2}\right). \tag{5}$$

It follows from Feldman [F99] that convex solutions of the equations (1) with F defined by (4), (5) are relevant in the study of variational evolution problems. This motivates our work. As an application of our existence results we obtain some information on the solutions of collapsing sandpiles and compression molding models, see Section 5 below.

We consider solutions of (1) with possibly nonsmooth initial data. The solution may remain nonsmooth for a finite time. We will interpret solutions in the viscosity sense.

We make the following assumptions on the function F in the right-hand side of (1):

(H1) $F \in C(\mathcal{A}\times[0,\infty))$;
$F \geq 0$ on $\mathcal{A}\times[0,\infty)$;
$F(\kappa,\gamma,t) \leq M|\gamma|$ on $\mathcal{A}\times[0,\infty)$ where M does not depend on κ, γ, or t.

(H2) For any κ_1, κ_2, γ, and t satisfying $(\kappa_1,\gamma,t)\in\mathcal{A}\times[0,\infty)$, $(\kappa_2,\gamma,t)\in\mathcal{A}\times[0,\infty)$,

$$F(\kappa_1,\gamma,t) \leq F(\kappa_2,\gamma,t) \quad \text{if} \quad \kappa_1 \geq \kappa_2.$$

(H3) For any $\kappa,\gamma_1,\gamma_2,t$ satisfying $(\kappa,\gamma_1,t)\in\mathcal{A}\times[0,\infty)$, $(\kappa,\gamma_2,t)\in\mathcal{A}\times[0,\infty)$,

$$F(\kappa,\gamma_1,t) \geq F(\kappa,\gamma_2,t) \quad \text{if} \quad \gamma_1 \geq \gamma_2.$$

(H4) For any $t \geq 0$ the function $\gamma \to F(0,\gamma,t)$ is concave on $[0,\infty)$.

Remark 1 *The hypothesis (H1) implies in particular that the solution is an expanding curve. The hypothesis (H2) means "parabolicity" of the equation (1). Combining (H2) and (H3) we see that solutions of the equation (1) satisfy a formal comparison principle: Let $\Gamma^i_t = \partial\Omega^i_t$, $i=1,2$, be two smooth solutions of (1). Let $\Omega^1_{t^*}\subseteq\Omega^2_{t^*}$ for some $t^*>0$, and let $x^*\in\Gamma^1_{t^*}\cap\Gamma^2_{t^*}$. Then by (H2) and (H3),*

$$V_{\Gamma^1_{t^*}}(x^*) \leq V_{\Gamma^2_{t^*}}(x^*),$$

where $V_{\Gamma^i_{t^*}}(x^*)$ is the outer normal velocity of the curve $\Gamma^i_{t^*}$ at x^*. To see this, we note that, since $\Omega^1_{t^*}\subseteq\Omega^2_{t^*}$, we have at x^*:

$$\kappa_{\Gamma^1_{t^*}}(x^*) \geq \kappa_{\Gamma^2_{t^*}}(x^*),$$
$$\gamma_{\Gamma^1_{t^*}}(x^*) \leq \gamma_{\Gamma^2_{t^*}}(x^*).$$

We do not expect that conditions (H1) - (H3) alone will guarantee the convexity preserving property, since they only imply parabolicity of Equation (1). The additional condition we need is the hypothesis (H4).

Remark 2 *Functions (4) and (5) satisfy conditions (H1) - (H4). The function (4) is smooth and bounded on the set $\mathcal{A}\times[0,\infty)$.*

Equations that describe expansion of a convex planar curve by its curvature, i.e. equations of the form

$$V = F(\kappa) \quad \text{on } \Gamma_t, \tag{6}$$

were considered in a number of papers. In particular, B. Chow and D.-H. Tsai [CT96] constructed a smooth convex solution given a smooth convex initial curve. B. Andrews [An98] constructed solutions both in the cases of smooth and nonsmooth initial convex curves, for a class of functions F in (6). In the present paper we do not assume that the initial data is smooth. Note that methods of [CT96] and [An98] are not readily applicable to the equation (1) even with smooth initial data, because of the dependence on γ in (1).

In this paper we define viscosity solutions of the equation (1), and prove existence of a convexity preserving viscosity solution given convex initial data. Viscosity (sub-, super-) solutions are families of sets $\{\Omega_t\}$, see Definition 15 below. The main result is the following

Theorem 3 *Let the function F satisfy (H1)-(H4). Let $\Omega \subset \mathbf{R}^2$ be an open bounded convex set. Then there exists a Lipschitz continuous family of open bounded convex sets $\{\Omega_t\}_{t\geq 0}$ such that $\{\Omega_t\}$ is a viscosity solution of (1) and $\Omega_0 = \Omega$.*

In order to prove Theorem 3 we develop a new version of Perron's method which produces a convexity preserving viscosity solution. Perron's method for viscosity solutions was developed by H. Ishii [I87] in the case of Hamilton-Jacobi equations and was used thereafter by many authors to construct viscosity solutions of elliptic, parabolic and geometric equations. However, convexity introduces substantially new features into the method. Indeed, we obtain a solution $\{\Omega_t\}$ as an intersection of all supersolutions which are families of convex sets and which contain a given subsolution. Thus we obtain a family of convex sets by construction. A standard argument similar to the one in [I87] allows us to show that the constructed family is a supersolution. It remains thus to prove that the constructed family is a subsolution. This step is more involved in our case since we have restricted the class of supersolutions to those which are convex in the spatial variables. For this reason we need the additional structure condition (H4).

Convexity of solutions of certain elliptic, parabolic and geometric equations was proved in a number of works, see [ALL97] and references therein. In all these proofs the comparison (and uniqueness) properties of solutions are essential. In this paper we took a different approach – our solution is convex by construction. In particular, we avoid the use of comparison results. However, the proof of the crucial Proposition 28 (essentially due to Alvarez-Lasry-Lions [ALL97]) uses the maximum principle for semicontinuous functions by M. G. Crandall and H. Ishii [CI90], which is the main technical tool for proofs of comparison results for viscosity solutions of elliptic and parabolic equations (see e.g. [CIL92]).

The paper is organized as follows: In Sections 1 - 3 we discuss Equation (1) and prove Theorem 3. In Section 4 we prove that the solutions become circular as they expand to infinite size (for a slightly restricted class of speeds F). In Section 5 we apply our results to the models of collapsing sandpiles and compression molding mentioned above. An Appendix gives the proof of a technical result (Lemma 26) required in proving that our constructed solutions are subsolutions.

1 Preliminaries: Properties of distance functions on open sets and families of open sets

In this section we study some properties of the distance function of an open bounded set. We do this for any dimension $n \geq 2$. In later sections we restrict to $n = 2$.

Let $\Omega \subset \mathbf{R}^n$ be an open bounded set. Denote by $d_\Omega(\cdot)$ the interior distance function of Ω, i.e.

$$d_\Omega(x) = \begin{cases} \text{dist}\,(x, \partial\Omega), & x \in \Omega, \\ 0, & x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

It is well-known that $d_\Omega(\cdot)$ is Lipschitz with constant one.

Let $x \in \mathbf{R}^n$. We denote by $\mathcal{N}_{\partial\Omega}(x)$ the set of all points of $\partial\Omega$ nearest to x, i.e.

$$\mathcal{N}_{\partial\Omega}(x) = \left\{ y \in \partial\Omega, \quad |x-y| = \inf_{z\in\partial\Omega} |x-z| \right\}.$$

Note that $\mathcal{N}_{\partial\Omega}(x)$ may contain more than one point.

Let $x \in \Omega$ and $y \in \mathcal{N}_{\partial\Omega}(x)$. Let the point z lie between x and y, i.e. $z = \lambda x + (1-\lambda)y$, where $\lambda \in (0,1)$. Then by [EH87, Lemma 3.4]

$$\mathcal{N}_{\partial\Omega}(z) = \{\, y \,\}.$$

It follows that for such x and y the set

$$R_x = \{\, z \in \Omega \;\mid\; z = \lambda x + (1-\lambda)y,\ \lambda \geq 0,\ y \in \mathcal{N}_{\partial\Omega}(z) \,\}$$

is an interval of the half-line $\{\, z \mid z = \lambda x + (1-\lambda)y,\ \lambda \geq 0 \,\}$, and $x \in R_x$. We call R_x the distance ray of x (through y). Note that R_x is the longest line segment through x on which the distance to $\partial\Omega$ changes linearly with slope 1. A point $x \in \Omega$ can have more than one distance ray, in which case we allow any of them to be denoted by R_x. All such rays R_x have equal length. We also can define distance rays for points of $\partial\Omega$. If $x \in \partial\Omega$ and there exists a distance ray R of Ω such that x is an endpoint of R, then R is a distance ray of x. If $x \in \partial\Omega$ and there is no distance ray R of Ω such that $x \in R$ (i.e. there is no $z \in \Omega$ such that $x \in \mathcal{N}_{\partial\Omega}(z)$), then the “ray” R_x consists of one point x. A point $x \in \partial\Omega$ can have more than one distance ray, and these rays can have different lengths.

For any $x \in \Omega$ one endpoint of R_x lies on $\partial\Omega$ and belongs to the set $\mathcal{N}_{\partial\Omega}(x)$. Call this endpoint the lower end of R_x. Call another endpoint of R_x the upper end of R_x (the names “lower” and “upper” correspond to the positions of the endpoints of R_x on the graph of $y = d_\Omega(x)$). Let y, z be upper and lower ends of the ray R_x, and let $w = \lambda y + (1-\lambda)z$ where $\lambda \in (0,\ 1)$. Then we say that the point w lies in the relative interior of the ray R_x.

Remark 4 *If $x \in \Omega$ lies in the relative interior of a distance ray R_x, then the ray R_x is the unique distance ray that contains x, and the set $\mathcal{N}_{\partial\Omega}(x)$ consists of one point, see [EH87, Lemma 3.4].*

The following definition extends the function $\gamma(\cdot)$ in (2) from $\partial\Omega$ to $\mathbf{R}^n$.

Definition 5 *The ridge function γ_Ω of the set Ω is defined as follows:*

$$\gamma(x) = \gamma_\Omega(x) = \begin{cases} 0 & x \in \mathbf{R}^n \setminus \overline{\Omega}, \\ \mid R_x \mid & x \in \Omega, \\ \sup \mid R_x \mid & x \in \partial\Omega, \end{cases} \tag{7}$$

where in the last line of (7) the supremum is taken over all distance rays beginning at x.

Lemma 6 *The function $\gamma_\Omega(\cdot)$ is uppersemicontinuous.*

Proof. Uppersemicontinuity of $\gamma_\Omega(\cdot)$ inside Ω is proved in [EH87, Proposition 3.7]. By definition of $\gamma_\Omega(\cdot)$ on $\partial\Omega$ in (7) one can see that the same proof implies that $\gamma_\Omega(\cdot)$ is uppersemicontinuous on $\overline{\Omega}$. Since $\gamma_\Omega \geq 0$, the lemma is proved. □

The function $\gamma_\Omega(\cdot)$ is not continuous in general. There are examples $\Omega \subset \mathbf{R}^2$, Ω convex with $C^{1,1}$-smooth boundary, but with $\gamma_\Omega(\cdot)$ not continuous. But C^2 (local) smoothness of $\partial\Omega$ implies the (local) continuity of $\gamma_\Omega(\cdot)$, as the next lemma shows.

Lemma 7 *Let $y \in \partial\Omega$, and suppose that for some $r > 0$ in an appropriate coordinate system $(x_1, ..., x_{n-1}, x_n) = (x', x_n)$ on $\mathbf{R}^n$*

$$\Omega \cap B_r(y) = \{x_n > \phi(x')\} \cap B_r(y), \quad \textit{where} \quad \phi \in C^2(\mathbf{R}^{n-1}).$$

Then $\gamma_\Omega(\cdot)$ is a continuous function on $B_{\frac{r}{2}}(y) \cap \overline{\Omega}$.

Remark 8 *The proof of Lemma 7 below follows the standard proof of continuity of geodesic distance to the cut locus of a point on a smooth Riemannian manifold, see e.g. [dC92].*

Proof. Uppersemicontinuity of γ_Ω has already been proven, and it remains to prove only lowersemicontinuity of γ_Ω on $B_{\frac{r}{2}}(y) \cap \bar{\Omega}$.

Let $\tilde{x}_i \to \tilde{x}$ as $i \to \infty$ where $\tilde{x}_i \in \overline{\Omega}$ and $\tilde{x} \in \overline{\Omega} \cap B_{\frac{r}{2}}(y)$. Then, since $y \in \partial\Omega$ and dist $(\tilde{x}, y) < \frac{r}{2}$, it follows that

$$\text{there exists } z \in \mathcal{N}_{\partial\Omega}(\tilde{x}) \text{ such that } z \in \Gamma_r, \tag{8}$$

where $\Gamma_r = \partial\Omega \cap B_r(y)$. We need to show that $\liminf_{i\to\infty} \gamma_\Omega(\tilde{x}_i) \geq \gamma_\Omega(\tilde{x})$.

Let $R_{\tilde{x}_i}$ be the distance ray containing $\tilde{x}_i$ (or any such ray if $\tilde{x}_i$ lies on more than one ray). Let x_i be the upper end of $R_{\tilde{x}_i}$. Then $\gamma_\Omega(\tilde{x}_i) \geq d_\Omega(x_i)$. Thus it is enough to prove

$$\liminf_{i\to\infty} d_\Omega(x_i) \geq \gamma_\Omega(\tilde{x}). \tag{9}$$

Let a subsequence of $\{d_\Omega(x_i)\}$ converges to a number A. Then a further subsequence of $\{x_i\}$ (which we will continue to denote $\{x_i\}$) converges to a point x. Then $x \in \overline{\Omega}$, and

$$\lim_{i\to\infty} d_\Omega(x_i) = d_\Omega(x) = A. \tag{10}$$

We have (using that x_i is the upper end of the ray $R_{\tilde{x}_i}$)

$$\text{dist } (x, \tilde{x}) = \lim_{i\to\infty} \text{dist } (x_i, \tilde{x}_i) = \lim_{i\to\infty} [d_\Omega(x_i) - d_\Omega(\tilde{x}_i)] = d_\Omega(x) - d_\Omega(\tilde{x}).$$

It follows that x and $\tilde{x}$ lie on the same distance ray. Also, if z is the point defined in (8), then z and $\tilde{x}$ lie on the same distance ray, and dist $(\tilde{x}, z) = d_\Omega(\tilde{x})$. Thus the points x, $\tilde{x}$ and z lie on the same distance ray, since otherwise we have
$d_\Omega(x) <$ dist $(x, z) \leq$ dist $(x, \tilde{x})$ + dist $(\tilde{x}, z) = [d_\Omega(x) - d_\Omega(\tilde{x})] + d_\Omega(\tilde{x}) = d_\Omega(x)$,
a contradiction. Since $\partial\Omega$ is smooth at z, there exists only one distance ray starting at z. Thus

$$\gamma_\Omega(\tilde{x}) = \gamma_\Omega(x) = \gamma_\Omega(z). \tag{11}$$

It remains to prove that

$$\gamma_\Omega(x) \leq d_\Omega(x). \tag{12}$$

Then (9) follows from (10) and (11), and then continuity of $\gamma(\cdot)$ at $\tilde{x}$ follows.

If there exists a subsequence $\{x_{i_j}\}$ and points $z_{i_j} \in \partial\Omega$ such that $z_{i_j} \in \mathcal{N}_{\partial\Omega}(x_{i_j})$ and $z_{i_j} \to z^* \neq z$ then x is the point of intersection of two distance rays, and it follows (see e.g. [EH87, section 3]) that x is the upper end of the both distance rays. Then $\gamma_\Omega(z) = d_\Omega(z)$, and (12) follows. Thus we can assume that

$$\text{for any choice of} \quad z_i \in \mathcal{N}_{\partial\Omega}(x_i) \quad \text{we have} \quad z_i \to z. \tag{13}$$

Since x_i is the upper endpoint of the distance ray $R_{\tilde{x}_i}$, then from [EGan, Proposition 7.1, Steps 2,3] we have the following: Let

$$\mathcal{N}_{\partial\Omega}(x_i) = \{z_i\}, \tag{14}$$

and let $\partial\Omega$ be twice differentiable at z_i. Then $\max_{m=1,\ldots,n-1} \kappa_m(z_i) > 0$, where $\kappa_1(z_i), \ldots \kappa_{n-1}(z_i)$ are the principal curvatures of $\partial\Omega$ at z_i, and

$$d_\Omega(x_i) = \left[\max_{m=1,\ldots,n-1} \kappa_m(z_i)\right]^{-1}.$$

Now suppose that there exists a subsequence of $\{x_i\}$ (we denote it also $\{x_i\}$) such that (14) holds for all i. Then by (13) and since Γ_r is a C^2 surface, we get

$$d_\Omega(x) = \left[\max_{m=1,\ldots,n-1} \kappa_m(z)\right]^{-1}.$$

Then (12) follows since $\gamma_\Omega(x)$ cannot exceed any positive radius of curvature at $z \in \mathcal{N}_{\partial\Omega}(x)$.

It remains to consider the case

$$\text{for each } i \text{ the set } \mathcal{N}_{\partial\Omega}(x_i) \text{ contains at least two points.} \tag{15}$$

Let z be the point (8). By assumptions of the lemma, for small $\rho > 0$ we have

$$\Omega \cap B_\rho(z) = \{x_n > \phi(x')\} \cap B_\rho(z), \quad \text{where} \quad \phi \in C^2(\mathbf{R}^{n-1}). \tag{16}$$

Let $z = (z', z_n)$ in the coordinates. By shifting and rotation the coordinates we can assume that $z_n = \phi(z') = 0$ and $D\phi(z') = 0$, and (16) holds if $\rho > 0$ is small enough. Let $U \subset \mathbf{R}^{n-1}$ be be the projection of $\Gamma \cap B_\rho(z)$ onto the hyperplane $\{x_n = 0\}$. Then $U = \{x' \mid (x', \phi(x')) \in B_\rho(z)\}$. Then U is an open set. Define the map $\Phi : U \times \mathbf{R}^1 \to \mathbf{R}^n$ by

$$\Phi(x', s) = \tilde{x} + s\,\vec{n}(\tilde{x}) \tag{17}$$

where $\tilde{x} = (x', \phi(x')) \in \Gamma$, and $\vec{n}(\tilde{x})$ is the unit inner normal to $\Gamma = \partial\Omega$ at $\tilde{x}$. Since Γ_r is a C^2 manifold, the map Φ is C^1. Let $d = |x - z|$. The Jacobian of the map Φ at (z', d) is given by the expression ([GT83, (14.98)])

$$\det D\Phi(z', d) = \sqrt{1 + |D\phi(z')|^2} \prod_{m=1}^{n-1} (1 - d\,\kappa_m(z)). \tag{18}$$

Suppose

$$d < \min_{m=1,\ldots,n-1} \frac{1}{\kappa_m(z)}.$$

Then $\det D\Phi(z', d) > 0$ and by the inverse function theorem Φ is a diffeomorphism between a neighborhood of (z', d) in $U \times \mathbf{R}^1 \subset \mathbf{R}^{n-1} \times \mathbf{R}^1$ and an open set containing x. This contradicts (13), (15). $\square$

We will consider below families of sets $\{\Omega_t\}$. Thus we need to study properties of the function

$$(x, t) \to \gamma_{\Omega_t}(x). \tag{19}$$

Definition 9 *Let $\{\Omega_t\}$, $t \in [0, \infty)$, be a family of open bounded sets in $\mathbf{R}^n$. We call the family of sets $\{\Omega_t\}$ Lipschitz (with respect to t) if there exists a constant $M > 0$ such that for any t_1, t_2 we have*

$$(\Omega_{t_1} \setminus \Omega_{t_2}) \cup (\Omega_{t_2} \setminus \Omega_{t_1}) \subseteq N_{M|t_1 - t_2|}(\partial\Omega_{t_1}),$$
$$(\Omega_{t_1} \setminus \Omega_{t_2}) \cup (\Omega_{t_2} \setminus \Omega_{t_1}) \subseteq N_{M|t_1 - t_2|}(\partial\Omega_{t_2}),$$

where $N_r(A)$ denotes the r-neighborhood (in $\mathbf{R}^n$) of the set $A \subset \mathbf{R}^n$. The family $\{\Omega_t\}$ is locally Lipschitz if for any $T_2 > T_1 \geq 0$ the family $\{\Omega_t\}$, $t \in [T_1, T_2)$ is Lipschitz.

Let

$$\begin{aligned} E &= \cup_t \,(\Omega_t \times \{t\}) \subset \mathbf{R}^n \times \mathbf{R}^1_+, \\ \mathbf{\Gamma} &= \cup_t \,(\partial\Omega_t \times \{t\}) \subset \mathbf{R}^n \times \mathbf{R}^1_+, \\ \Gamma_t &= \partial\Omega_t \subset \mathbf{R}^n. \end{aligned} \tag{20}$$

Lemma 10 *Let $\{\Omega_t\}$ be a locally Lipschitz family of open bounded sets. Then*
(a) The function $(x, t) \to d_{\Omega_t}(x)$ is Lipschitz;
(b) The function (19) is uppersemicontinuous.

Proof.
(a) Let $x_i \in \Omega_{t_i}$, $i = 1, 2$, and let $y \in \mathcal{N}_{\partial\Omega_{t_1}}(x_1)$. If $y \in \Omega_{t_2}$, then $y \in \Omega_{t_2} \setminus \Omega_{t_1}$; otherwise $y \notin \Omega_{t_2}$ and so $y \in \partial(\Omega_{t_1} \setminus \Omega_{t_2})$. Thus by Definition 9 there exists $w \in \partial\Omega_{t_2}$ such that $|w - y| \leq M|t_1 - t_2|$. Then

$$d_{\Omega_{t_1}}(x_1) = |x_1 - y| \leq |x_1 - x_2| + |x_2 - w| + |w - y| \leq |x_1 - x_2| + d_{\Omega_{t_2}}(x_2) + M|t_1 - t_2|,$$

from which

$$d_{\Omega_{t_1}}(x_1) - d_{\Omega_{t_2}}(x_2) \le (M+1)|(x_1,t_1) - (x_2,t_2)|.$$

The same holds if we interchange (x_1, t_1) and (x_2, t_2). The claim (a) follows.

(b) It is enough to prove that if $(x_i, t_i) \to (x, t)$ where $x_i \in \overline{\Omega}_{t_i}$, then

$$\gamma_{\Omega_t}(x) \ge \limsup \gamma_{\Omega_{t_i}}(x_i) \tag{21}$$

Let $y_i \in \mathcal{N}_{\partial\Omega_{t_i}}(x_i)$. Selecting a subsequence if necessary, we can assume that $y_i \to y \in \partial\Omega_t$. By (a) we get $y \in \mathcal{N}_{\partial\Omega_t}(x)$. Let z_i be the upper end of the distance ray R_i of Ω_{t_i} such that R_i contains x_i and y_i (such a ray R_i exists by choice of y_i). Then $d_{\Omega_{t_i}}(z_i) = \gamma_{\Omega_{t_i}}(x_i)$. For any converging subsequence $z_{i_j} \to z$ we get by (a) that $y \in \mathcal{N}_{\partial\Omega_t}(z)$. It follows that x lies in the interval connecting y and z. Thus

$$\gamma_{\Omega_t}(x) \ge |z - x| = \lim_{j\to\infty} \gamma_{\Omega_{t_{i_j}}}(x_{i_j})$$

This implies (21). □

Denote by $C^{2;1}(\mathbf{R}^n \times \mathbf{R}^1)$ the set of functions $f(x,t)$ which are twice continuously differentiable in x and once in t.

Lemma 11 *Let $\{\Omega_t\}$ be a locally Lipschitz family of open sets. Let E, $\mathbf{\Gamma}$ be defined by (20). Let $y \in \partial\Omega_{t_0}$ and suppose that for some $r > 0$ we have in an appropriate coordinate system $(x_1, ..., x_{n-1}, x_n) = (x', x_n)$ on $\mathbf{R}^n$*

$$E \cap B^{n+1}_r(y, t_0) = \{x_n > \phi(x', t)\} \cap B^{n+1}_r(y, t_0), \quad \textit{where} \quad \phi \in C^{2;1}(\mathbf{R}^n \times \mathbf{R}^1).$$

Then the function (19) is continuous on $\mathbf{\Gamma} \cap B^{n+1}_{\frac{r}{2}}(y, t_0)$

Proof. The proof of Lemma 7 applies with obvious modifications. We sketch the argument.

It is enough to show continuity of the function (19) at (y, t_0). Using Lemma 10(a), we show similar to the proof of Lemma 7 that we need only to prove the following: *Let $(x_i, t_i) \to (x, t_0)$ as $i \to \infty$ where $i = 1, 2, \ldots$ and $x_i \in \Omega_{t_i}$, and each x_i is the upper end of its distance ray in Ω_{t_i}, and $y \in \mathcal{N}_{\partial\Omega}(x)$. Then*

$$\gamma_{\Omega_t}(x) \le d_{\Omega_t}(x) \tag{22}$$

We can assume that

$$\text{for any choice of} \quad z_i \in \mathcal{N}_{\partial\Omega_{t_i}}(x_i) \quad \text{we have} \quad z_i \to y. \tag{23}$$

Otherwise, as in the proof of Lemma 7 (and using Lemma 10 (a)) we show that x is a point of intersection of two distance rays of Ω_t and thus (22) follows.

Denote by $\kappa_1(x_i, t_i), ..., \kappa_{n-1}(x_i, t_i)$ the principal curvatures of $\partial\Omega_{t_i}$ at x_i. It follows from the assumptions of the Lemma that the principal curvatures depend continuously on (x,t) in $B^{n+1}_r(y, t_0) \cap \Gamma$. Thus, as in Lemma 7, we can reduce to the following case:

$$\text{for each } i \text{ the set } \mathcal{N}_{\partial\Omega_{t_i}}(x_i) \text{ contains at least two points.} \tag{24}$$

There exists $U \subset \mathbf{R}^{n-1}$ and $\rho > 0$ such that in the (x', x_n) coordinates introduced in the formulation of Lemma 11, we have the following: For any $x' \in U$ and $t \in (t_0 - \rho, t_0 + \rho)$, $(x', \phi(x'), t) \in B^{n+1}_{\frac{r}{100}}(y, t_0)$. For each $t \in (t_0 - \rho, t_0 + \rho)$ define a map $\Phi_t : U \times \mathbf{R}^1 \to \mathbf{R}^n$ as follows:

$$\Phi_t(x', s) = \tilde{x} + s\,\vec{n}_t(\tilde{x}) \tag{25}$$

where $\tilde{x} = (x', \phi(x', t)) \in \partial\Omega_t$, and $\vec{n}_t(\tilde{x})$ is the unit inner normal to $\Gamma_t = \partial\Omega_t \subset \mathbf{R}^n$ at $\tilde{x}$. The Jacobian of the map Φ_t is given by the expression

$$\det D\Phi_t(x', s) = \sqrt{1 + |D_{x'}\phi(x', t)|^2} \prod_{m=1}^{n-1} (1 - s\,\kappa_m(\tilde{x}, t)).$$

Thus it follows from the assumptions of the Lemma that the function

$$(x', s, t) \to \det D\Phi_t(x', s)$$

is continuous on $U \times \mathbf{R}^1 \times (t_0 - \rho, t_0 + \rho)$. Let $d = |x - y|$. If $\det D\Phi_{t_0}(y', d) > 0$ then the same holds for any $(x', s, t) \in B_\varepsilon^{n-1}(y') \times (d - \varepsilon, d + \varepsilon) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ for small $\varepsilon > 0$. Then there exists $\delta > 0$ such that for $t \in (t_0 - \delta, t_0 + \delta)$ the inverse map Φ_t^{-1} is defined on $B_\delta^n(y)$. This contradicts (23), (24). Thus $\det D\Phi_{t_0}(y', d) = 0$, and $d = [\max_{m=1,\ldots,n-1} \kappa_m(\tilde{y}, t_0)]^{-1}$, where $\tilde{y} = (y', \phi(y'))$. Since $d_{\Omega_{t_0}}(x) = d$, this implies (22). □

Lemma 12 *Let $\Omega_1 \subseteq \Omega_2$ and $y \in \partial\Omega_1 \cap \partial\Omega_2$. Then $\gamma_{\Omega_2}(y) \geq \gamma_{\Omega_1}(y)$.*

Proof. Let B be a ball of radius $\gamma_{\Omega_1}(y)$ such that $B \subseteq \Omega_1$ and $y \in \partial B$. Then we have $B \subseteq \Omega_2$ and thus $\gamma_{\Omega_2}(y) \geq \gamma_{\Omega_1}(y)$. □

The following lemma shows an important property of $\gamma_\Omega(\cdot)$ for convex Ω: $\gamma_\Omega(\cdot)$ is concave on the "flat" parts of $\partial\Omega$.

Lemma 13 *Let $\Omega \subset \mathbf{R}^n$ be a convex open bounded set, and let P be a hyperplane such that $P \cap \Omega = \emptyset$ and let the set $\tilde{\Gamma} = P \cap \partial\Omega$ is nonempty and contains more than one point. Then the function $\gamma_\Omega(\cdot)$ is concave on $\tilde{\Gamma}$.*

Proof. Since Ω is convex, so is $\tilde{\Gamma}$. If $z_1, z_2 \in \tilde{\Gamma}$ and $\lambda \in (0, 1)$, then $z^* = \lambda z_1 + (1 - \lambda) z_2 \in \tilde{\Gamma}$. Let B_i, $i = 1, 2$ be open balls of radii $\gamma_\Omega(z_i)$ such that $B_i \subset \Omega$ and $z_i \in \partial B_i$. Let A be the convex hull of B_1, B_2. Then $A \subset \Omega$. Let y_i be the center of the ball B_i, $i = 1, 2$. Let $y^* = \lambda y_1 + (1 - \lambda) y_2$ and $r^* = \lambda\gamma_\Omega(z_1) + (1 - \lambda)\gamma_\Omega(z_2)$. Then we have $B_{r^*}(y^*) \subset A \subset \Omega$ and $z^* \in \partial B_{r^*}(y^*)$. Thus

$$\gamma_\Omega(z^*) \geq r^* = \lambda\gamma_\Omega(z_1) + (1 - \lambda)\gamma_\Omega(z_2).$$

□

The following lemma shows that small perturbations of the boundary do not increase the ridge function too much.

Lemma 14 *Let $\{\Omega_t\}_{t \geq 0}$ be a continuous family of open bounded sets. Let $\tau > 0$ and $y \in \partial\Omega_\tau$. Then for any $\varepsilon > 0$ there exists $\rho > 0$ depending on $\{\Omega_t\}$, y, τ, ε such that for any continuous family of open bounded sets $\{\Lambda_t\}$ satisfying $\Lambda_t \subseteq \Omega_t$ for all t, and any (z, t) such that $z \in \partial\Lambda_t \cap B_\rho^{n+1}(y, \tau)$, we have*

$$\gamma_{\Lambda_t}(z) \leq \gamma_{\Omega_\tau}(y) + \varepsilon.$$

Proof. Suppose the assertion is false. Then there exist $\varepsilon > 0$, a sequence of positive numbers $\rho_i \to 0$, a sequence of continuous families of sets $\{\Lambda_t^i\}$ satisfying $\Lambda_t^i \subseteq \Omega_t$ for all i, t, and a sequence of points (z_i, t_i) satisfying $z_i \in \partial\Lambda_{t_i}^i \cap B_{\rho_i}^{n+1}(y, \tau)$ and

$$\gamma_{\Lambda_{t_i}^i}(z_i) > \gamma_{\Omega_\tau}(y) + \varepsilon.$$

We have $(z_i, t_i) \to (y, \tau)$ by definition of z_i, t_i. Let x_i be the center of the ball B_i in x-space of radius $\gamma_{\Lambda_{t_i}^i}(z_i)$ such that $B_i \subset \overline{\Lambda}_{t_i}^i$ and B_i touches $\partial\Lambda_{t_i}^i$ at z_i. Then, since $\{\Omega_t\}$ is a continuous family of bounded sets, there exists a subsequence $x_{i_k} \to x \in \overline{\Omega}_\tau$. Then $\gamma_{\Lambda_{t_{i_k}}^{i_k}}(z_{i_k}) \to |x - y|$, and so

$$|x - y| \geq \gamma_{\Omega_\tau}(y) + \varepsilon. \tag{26}$$

However since $B_i \subset \overline{\Omega}_{t_i}$ for all $i = 1, 2, \ldots$, it follows from continuity of $\{\Omega_t\}$ that if B is the ball in $\mathbf{R}^n$ with the center x and radius $|x - y|$ then $B \subset \Omega_\tau$. Since $y \in \partial\Omega_\tau \cap \partial B$, we have $\gamma_{\Omega_\tau}(y) \geq |x - y|$, contradicting (26). □

2 Definition and basic properties of viscosity solutions

From now on we assume $n = 2$. We write $Q_r^-(x,t) = B_r(x) \times (t-r,t]$ for any $r > 0$.

Definition 15 *A continuous family of open bounded sets $\{\Omega_t\}_{t\geq 0}$ is called a viscosity supersolution (resp. subsolution) of Eq. (1) if for any $t_0 > 0$ and $y_0 \in \partial\Omega_{t_0}$ and any continuous family of sets $\{\Lambda_t\}_{0\leq t\leq t_0}$ satisfying*

(a). *$\Lambda_t \subseteq \Omega_t$ (resp. $\Lambda_t \supseteq \Omega_t$ for subsolutions) for every $t \leq t_0$;*

(b). *$y_0 \in \partial\Lambda_{t_0} \cap \partial\Omega_{t_0}$, and $\{\Lambda_t\}$ is $C^{2;1}$ near (y_0,t_0), i.e. there exists $r > 0$ such that in suitable coordinates (x_1,x_2) for $\mathbf{R}^2$ with origin at y_0,*

$$[\cup_t(\Lambda_t \times \{t\})] \cap Q_r^-(y_0,t_0) = \{x_2 < \phi(x_1,t)\} \cap Q_r^-(y_0,t_0),$$

with $\phi \in C^{2;1}(Q_\rho^-(0,t_0))$ for some $\rho > r$, and $\phi(x_1,t) = v(t-t_0) - \frac{1}{2}\kappa x_1^2 + o(x_1^2 + |t-t_0|)$ as $(x_1,t) \to (0,t_0)$ in $Q_\rho^-(0,t_0)$ for some constants v and κ;

the following inequality also holds:

$$v \geq \ (\textit{resp.} \ \leq \textit{ for subsolutions}) \ F\big(\kappa,\ \gamma_{\Lambda_{t_0}}(y_0),\ t_0\big). \tag{27}$$

A family $\{\Omega_t\}_{t\geq 0}$ is a viscosity solution if it is both a subsolution and a supersolution.

Remark 16 *Note that the above definition contains a nonlocal condition (a).*

Proposition 17 *Let Ω_t be a viscosity solution of (1), and define E and $\mathbf{\Gamma}$ by (20). Let $\mathbf{\Gamma}$ be $C^{2;1}$ at $(y,t_0) \in \mathbf{\Gamma}$, so that in some coordinates (x_1,x_2) in $\mathbf{R}^2$ and some $\rho > 0$ we have*

$$y = (0,0), \quad E \cap Q_\rho^-(y,t_0) = \{x_2 < g(x_1,t)\} \cap Q_\rho^-(y,t_0),$$

where

$$g(x_1,t) = v(t-t_0) - \frac{1}{2}\kappa x_1^2 + o(x_1^2 + |t-t_0|). \tag{28}$$

Then

$$v = F(\kappa, \gamma_{\Omega_{t_0}}(y), t_0),$$

where v, κ are from (28).

Proof. Let $\varepsilon > 0$. For any $M > 0$, define $g_{M,\varepsilon}(x_1,t) = \max\{g(x_1,t), \tilde{g}(x_1,t)\}$, where

$$\tilde{g}(x_1,t) = (v-\varepsilon)(t-t_0) - \frac{1}{2}(\kappa-\varepsilon)x_1^2 - M(x_1^4 + (t-t_0)^2).$$

Then for M large enough,

$$g_{M,\varepsilon}(x_1,t) = \begin{cases} g(x_1,t) & \text{for } (x_1,t) \in Q_\rho^-(0,t_0) \backslash Q_{R(M,\varepsilon)}^-(0,t_0); \\ \tilde{g}(x_1,t) & \text{for } (x_1,t) \in Q_{r(M,\varepsilon)}^-(0,t_0); \end{cases}$$

where $0 < r(M,\varepsilon) < R(M,\varepsilon) < \rho$ and $R(M,\varepsilon) \to 0$ as $M \to \infty$.

Let $\{\Lambda_t^{M,\varepsilon}\}$ be the family of sets defined by

$$\Lambda_t^{M,\varepsilon} = \Omega_t \cup \left(\{x_2 < g_{M,\varepsilon}(x_1,t)\} \cap Q_\rho^-(y,t_0)\right).$$

Then $\{\Lambda_t^{M,\varepsilon}\}$ satisfies conditions (a) and (b) in Definition 15 for the subsolution $\{\Omega_t\}$, so

$$V_{\partial\Lambda_{t_0}^{M,\varepsilon}}(y) \leq F\left(\kappa_{\partial\Lambda_{t_0}^{M,\varepsilon}}(y), \gamma_{\Lambda_{t_0}^{M,\varepsilon}}(y), t_0\right).$$

We have $V_{\Lambda_{t_0}^{M,\varepsilon}}(y) = v - \varepsilon$ and $\kappa_{\Lambda_{t_0}^{M,\varepsilon}}(y) = \kappa - \varepsilon$. It remains to estimate $\gamma_{\Lambda_{t_0}^{M,\varepsilon}}(y)$. We need to prove that $\gamma_{\Lambda_{t_0}^{M,\varepsilon}}(y)$ is not too much larger than $\gamma_{\Omega_{t_0}}(y)$ for ε small and M large. Let $B \subset \Omega_{t_0}$ be the disk of radius $\gamma_{\Omega_{t_0}}(y)$ with $y \in \partial B$, and let z be the center of B. We consider two cases: Either $\mathcal{N}_{\partial\Omega_{t_0}}(z) = \{y\}$, or there is another point $y' \in \partial\Omega_{t_0}$ with $|z - y'| = |z = y|$. In the first case, as in the proof of Lemma 7, it follows from [EGan, Proposition 7.1, steps 2 and 3] that

$$\gamma_{\Omega_{t_0}}(y) = \frac{1}{\kappa}.$$

In particular $\kappa > 0$, so for ε sufficiently small we have $\kappa_{\Lambda_{t_0}^{M,\varepsilon}}(y) = \kappa - \varepsilon > 0$. Since $\kappa_{\Lambda_{t_0}^{M,\varepsilon}}(y)\gamma_{\Lambda_{t_0}^{M,\varepsilon}}(y) \leq 1$ this implies

$$\gamma_{\Lambda_{t_0}^{M}}(y) \leq \frac{1}{\kappa - \varepsilon} = \frac{\kappa}{\kappa - \varepsilon}\gamma_{\Omega_{t_0}}(y).$$

This gives

$$v \leq F\left(\kappa - \varepsilon, \frac{\kappa}{\kappa - \varepsilon}\gamma_{\Omega_{t_0}}(y), t_0\right) + \varepsilon \ \to \ F(\kappa, \gamma_{\Omega_{t_0}}(y), t_0) \quad \text{as } \varepsilon \to 0.$$

In the second case we have

$$|z - y| = |z - y'|.$$

But for all large enough M we have $y' \in \partial\Lambda_{t_0}^{M,\varepsilon}$ and thus for such M

$$\gamma_{\Lambda_{t_0}^{M,\varepsilon}}(y) = |z - y| = \gamma_{\Omega_{t_0}}(y).$$

Thus in this case,

$$v \leq F(\kappa - \varepsilon, \gamma_{\Omega_{t_0}}(y), t_0) \to F(\kappa, \gamma_{\Omega_{t_0}}(y), t_0) \quad \text{as } \varepsilon \to 0.$$

The opposite inequality is proved similarly by constructing a test family for the supersolution property of $\{\Omega_t\}$ in Definition 15. □

3 Perron's method for construction of convex solutions

In this section we prove Theorem 3. The first step towards the proof is to construct convex subsolutions and supersolutions with the initial data Ω.

Proposition 18 *Let $\Omega \subset \mathbf{R}^2$ be an open bounded convex set. Then there exist two locally Lipschitz families of open, bounded, convex sets $\{\Omega_t^1\}$ and $\{\Omega_t^2\}$ such that $\{\Omega_t^1\}$ is a subsolution and $\{\Omega_t^2\}$ is a supersolution of (1) and $\Omega_0^1 = \Omega_0^2 = \Omega$. The Lipschitz constant of $\{\Omega_t^2\}$ on $[0,T]$ depends only on diam(Ω) and T.*

Proof.
The family $\Omega_t^1 \equiv \Omega$ is a subsolution since $F \geq 0$.

It remains to construct a supersolution. Let $N_r(\Omega)$ be the r-neighborhood of Ω. We will use the following properties of r-neighborhoods of convex sets:

Lemma 19 *([F99, Section 3]) Let Ω be a convex set. Then the set $N_r(\Omega)$ has the following properties.*

(a). *The boundary of $N_r(\Omega)$ is $C^{1,1}$ smooth.*

(b). *For any $r_1, r_2 \geq 0$ we have*

$$N_{r_1+r_2} = N_{r_1}(N_{r_2}(\Omega)). \tag{29}$$

(c). *Any point on $\partial N_r(\Omega)$ has a unique nearest point on $\partial\Omega$.*

(d). *Let $x_r \in \partial N_r(\Omega)$ and $\mathcal{N}_{\partial\Omega}(x_r) = \{x\}$ where $x \in \partial\Omega$. Then the segment connecting x and x_r is orthogonal to $\partial N_r(\Omega)$ at x_r (this is well-defined since $\partial N_r(\Omega)$ is $C^{1,1}$).*

(e). *For any point $x \in \partial\Omega$ there exists $x_r \in \partial N_r(\Omega)$ such that $\mathcal{N}_{\partial\Omega}(x_r) = \{x\}$.*

Let $r > 0$. Connect each point $x_r \in \partial N_r(\Omega)$ with its nearest point on $\partial\Omega$ and extend these intervals beyond x_r. We thus decompose $\mathbf{R}^2 \setminus \Omega$ into the set of nonintersecting rays by Lemma 19(c). Each of these rays is orthogonal to the surface $\partial N_R(\Omega)$ at the point of intersection for any $R > 0$ by Lemma 19(d).

Let M be the constant from the inequality in (H1). Let us define

$$t_0 = 0, \;\; t_k = \frac{1}{2M}\sum_{i=1}^{k}\frac{1}{i+1} \quad \text{for} \;\; k = 1, 2, ...$$

Then $\{t_k\}$ is an increasing sequence and $t_k \to \infty$ as $k \to \infty$. Let $D = \text{diam}(\Omega)$. Define

$$\begin{aligned} r(t) &= \frac{k-1}{2}D + (k+1)MD(t - t_{k-1}) \quad \text{for} \quad t \in [t_{k-1}, t_k), \quad \text{where} \quad k = 1, 2, ..., \\ \Omega_t^2 &= N_{r(t)}(\Omega). \end{aligned}$$

Ω_t^2 is defined for all $t \geq 0$, $\Omega_0^2 = \Omega$, and the family $\{\Omega_t^2\}$ is locally Lipschitz, and the Lipschitz constant on any interval $[0, T]$ depends only on D and T. Using Lemma 19, we calculate that for $t \in (t_{k-1}, t_k)$ the outer normal velocity at the every point of $\partial\Omega_t^2$ is $(k+1)MD$. Thus, if a family $\{\Lambda_t\}$ satisfies $\Lambda_t \subseteq \Omega_t$ for each t, if $y \in \partial\Omega_{t^*}^2 \cap \partial\Lambda_{t^*}$ where $t^* \in (t_{k-1}, t_k]$ and if $\{\Lambda_t\}$ is $C^{2,1}$ in the neighborhood of (y, t^*), then

$$(k+2)MD \geq V_{\partial\Lambda_{t^*}}(y) \geq (k+1)MD.$$

We also have:

$$\gamma_{\Lambda_{t^*}}(y) \leq \frac{1}{2}\text{diam}(\Lambda_{t^*}) \leq \frac{k}{4}D.$$

Thus it follows from (H1) that

$$V_{\partial\Lambda_{t^*}}(y) \geq F(\kappa_{\partial\Lambda_{t^*}}(y), \gamma_{\Lambda_{t^*}}(y), t^*),$$

i.e. the family $\{\Omega_t^2\}$ is a supersolution of (1). □

Construction 20 *(Perron's method). Let $\mathcal{S}$ be the set of all locally Lipschitz families of convex sets $\{U_t\}$ satisfying the following:*

(a). *$\{U_t\}$ is a supersolution of (1).*

(b). *For every $t \geq 0$*

$$\Omega_t^1 \subseteq U_t \subseteq \Omega_t^2,$$

where Ω_t^1, Ω_t^2 are sub- and super- solutions constructed in Proposition 18.

Define a family of sets $\{\Omega_t\}$ as follows: For each $s \geq 0$

$$\Omega_s = Int\left[\cap_{\{U_t\}\in\mathcal{S}} U_s\right],$$

where $Int(A)$ denotes the interior of the set A.

Remark 21 *$\{\Omega_t^2\} \in \mathcal{S}$ and thus $\mathcal{S}$ is nonempty set. By construction, Ω_t is an open bounded convex set for each $t \geq 0$, nonempty if $\Omega \neq \emptyset$ (since $\Omega = \Omega_t^1 \subset \Omega_t$), and $\Omega_0 = \Omega$.*

Proposition 22 *$\{\Omega_t\}$ is a locally Lipschitz family.*

Proof. First we observe that supersolutions always expand outwards:

Lemma 23 *Any supersolution $\{U_t\}_{t\geq 0}$ of Eq. (1) is an expanding family, in the sense that $U_{t_1} \subseteq U_{t_2}$ for any $t_2 > t_1 \geq 0$.*

Proof. Suppose the contrary. Then there exists $t_1 \geq 0$ and $t_2 > t_1$ such that $U_{t_1} \backslash U_{t_2} \neq \emptyset$. Let $U^{(i)}$ be a sequence of smoothly bounded open sets with $\overline{U^{(i)}} \subseteq U^{(i+1)}$ and $\cup_i U^{(i)} = U_{t_1}$. For any $r > 0$ let $U_r^{(i)} = \{y \in U^{(i)} : d(y, \partial U^{(i)}) > r\}$. Note that $U_r^{(i)}$ is a smoothly bounded open set for r sufficiently small (depending on i). For each i and each $\varepsilon > 0$ define a family $\{U_t^{(i,\varepsilon)}\}_{t\geq 0}$ by $U_t^{(i,\varepsilon)} = U^{(i)}_{\varepsilon(t-t_1)} \cap U_t$ for $t \geq t_1$ and $U_t^{(i,\varepsilon)} = U^i \cap U_t$ for $t \in [0, t_1)$. Choose i sufficiently large to ensure that $U^{(i)} \backslash U_{t_2} \neq \emptyset$. Then choose $\varepsilon > 0$ sufficiently small to ensure that $U^{(i)}_{\varepsilon(t-t_1)}$ is smoothly bounded for all $t \in [t_1, t_2]$ and $U^{(i)}_{\varepsilon(t_2-t_1)} \backslash U_{t_2} \neq \emptyset$. Let $t_3 = \sup\{t \geq t_1 : U^{(i)}_{\varepsilon(s-t_1)} \backslash U_s = \emptyset \text{ for } t_1 \leq s \leq t\}$. Note that $t_2 > t_3 > t_1$ by the continuity of the family $\{U_t\}$. Then we have $U_t^{(i,\varepsilon)} \subseteq U_t$ for each $t \in [0, t_3]$, $U_t^{(i,\varepsilon)} = U^{(i)}_{\varepsilon(t-t_1)}$ for $t \in [t_1, t_3]$, and there exists $y_0 \in \partial U_{t_3}^{(i,\varepsilon)} \cap \partial U_{t_3}$. Therefore by Definition 15 we have

$$-\varepsilon = V_{\partial U_{t_3}^{(i,\varepsilon)}}(y_0) \geq F\left(\kappa_{\partial U^{(i)}_{\varepsilon(t_3-t_1)}}(y_0), \gamma_{U^{(i)}_{\varepsilon(t_3-t_1)}}(y_0), t_3\right) \geq 0,$$

by condition (H1). This is a contradiction which completes the proof of the Lemma. $\square$

By the Lemma, $\{\Omega_t\}$ is an intersection of expanding families of sets, and so

$$\Omega_{t_1} \subseteq \Omega_{t_2} \quad \text{if} \quad t_1 \leq t_2.$$

We need to show that for every $T_2 > T_1 \geq 0$ there exists a constant L such that for all $t_1, t_2 \in [T_1, T_2]$ with $t_1 < t_2$, we have

$$\Omega_{t_2} \subseteq N_{L|t_2-t_1|}(\Omega_{t_1}). \tag{30}$$

It follows from Construction 20 that for every $\varepsilon > 0$ there exist a finite number of families of sets $\{U_t^k\} \in \mathcal{S}$, $k = 1, ..., K_\varepsilon$ such that

$$\Omega_{t_1} \subseteq \cap_{k=1}^{K_\varepsilon} U_{t_1}^k \subseteq N_\varepsilon(\Omega_{t_1}).$$

Let $\{U_t\}$ be defined by $U_t = \cap_{k=1}^{K_\varepsilon} U_{t_1}^k$ for all $t > 0$. The intersection of a finite number of supersolutions is a supersolution, so we have

$$\{U_t\} \in \mathcal{S}, \quad U_{t_1} \subseteq N_\varepsilon(\Omega_{t_1}).$$

Since U_{t_1} is convex, we can use Proposition 18 to construct a supersolution $\{W_t\}_{t\geq t_1}$ satisfying $W_{t_1} = U_{t_1}$, with Lipschitz constant bounded in terms of $T_2 - T_1$ and $\operatorname{diam}(\Omega_{t_1})$ (for ε sufficiently small). The family of sets $\{\tilde{U}_t\}_{t\geq 0}$ defined by

$$\tilde{U}_t = \begin{cases} U_t, & t < t_1, \\ W_t \cap \Omega_t^2, & t \geq t_1 \end{cases}$$

satisfies $\{\tilde{U}_t\} \in \mathcal{S}$. Thus

$$\Omega_{t_2} \subseteq \tilde{U}_{t_2}.$$

Then it follows from the properties of $\{W_t\}$ and from the property (29) of convex sets that

$$\Omega_{t_2} \subseteq N_{L|t_2-t_1|+\varepsilon}(\Omega_{t_1}).$$

Sending ε to 0 we get (30), completing the proof of Proposition 22. $\square$

Proposition 24 *$\{\Omega_t\}$ is a supersolution of (1).*

Proof. We follow [I87, Proposition 2.4], apart from some modifications to allow nonlocality.

Let $\{\Lambda_t\}$ satisfy conditions (a) and (b) of Definition 15 for supersolutions. That is, $\Lambda_t \subset \Omega_t$ for $t \in [0, t_0]$, $y_0 \in \partial\Lambda_{t_0} \cap \partial\Omega_{t_0}$, and $\{\Lambda_t\}$ is a $C^{2;1}$ family near (y_0, t_0).

We approximate $\{\Lambda_t\}$ by families which stay away from $\{\partial\Omega_t\}$ except near (y_0, t_0). Choose the origin of $\mathbf{R}^2$ to lie in Ω_0. Since $\{\Omega_t\}$ is expanding, there exists $R \geq r > 0$ such that $R \geq |y| \geq r$ for all $y \in \partial\Omega_t$, $0 \leq t \leq t_0$. Choose an orthonormal basis $\{e_1, e_2\}$ for $\mathbf{R}^2$ such that e_2 is the outer unit normal to Λ_{t_0} at y_0, and write $y_0 = y_1 e_2 + y_2 e_2 = (y_1, y_2)$. Let κ be the curvature and v the outward speed of $\{\Lambda_t\}$ at (y_0, t_0). Note that $\kappa \geq 0$ and $v \geq 0$ since $\{\Omega_t\}$ is convex and expanding.

Let $\varepsilon > 0$, and write

$$\phi_\varepsilon(\xi, t) = y_2 + (v + \varepsilon)(t - t_0) - \frac{1}{2}(\kappa + \varepsilon)\xi^2.$$

There exists $\rho(\varepsilon) > 0$ such that

$$\{x_1 e_1 + x_2 e_2 \mid x_2 = \phi_\varepsilon(x_1 - y_1, t),\ |x_1 - y_1| < \rho\} \subset \overline{\Lambda_t}$$

for $t_0 - \rho^2 < t \leq t_0$, and

$$\frac{\varepsilon}{2}\left(|t - t_0| + (x_1 - y_1)^2\right) \leq d\left((x_1, \phi_\varepsilon(x_1 - y_1, t)), \partial\Lambda_t\right) \leq 2\varepsilon\left(|t - t_0| + (x_1 - y_1)^2\right)$$

for $|x_1 - y_1| < \rho$ and $t_0 - \rho^2 < t \leq t_0$. In particular we have

$$(y_1 \pm \rho, \phi_\varepsilon(\rho, t)) \in \left(1 - \frac{\varepsilon\rho^2}{4R}\right)\Lambda_t$$

for $t_0 - \rho^2 \leq t \leq t_0$, and

$$(x_1, \phi_\varepsilon(x_1 - y_1, t_0 - \rho^2)) \in \left(1 - \frac{\varepsilon\rho^2}{4R}\right)\Lambda_{t_0 - \rho^2}$$

for $|x_1 - y_1| \leq \rho$. Define

$$\tilde{\Lambda}_t^\varepsilon = \left(1 - \frac{\varepsilon\rho^2 r}{4R}\right)\Lambda_t \ \cup\ \{(x_1, x_2) \mid |x_1 - y_1| < \rho,\ \phi_\varepsilon(\rho, t) < x_2 < \phi_\varepsilon(x_1 - y_1, t)\}.$$

Our estimates imply that $\{\tilde{\Lambda}_t^\varepsilon\}$ is a continuous family, $C^{2;1}$ in a neighbourhood of (y_0, t_0), lying inside $\{\Omega_t\}$ and touching at (y_0, t_0), with curvature $\kappa + \varepsilon$ and outward speed $v + \varepsilon$ at (y_0, t_0). Also, we have the estimate

$$d(y, \partial\Omega_t) \geq \min\left\{\frac{\varepsilon\rho(\varepsilon)^2 r}{4R},\ \frac{\varepsilon}{2}(|t - t_0| + |y - y_0|^2)\right\} \quad \text{for } y \in \partial\tilde{\Lambda}_t^\varepsilon,\ 0 \leq t \leq t_0. \tag{31}$$

By the construction of $\{\Omega_t\}$ (noting that the intersection of a finite number of supersolutions is a supersolution), there exist families $\{U_t^k\} \in \mathcal{S}$ for $k = 1, 2, \ldots$ such that

$$\Omega_t \subseteq U_t^k \subseteq N_{1/k}\Omega_t$$

for $0 \leq t \leq t_0$.

For each k, let $h(k) = \sup\{h : \tilde{\Lambda}_t^\varepsilon + h e_2 \subseteq U_t^k \text{ for } 0 \leq t \leq t_0\}$. Then there exist points (y_k, t_k) with $y_k \in \partial U_{t_k}^k \cap \partial\left(\tilde{\Lambda}_{t_k}^\varepsilon + h(k)e_2\right)$. $\{U_t^k\}$ is a supersolution, so by Definition 15

$$V_{\partial\tilde{\Lambda}_{t_k}^\varepsilon}(y_k) \geq F(\kappa_{\partial\tilde{\Lambda}_{t_k}^\varepsilon}(y_k), \gamma_{\tilde{\Lambda}_{t_k}^\varepsilon}(y_k), t_k).$$

Since $d(y_k, \partial\Omega_{t_k}) \leq d(y_k, \partial U_{t_k}^k) \leq h(k) \leq 1/k$, the estimate (31) implies for $1/k < \varepsilon\rho(\varepsilon)^2 r/4R$,

$$|t_k - t_0| + |y_k - y_0|^2 \leq \frac{2}{k\varepsilon} \to 0 \text{ as } k \to \infty.$$

Since $\tilde{\Lambda}_t^\varepsilon$ is $C^{2;1}$ near (y_0, t_0), we have by taking $k \to \infty$

$$v + \varepsilon \geq F(\kappa + \varepsilon, \gamma_{\tilde{\Lambda}_{t_0}^\varepsilon}(y_0), t_0),$$

where we used Lemma 11 in the second argument. Lemma 14 and the construction above show that $\gamma_{\tilde{\Lambda}_{t_0}^\varepsilon}(y_0) \to \gamma_{\Lambda_{t_0}}(y_0)$ as $\varepsilon \to 0$. Therefore we have

$$v \geq F(\kappa, \gamma_{\Lambda_{t_0}}(y_0), t_0)$$

as required. □

In order to complete the proof of Theorem 3 it remains to prove the following:

Proposition 25 *$\{\Omega_t\}$ is a subsolution of (1).*

Proof. Suppose $\{\Omega_t\}$ is not a subsolution of (1). Then there exists $t_0 > 0$, a continuous family of sets $\{\Lambda_t\}$, and a point y_0 such that $\Lambda_t \supseteq \Omega_t$ for every $t \in [0, t_0]$, $y_0 \in \partial\Lambda_{t_0} \cap \partial\Omega_{t_0}$, $\{\Lambda_t\}$ is $C^{2;1}$ in a neighborhood of (y_0, t_0), and

$$V_{\partial\Lambda_{t_0}}(y_0) > F(\kappa_{\partial\Lambda_{t_0}}(y_0), \gamma_{\Lambda_{t_0}}(y_0), t_0). \tag{32}$$

Note that

$$y \notin \partial\Omega^1_{t_0}, \tag{33}$$

where Ω^1_t is a subsolution of (1) constructed in Proposition 18. Indeed, if $y \in \partial\Omega^1_{t_0}$ then $\{\Lambda_t\}$ is a test family for the subsolution property of $\{\Omega^1_t\}$, which contradicts (32).

From (32), property (H2) of F and Lemma 12 we get

$$V_{\partial\Lambda_{t_0}}(y_0) > F(\kappa_{\partial\Lambda_{t_0}}(y_0), \gamma_{\Omega_{t_0}}(y_0), t_0). \tag{34}$$

It suffices to show that the weaker inequality (34) leads to a contradiction.

Consider two cases.

Case 1. y_0 is an extreme point of $\overline{\Omega}_{t_0}$, i.e. there do not exist distinct points $y', y'' \in \overline{\Omega}_{t_0}$ and a number $\lambda \in (0,1)$ such that $y_0 = \lambda y' + (1-\lambda)y''$.

In the appropriate coordinate system in $\mathbf{R}^2$ we have $y_0 = (0,0)$, and there exist functions $f \in C^{2;1}(\mathbf{R}^1 \times [0, t_0])$, $g \in C(\mathbf{R}^1 \times [0, t_0])$ and $r > 0$ such that

$$\Lambda_t \cap B_r(y_0) = \{x_2 < f(x_1, t)\} \cap B_r(y_0) \quad \text{for any } t \in [t_0 - r^2, t_0], \tag{35}$$
$$\Omega_t \cap B_r(y_0) = \{x_2 < g(x_1, t)\} \cap B_r(y_0) \quad \text{for any } t \in [t_0 - r^2, t_0], \tag{36}$$

and the functions $f(z,t)$ and $g(z,t)$ satisfy

$$\begin{aligned}
&f \geq g \qquad \text{on } Q_r^-(y_0, t_0);\\
&f(0, t_0) = g(0, t_0) = 0, \quad \partial_z f(0, t_0) = 0;\\
&|g(z, t_1) - g(z, t_2)| \leq M|t_1 - t_2| \quad \text{for any } \ z \in [-2r_0,\, 2r_0] \ \text{ and } \ t_1, t_2 \in [0, t_0];\\
&z \to g(z,t) \quad \text{is concave for every} \quad t \in \mathbf{R}^1_+.
\end{aligned} \tag{37}$$

Then we clearly have

$$g(z, t_0) \leq 0 \quad \text{for all} \quad z \in \mathbf{R}^1. \tag{38}$$

The condition of Case 1 can be expressed as the following (possibly changing x_1 to $-x_1$):

$$g(z, t_0) < 0 \quad \text{for all} \quad z > 0. \tag{39}$$

We can now rewrite (34) as

$$\partial_t f(0, t_0) > F(-\partial^2_{zz} f(0, t_0), \gamma_{\Omega_{t_0}}(y_0), t_0). \tag{40}$$

In Case 1, as in [I87], we use (40) to locally perturb $\{\Omega_t\}$ near (y_0, t_0) to produce a new supersolution which is a strict subset of $\{\Omega_t\}$ for some t, contradicting Construction 20. However, since this new supersolution has to be a family of convex sets, the argument has new features, in particular in Case 1b below.

Consider two cases.

Case 1a.

$$\partial^2_{zz} f(0, t_0) < 0.$$

Proof in Case 1a. Let $\varepsilon > 0$ be such that

$$0 < \varepsilon < -\partial^2_{zz} f(0, t_0).$$

Define the function

$$\varphi(z, t) = f(z, t) + \varepsilon \left(z^2 + t_0 - t\right).$$

Then $\varphi \in C^{2;1}(\mathbf{R}^1 \times [0, t_0])$ and $\partial^2_{zz}\varphi(0, t_0) < 0$, so φ is concave in z for every t for (z, t) in a small neighborhood of $(0, t_0)$. Also,

$$\varphi > g \quad \text{in} \quad \left(\mathbf{R}^1 \times [0, t_0]\right) \setminus (0, t_0).$$

Given $r > 0$ we can choose $\delta > 0$ such that

$$\varphi > g + \delta \quad \text{on} \quad \partial Q_r^-(0, t_0) \cap \left(\mathbf{R}^1 \times [0, t_0)\right).$$

Let

$$\tilde{g}(z, t) = \min(\, g(z, t),\ \varphi(z, t) - \delta\,).$$

Then

$$\tilde{g} \equiv g \quad \text{on} \quad \partial Q_r^-(0, t_0) \cap \left(\mathbf{R}^1 \times [0, t_0)\right); \qquad \tilde{g}(0, t_0) < g(0, t_0).$$

Let

$$\tilde{\Omega}_t = \left[\Omega_t \setminus Q_r^-(y_0, t_0)\right] \cup \left[\{x_2 < \tilde{g}(x_1, t)\} \cap Q_r^-(y_0, t_0)\right].$$

Then $\{\tilde{\Omega}_t\}$ is a continuous family of convex sets for r small enough. Using (33), we have for small $r > 0$

$$\Omega_t^1 \subseteq \tilde{\Omega}_t \subseteq \Omega_t \quad \text{for all} \quad t \in [0, t_0]; \qquad \tilde{\Omega}_{t_0} \neq \Omega_{t_0}.$$

We continue the family $\{\tilde{\Omega}_t\}$ for $t > t_0$ using the supersolution construction of Proposition 18. It remains to show that $\{\tilde{\Omega}_t\}$ is a supersolution if $r > 0$ above is chosen small enough.

From Lemma 14 we see that for any $\sigma > 0$ we can choose $r > 0$ such that

$$\gamma_{\tilde{\Omega}_t}(z) \leq \gamma_{\Omega_{t_0}}(y_0) + \sigma \quad \text{for any} \quad (z, t) \in \tilde{\mathbf{\Gamma}} \cap Q_r^-(y_0, t_0). \tag{41}$$

where

$$\tilde{\mathbf{\Gamma}} = \cup_s \partial \left(\tilde{\Omega}_s \times \{s\}\right). \tag{42}$$

In addition let $r < r_0$. We have

$$\tilde{\mathbf{\Gamma}} \setminus \mathbf{\Gamma} \subset Q_r^-(y, t_0) \cap \{x_2 = \varphi(x_1, t)\}. \tag{43}$$

Since the function φ is $C^{2;1}$, choosing small enough $\sigma > 0$ and taking corresponding $r(\sigma)$ and $\delta(r)$, we have by (41), (40) and continuity of F

$$V_{\partial\tilde{\Omega}_t}(x) > F(\kappa_{\partial\tilde{\Omega}_t}(x), \gamma_{\tilde{\Omega}_t}(x), t) \quad \text{for any} \quad (x, t) \in \tilde{\mathbf{\Gamma}} \setminus \mathbf{\Gamma}. \tag{44}$$

Now consider a family $\{U_t\}$ with $U_t \subset \tilde{\Omega}_t$ for any t, and suppose $y_1 \in \partial\tilde{\Omega}_{t_1} \cap \partial U_{t_1}$, with $\{U_t\}$ a $C^{2;1}$ family near (y_1, t_1). We need to show that

$$V_{\partial U_{t_1}}(y_1) \geq F(\kappa_{\partial U_{t_1}}(y_1), \gamma_{U_{t_1}}(y_1), t_1). \tag{45}$$

If $t_1 > t_0$ this is true by construction. If $y_1 \in \partial\Omega_{t_1}$ then the inequality (45) follows from the fact that $\{\Omega_t\}$ is a supersolution. If $y_1 \in \partial\tilde{\Omega}_{t_1} \setminus \partial\Omega_{t_1}$, then $\tilde{\mathbf{\Gamma}}$ is $C^{2,1}$ near (y_1, t_1) and we have $\kappa_{\partial U_{t_1}}(y_1) \geq \kappa_{\partial\tilde{\Omega}_{t_1}}(y_1)$. Inequality (45) follows from (44), Lemma 12, and properties (H2), (H3) of F. So $\{\tilde{\Omega}_t\}$ is a supersolution with $\tilde{\Omega}_{t_0}$ a strict subset of Ω_{t_0}. This is impossible by the construction of $\{\Omega_t\}$, and the proof of Case 1a is complete.

Case 1b.

$$\partial^2_{zz} f(0, t_0) \geq 0. \tag{46}$$

Proof in Case 1b. Since $\kappa_{\Lambda_{t_0}}(y_0) = -\partial^2_{zz} f(0, t_0)$, we have from (34), (46) and (H2)

$$V_{\{\partial\Lambda_{t_0}\}}(y_0) > F(0, \gamma_{\Omega_{t_0}}(y_0), t_0). \tag{47}$$

In this case the function $\varphi(x, t)$ may be nonconvex in x even near $(0, t_0)$, so we cannot repeat directly the construction of Case 1a. Instead, we use the following:

Lemma 26 *Let $r_0 \in (0, 1/2)$, and write $I_{2r_0} = (-2r_0, 2r_0)$ and $Q^-_{2r_0} = I_{2r_0} \times (-2r_0, 0]$. Let $U \in C(Q^-_{2r_0})$ satisfy*

$$|U(x, t_1) - U(x, t_2)| \leq M|t_1 - t_2| \quad \textit{for all } x \in I_{2r_0} \textit{ and } t_1,\ t_2 \in (-2r_0, 0]. \tag{48}$$

Assume in addition the following:

$$x \to U(x,t) \quad \textit{is concave for any } t \in (-2r_0, 0];$$

$$U(0,0) = 0; \tag{49}$$

$$U(x, 0) \leq 0 \qquad \textit{for all } x \in I_{2r_0}; \tag{50}$$

$$U(x, 0) < 0 \qquad \textit{for all } x \in (r_*, 2r_0) \tag{51}$$

where $r_* \in [0, (r_0/4)^2)$, and suppose there exists $\hat{f} \in C^{2;1}(Q^-_{2r_0})$ such that

$$\hat{f}(x,t) \geq U(x,t) \quad \textit{for all } (x,t) \in Q^-_{2r_0}; \tag{52}$$

$$\hat{f}(0,0) = \partial_x \hat{f}(0,0) = \partial_t \hat{f}(0,0) = 0. \tag{53}$$

Then for any $\varepsilon > 0$, $r \in (2\sqrt{r_*}, r_0)$ there exists a function $V \in C(Q^-_{2r_0})$ satisfying (48) and such that

$$V(\cdot, t) \textit{ is concave for any } t \in (-2r_0, 0],$$

and

$$V(x,t) \leq U(x,t) \qquad \textit{for all } (x,t) \in Q^-_{2r_0}; \tag{54}$$

$$V(x,t) = U(x,t) \qquad \textit{for all } (x,t) \in Q^-_{2r_0} \setminus Q^-_r; \tag{55}$$

$$V(\tilde{x},\tilde{t}) < U(\tilde{x},\tilde{t}) \qquad \textit{for some } (\tilde{x},\tilde{t}) \in Q^-_r; \tag{56}$$

and for any $(\tilde{x},\tilde{t})$ at which (56) holds there exist $\rho > 0$ and $\varphi \in C^{2;1}(B_\rho(\tilde{x},\tilde{t}) \cap Q^-_r)$ such that

$$\begin{aligned} &x \to \varphi(x,t) \quad \textit{is linear for any } t; \\ &V = \varphi \quad \textit{in } B_\rho(\tilde{x},\tilde{t}) \cap Q^-_r, \end{aligned} \tag{57}$$

and for any $(x,t) \in B_\rho(\tilde{x},\tilde{t}) \cap Q^-_r$

$$\partial_t \varphi(x,t) \geq -2\varepsilon - (M+1)\sqrt{r_*}, \quad |\partial_x \varphi(x,t)| \leq \varepsilon. \tag{58}$$

We give the proof of Lemma 26 in an Appendix.

Remark 27 *Lemma 26 is the only place in the proof of Theorem 3 which is specific for curves on a plane, and does not work in higher dimensions.*

Now we can finish the proof of Case 1b. It follows from (37), (38), (39), (46) that the function

$$U(z,t) = g(z,t+t_0) - \partial_t f(0,t_0)t \tag{59}$$

satisfies the conditions of Lemma 26 with $r_* = 0$ and

$$\hat{f}(z,t) = f(z,t+t_0) - \partial_t f(0,t_0)t.$$

Then let $V(z,t)$ be the function constructed in Lemma 26 for the function (59) for ε, r to be chosen below. Define

$$\tilde{g}(z,t) = V(z,t-t_0) + \partial_t f(0,t_0)(t-t_0).$$

In analogy with Case 1a, define

$$\tilde{\Omega}_t = (\Omega_t \setminus Q_r^-(y_0,t_0)) \cup (\{\, x_2 < \tilde{g}(x_1,t)\,\} \cap Q_r^-(y_0,t_0)), \tag{60}$$

and extend this family beyond $t = t_0$ using the construction of supersolutions from Proposition 18. The properties of $V(z,t)$ proved in Lemma 26 imply that $\{\tilde{\Omega}_t\}$ is a Lipschitz family of convex sets, and

$$\tilde{\Omega}_t \subseteq \Omega_t \quad \text{for all} \quad t \geq 0; \quad \tilde{\Omega}_{\tilde{t}} \neq \Omega_{\tilde{t}} \quad \text{for some} \quad \tilde{t} \in (t_0 - r, t_0];$$

(we assume that, possibly making r smaller, we have $t_0 - r > 0$). Define $\tilde{\mathbf{\Gamma}}$ by (42), where $\tilde{\Omega}_t$ is defined by (60). Then properties of $V(z,t)$ imply that (41) and (43) hold true if r, ε are small enough.

We can rewrite (47) as

$$\partial_t f(0,t_0) > F(0, \gamma_{\Omega_{t_0}}(y), t_0). \tag{61}$$

From this, as in Case 1a, using (41), properties of $V(z,t)$ (in particular (57) and (58)) and continuity of F, and making δ, r, ε smaller if necessary we get the inequality

$$\partial_t \varphi(x) > F(0, \gamma_{\tilde{\Omega}_t}(x), t) \quad \text{for any} \quad (x,t) \in \tilde{\mathbf{\Gamma}} \setminus \mathbf{\Gamma}, \tag{62}$$

where φ is the function from (57).

Now let the family of sets $\{U_t\}$ satisfy $U_t \subseteq \tilde{\Omega}_t$ for any $t \in [0,t_0]$, let $y_1 \in \partial\tilde{\Omega}_{t_1} \cap \partial U_{t_1}$, and assume $\{U_t\}$ is a $C^{2;1}$ family near (y_1,t_1). We need to show that

$$V_{\partial U_{t_1}}(y_1) \geq F(\kappa_{\partial U_{t_1}}(y_1), \gamma_{U_{t_1}}(y_1), t_1). \tag{63}$$

If $y_1 \in \partial\Omega_{t_1}$ then the inequality (63) follows from the fact that $\{\Omega_t\}$ is a supersolution. Let $y_1 \in \partial\tilde{\Omega}_{t_1} \setminus \partial\Omega_{t_1}$. Then (62) holds at (y_1,t_1). Since $\tilde{\Omega}_{t_1}$ is convex and $U_{t_1} \subseteq \tilde{\Omega}_{t_1}$, we get

$$\kappa_{\partial U_{t_1}}(y_1) \geq 0.$$

Now the inequality (63) follows from (62), Lemma 12 and properties (H2), (H3) of F. So $\{\tilde{\Omega}_t\}$ is a supersolution. Also, from (33), $\Omega_t^1 \subseteq \tilde{\Omega}_t$ for all $t > 0$ if r is small. Thus $\{\tilde{\Omega}_t\} \in \mathcal{S}$. This is a contradiction since $\tilde{\Omega}_{t_1}$ is a strict subset of Ω_{t_1}. Thus (34) can not hold, and we have completed the proof of Case 1b, hence Case 1.

Case 2. y_0 is not an extreme point of $\overline{\Omega}_{t_0}$, i.e. there exist distinct points y_1, $y_2 \in \overline{\Omega}_{t_0}$ and a number $\lambda \in (0,1)$ such that $y_0 = \lambda y_1 + (1-\lambda)y_2$.

Proof in Case 2. Since $\overline{\Omega}_{t_0}$ is a closed convex bounded set, $\overline{\Omega}_{t_0}$ is the convex hull of its extreme points, see e.g. [R72, Theorem 18.4]. Thus we can assume y_1, y_2, λ that y_1 and y_2 are extreme points of $\overline{\Omega}_{t_0}$. We will always assume this below.

The condition of Case 2 means that y_0 lies in the relative interior of a linear segment $I \subset \partial\Omega_{t_0}$. It is clear that in this case we cannot modify the family of sets $\{\Omega_t\}$ locally near (y_0,t_0) so that the resulting family consists of convex sets. Thus we cannot use the method of the proof from [I87] in the proof below.

The idea of the proof in Case 2 is to show that if (34) holds at (y_0,t_0), then similar conditions hold either at (y_1,t_0) or at (y_2,t_0) if we replace $\{\Lambda_t\}$ by some other smooth test family of sets.

Then, since y_1 and y_2 are extreme points of Ω_{t_0}, we are under the conditions of the Case 1, and Proposition 25 is proved.

Proposition 28 below is a modification and an extension to the parabolic case of [ALL97, Proposition 1]. We give the general n-dimensional version although in this paper we will use only the case $n = 1$.

Introduce first some notation. Let $\Omega \subset \mathbf{R}^n$ be a convex open set, and let $u \in C(\overline{\Omega} \times \mathbf{R}^1)$. Let $\mathcal{S}^n$ be the set of symmetric $n \times n$ matrices. Let $(x,t) \in \overline{\Omega} \times \mathbf{R}^1$. The parabolic (2,1)-subjet $J_{-}^{2,1}u(x,t) \subset \mathbf{R}^1 \times \mathbf{R}^n \times \mathcal{S}^n$ of u at (x,t) is defined as follows: $(\tau,\, p,\, A) \in J_{-}^{2,1}u(x,t)$ if and only if there exists $\varphi \in C^{2;1}(\mathbf{R}^n \times \mathbf{R}^1)$ such that (x,t) is a local minimum point of $u - \varphi$ on $\overline{\Omega} \times (-\infty, t]$ and $(\tau,\, p,\, A) = (\partial_t \varphi(x,t),\; D\varphi(x,t),\; D^2\varphi(x,t))$. Note that $J_{-}^{2,1}u(x,t)$ can be the empty set. Define:

$$\overline{J}_{-}^{2,1}u(x,t) = \{(\tau,\, p,\, A) \in \mathbf{R}^1 \times \mathbf{R}^n \times \mathcal{S}^n \;:\; \text{there exist } (x_n, t_n) \in \overline{\Omega} \times \mathbf{R}^1 \text{ and } (\tau_n,\, p_n,\, A_n) \in J_{-}^{2,1}u(x_n, t_n) \text{ such that } (x_n,\, t_n,\, \tau_n,\, p_n,\, A_n) \to (x,\, t,\, \tau,\, p,\, A)\}$$

Proposition 28 *Let $u \in C(\overline{\Omega} \times (-\infty, 0])$ be a function satisfying:*

$$\begin{aligned} &x \to u(x,t) \quad \textit{is convex for each } t; \\ &|u(x,t_1) - u(x,t_2)| \le M|t_1 - t_2| \quad \textit{for each} \quad x \in \overline{\Omega},\;\; t_1,\, t_2 \le 0. \end{aligned} \tag{64}$$

Let $k \le n+1$ and let $x^*, x_1, ..., x_k \in \overline{\Omega},\, ,\ \lambda_1, ...\lambda_k \in \mathbf{R}^1$ satisfy

$$\lambda_i > 0, \quad \textit{for} \;\; i = 1, ..., k; \qquad \sum_{i=1}^{k} \lambda_i = 1;$$

$$x^* = \sum_{i=1}^{k} \lambda_i x_i;$$

$$u(x^*, 0) = \sum_{i=1}^{k} \lambda_i u(x_i, 0);$$

Let $(\tau, p, A) \in J_{-}^{2,1}u(x^*, 0)$. Then there exist $A_1, ..., A_k \in \mathcal{S}^n$ and $\tau_1, ..., \tau_k \in \mathbf{R}^1$ such that

$$(\tau_i, p, A_i) \in \overline{J}_{-}^{2,1}u(x_i, 0) \;\textit{for}\; i = 1, ..., k; \tag{65}$$

$$\tau = \sum_{i=1}^{k} \lambda_i \tau_i. \tag{66}$$

In addition, if $A \ge 0$ (i.e. A is nonnegative definite), then for any $\varepsilon > 0$ small enough the matrices $A_1, ..., A_k$ and the numbers $\tau_1, ..., \tau_k$ can be chosen to satisfy (65), (66) and the following additional properties:

$$\begin{aligned} &A_i \ge 0; \\ &A - \varepsilon A^2 \le (\lambda_1 A_1^{-1} + ... + \lambda_k A_k^{-k})^{-1}. \end{aligned} \tag{67}$$

Remark 29 *If any of the matrices A_i is degenerate, then the right-hand side of the inequality (67) should be obtained as follows: Use $A_i + \frac{1}{m}I$ instead of A_i in the expression $(\lambda_1 A_1^{-1} + ... + \lambda_k A_k^{-k})^{-1}$ and send $m \to \infty$. The sequence converges (see the proof in [ALL97]), and the inequality (67) holds with this limit in the right-hand side.*

Proof. We define another, a more restrictive, parabolic (2,1)-subjet $\mathcal{P}_{-}^{2,1}u(x,t) \subset \mathbf{R}^1 \times \mathbf{R}^n \times \mathcal{S}^n$ of u at (x,t) as follows: $(\tau,\, p,\, A) \in \mathcal{P}_{-}^{2,1}u(x,t)$ if and only if there exists $\varphi \in C^{2;1}(\mathbf{R}^n \times \mathbf{R}^1)$ such that (x,t) is a local minimum point of $u - \varphi$ on $\overline{\Omega} \times (-\infty, \infty)$ and $(\tau,\, p,\, A) = (\partial_t\varphi(x,t),\; D\varphi(x,t),\; D^2\varphi(x,t))$. Define

$$\overline{\mathcal{P}}_{-}^{2,1}u(x,t) = \{(\tau,\, p,\, A) \in \mathbf{R}^1 \times \mathbf{R}^n \times \mathcal{S}^n \;:\; \text{there exist } (x_n, t_n) \in \overline{\Omega} \times \mathbf{R}^1 \text{ and } (\tau_n,\, p_n,\, A_n) \in \mathcal{P}_{-}^{2,1}u(x_n, t_n) \text{ such that } (x_n,\, t_n,\, \tau_n,\, p_n,\, A_n) \to (x,\, t,\, \tau,\, p,\, A)\}$$

Step 1. Assume first that $(\tau, p, A) \in \mathcal{P}_-^{2,1}u(x^*,0)$. Then there exists $\varphi \in C^{2;1}(\mathbf{R}^n \times \mathbf{R}^1)$, such that $u - \varphi$ achieves a (global) minimum on $\overline{\Omega} \times (-\infty, \infty)$ at $(x^*, 0)$ and

$$(\tau,\ p,\ A) = \left(\partial_t\varphi(x^*,0),\ D\varphi(x^*,0),\ D^2\varphi(x^*,0)\right),$$

see e.g. [Son93, Lemma 14.1]. Then for any $y_1, ..., y_k \in \overline{\Omega}$, $t \in \mathbf{R}^1$ we have

$$\begin{array}{cc}
\lambda_1 u(y_1,t) + ... + \lambda_k u(y_k,t) - \varphi(\lambda_1 y_1 + ... + \lambda_k y_k,\ t) & \geq \\
u(\lambda_1 y_1 + ... + \lambda_k y_k,\ t) - \varphi(\lambda_1 y_1 + ... + \lambda_k y_k,\ t) & \geq \\
u(\lambda_1 x_1 + ... + \lambda_k x_k,\ 0) - \varphi(\lambda_1 x_1 + ... + \lambda_k x_k,\ 0) & = \\
\lambda_1 u(x_1,0) + ... + \lambda_k u(x_k,0) - \varphi(\lambda_1 x_1 + ... + \lambda_k x_k,\ 0). &
\end{array}$$

Thus the point $(x_1, ..., x_k, 0) \in \mathbf{R}^{nk} \times (-\infty,\infty)$ is a point of minimum in $(\overline{\Omega})^k \times (-\infty,\infty)$ of the function

$$(y_1, ..., y_k, t) \to \lambda_1 u(y_1,t) + ... + \lambda_k u(y_k,t) - \varphi(\lambda_1 y_1 + ... + \lambda_k y_k,\ t).$$

The conditions of Proposition 28 (in particular, (64) imply that for small $\varepsilon > 0$ we can use [CIL92, Theorem 8.3] to get $A_1, ..., A_k \in \mathcal{S}^n$ and $\tau_1, ..., \tau_k \in \mathbf{R}^1$ such that $(\tau_i, p, A_i) \in \overline{\mathcal{P}}_-^{2,1}u(x_i,0)$ for $i = 1, ..., k$ and (66) are satisfied and

$$\begin{pmatrix}
\lambda_1 A_1 & . & . & . & 0 \\
. & & . & & . \\
. & & & . & . \\
. & & & . & . \\
0 & . & . & . & \lambda_k A_k
\end{pmatrix} \geq \begin{pmatrix}
\lambda_1^2 B & . & . & . & \lambda_1\lambda_k B \\
. & & . & & . \\
. & & & . & . \\
. & & & . & . \\
\lambda_1\lambda_k B & . & . & . & \lambda_k^2 B
\end{pmatrix}, \tag{68}$$

where $B = A - \varepsilon A^2$.

If $A \geq 0$ then (68) implies $A_i \geq 0$ and (67) as in Proposition 1 of [ALL97].

Step 2. Now let $(\tau, p, A) \in J_-^{2,1}u(x^*,0)$. Let $\varphi \in C^{2;1}(\mathbf{R}^n \times \mathbf{R}^1)$, and $(x^*, 0)$ is a local minimum point of $u - \varphi$ on $\overline{\Omega} \times (-\infty, 0]$, and $(\tau,\ p,\ A) = (\partial_t\varphi(x^*,0),\ D\varphi(x^*,0),\ D^2\varphi(x^*,0))$. Then by (64) $\partial_t\varphi(x^*,0) \leq M$ and thus for some $\rho > 0$

$$\partial_t\varphi < 2M \qquad \text{in}\ \ B_\rho^n(x^*) \times (-\rho, \rho). \tag{69}$$

Define for $(x,t) \in \overline{\Omega} \times \mathbf{R}^1$

$$\hat{u}(x,t) = \begin{cases} u(x,t) & \text{if}\ \ t \leq 0; \\ u(x,t) + 2Mt & \text{if}\ \ t > 0. \end{cases}$$

Then $\hat{u} \in C(\overline{\Omega} \times \mathbf{R}^1)$ and, using (64), (69), we get for any $(x,t) \in B_\rho^n(x^*) \times (0, \rho)$:

$$\begin{array}{rcl}
\hat{u}(x,t) - \varphi(x,t) & = & u(x,t) + 2Mt - \varphi(x,t) \\
& > & u(x,0) - Mt + 2Mt - [\varphi(x,0) + Mt] \geq \hat{u}(x^*,0) - \varphi(x^*,0).
\end{array}$$

Thus $(x^*, 0)$ is a local minimum point of $\hat{u} - \varphi$ on $\overline{\Omega} \times (-\infty,\infty)$. Thus $(\tau, p, A) \in \mathcal{P}_-^{2,1}\hat{u}(x^*,0)$. Note that $\hat{u}$ satisfies all conditions of Proposition 28, in particular (64) holds with constant $3M$. Then, by Step 1, there exist $A_1, ..., A_k \in \mathcal{S}^n$ and $\tau_1, ..., \tau_k \in \mathbf{R}^1$ such that $(\tau_i, p, A_i) \in \overline{\mathcal{P}}_-^{2,1}\hat{u}(x_i,0)$ for $i = 1, ..., k$ and (66) holds, and $A_i \geq 0$ and satisfy (67) if $A \geq 0$. It follows that $(\tau_i, p, A_i) \in \overline{J}_-^{2,1}u(x_i,0)$ for $i = 1, ..., k$. Indeed, if $(x,t) \in \overline{\Omega} \times \mathbf{R}^1$ and $(\theta, q, X) \in \mathcal{P}_-^{2,1}\hat{u}(x,t)$ then $(\theta, q, X) \in J_-^{2,1}\hat{u}(x,t)$: this follows directly from the definitions if $t \leq 0$, and an easy argument shows that it is also true for $t > 0$. Proposition 28 is proved. $\square$

We return to the proof of Case 2 of Proposition 25.

The condition of Case 2 implies that in an appropriate coordinate system in $\mathbf{R}^2$

$$y = (0,0), \quad y_1 = (z_1, 0), \quad y_2 = (z_2, 0), \quad z_1 < 0 < z_2,$$

and there exists $r_0 > 0$, $f \in C^{2;1}(\mathbf{R}^1 \times (-\infty, t_0])$, $g \in C(\mathbf{R}^1 \times (-\infty, t_0])$ satisfying (35), (36), (37) and the following:

$$\begin{aligned}
&\Omega_t \subseteq \{x_2 < g(x_1,t)\} \quad \text{for any } \ t \in (t_0 - r_0, t_0],\\
&\Omega_{t_0} \cap \mathcal{B} = \{x_2 < g(x_1,t_0)\} \cap \mathcal{B}, \quad \text{where} \quad \mathcal{B} = \{(x_1,x_2) \mid x_1 \in [z_1,z_2], \ |x_2| < r_0\},\\
&g(z,t_0) = 0 \quad \text{for} \quad z \in [z_1,z_2].
\end{aligned} \tag{70}$$

To construct such a function g, we first choose r small enough so that in the appropriate coordinates (x_1, x_2) the following is true: $\mathbf{\Gamma} \cap Q_r^-(y_0,t_0)$ is a graph of $x_2 = g(x_1,t)$. Such an r exists since $\{\Omega_t\}$ is a Lipschitz family of convex sets. Then for each $t \in (t_0 - r, t_0]$ extend $g(\cdot,t)$ from $\{z \mid (z,t) \in Q_r^-(y_0,t_0)\}$ to $\mathbf{R}^1$ by defining $g(\cdot,t)$ as the infimum of all supporting straight lines of the graph of the concave function $g(\cdot,t)$ at all points $z \in \{z \mid (z,t) \in Q_r^-(y_0,t_0)\}$. The constructed function $g(z,t)$ obviously satisfies all required properties.

We can apply Proposition 28 with $n = 1$ to the function $-g$ on $\mathbf{R}^1 \times (-\infty, t_0]$. We have:

$$0 = \lambda_1 z_1 + \lambda_2 z_2, \quad \text{where} \quad \lambda_1, \lambda_2 \in (0,1) \quad \text{and} \quad \lambda_1 + \lambda_2 = 1.$$

Also, by (70),

$$g(0,t_0) = \lambda_1 g(z_1,t_0) + \lambda_2 g(z_2,t_0).$$

By (37),

$$\big(-\partial_t f(0,t_0), 0, \partial^2_{zz} f(0,t_0)\big) \in J^{2,1}_-(-g)(0,t_0).$$

By Proposition 28 there exist A_1, A_2, τ_1, $\tau_2 \in \mathbf{R}^1$ such that

$$(-\tau_i, 0, -A_i) \in \overline{J}^{2,1}_-(-g)(z_i,t_0), \qquad i = 1,2; \tag{71}$$

$$\partial_t f(0,t_0) = \lambda_1 \tau_1 + \lambda_2 \tau_2. \tag{72}$$

By (70) we have

$$\partial^2_{zz} f(0,t_0) \ge 0. \tag{73}$$

By Lemma 13 and (70) we have

$$z \to \gamma_{\Omega_{t_0}}(z,0) \quad \text{is concave on} \quad [z_1,z_2]. \tag{74}$$

We estimate using (72)

$$\begin{array}{rcl}
\lambda_1\tau_1 + \lambda_2\tau_2 & = & \partial_t f(0,t_0)\\
\text{(by (34))} & > & F(-\partial^2_{zz} f(0,t_0), \gamma_{\Omega_{t_0}}(0,0), t_0)\\
\text{(by (73), (H2))} & \ge & F(0, \gamma_{\Omega_{t_0}}(0,0), t_0)\\
\text{(by (H3), (74))} & \ge & F(0, \lambda_1 \gamma_{\Omega_{t_0}}(z_1,0) + \lambda_2 \gamma_{\Omega_{t_0}}(z_2,0), t_0)\\
\text{(by (H4))} & \ge & \lambda_1 F(0, \gamma_{\Omega_{t_0}}(z_1,0), t_0) + \lambda_2 F(0, \gamma_{\Omega_{t_0}}(z_2,0), t_0)\\
 & = & \lambda_1 F(0, \gamma_{\Omega_{t_0}}(y_1), t_0) + \lambda_2 F(0, \gamma_{\Omega_{t_0}}(y_2), t_0).
\end{array}$$

Note the strict inequality on the second line of the estimate above. Thus we get

$$\text{either} \quad \tau_1 > F(0, \gamma_{\Omega_{t_0}}(y_1), t_0) \quad \text{or} \quad \tau_2 > F(0, \gamma_{\Omega_{t_0}}(y_2), t_0).$$

Suppose without loss of generality that in fact the first inequality above holds. Then using (71) we conclude that there exists a sequence $(z^i, t^i) \to (y_1, t_0)$as $i \to \infty$, and for each i a function $\varphi^i \in C^{2;1}(\mathbf{R}^1 \times (-\infty, t^i])$ such that

$$\varphi^i \ge g \quad \text{on} \quad \mathbf{R}^1 \times (-\infty, t^i], \qquad \varphi(z^i,t^i) = g(z^i,t^i),$$

and, denoting

$$\partial_t \varphi^i(z^i,t^i) = \tau^i, \quad \partial_z \varphi^i(z^i,t^i) = p^i, \quad \partial^2_{zz} \varphi^i(z^i,t^i) = A^i,$$

we have

$$(\tau^i, p^i, A^i) \to (\tau_1, 0, A_1) \quad \text{ as } \quad i \to \infty.$$

Then, denoting $y^i{:=}(z^i, g(z^i, t^i)) \in \mathbf{R}^2$, and using uppersemicontinuity of $(y,t) \to \gamma_{\Omega_t}(y)$, we get for large enough i

$$\partial_t \varphi^i(z^i, t^i) > F(0, \gamma_{\Omega_{t^i}}(y^i), t^i) + \delta, \tag{75}$$

where

$$\delta{:=}\frac{1}{2}\left[\tau_1 - F(0, \gamma_{\Omega_{t_0}}(y_1), t_0)\right] > 0.$$

Let large i be chosen below. Define

$$\tilde{\Lambda}_t = \{(x_1, x_2) \ | \ x_2 < \varphi^i(x_1, t)\}$$

Then $\{\tilde{\Lambda}_t\}$ is a family of sets with $C^{2;1}$ boundary, $\Omega_t \subset \tilde{\Lambda}_t$ for all t, and $y^i \in \partial\Omega_{t^i} \cap \partial\tilde{\Lambda}_{t^i}$. We can rewrite (75) as:

$$V_{\partial\tilde{\Lambda}_{t^i}}(y^i) > F(0, \gamma_{\Omega_{t^i}}(y^i), t^i) + \delta. \tag{76}$$

If $-\kappa_{\partial\tilde{\Lambda}_{t^i}}(y^i) = A^i < 0$ then by (H2) we get

$$V_{\partial\tilde{\Lambda}_{t^i}}(y^i) > F(\kappa_{\partial\tilde{\Lambda}_{t^i}}(y^i), \gamma_{\Omega_{t^i}}(y^i), t^i). \tag{77}$$

Now we argue similar to Case 1.

If $A^i < 0$ then we have (77), then y^i is an extreme point of $\partial\Omega_{t^i}$ and we are in the situation of Case 1a since the inequality (77) is just (34) moved to the point (y^i, t^i).

If $A_1 \geq 0$, then with (76) we are in the situation similar Case 1b since (76) is just (47) moved to the point (y^i, t^i) and we apply Lemma 26 with

$$\begin{aligned} U(z,t) &= g(z + z^i, t + t^i) - \partial_t \varphi^i(z^i, t^i)t - \partial_z \varphi^i(z^i, t^i)z, \\ \hat{f}(z,t) &= \varphi(z + z^i, t + t^i) - \partial_t \varphi^i(z^i, t^i)t - \partial_z \varphi^i(z^i, t^i)z. \end{aligned}$$

However we do not know whether y_i is an extreme point of $\overline{\Omega}_{t^i}$, and thus possibly $r_* > 0$ in Lemma 26, which restricts the choice of r in Lemma 26 and affects the estimate of $\partial_t \varphi$ in (58). We will show that

$$r_*^i \to 0 \qquad \text{as} \qquad i \to \infty, \tag{78}$$

where $r_*^i \geq 0$ is the value of r_* for (y^i, t^i). Once (78) is proved, then choosing i large so that $(M+1)\sqrt{r_*^i} < \dfrac{\delta}{2}$ we can follow the argument of Case 1b, further reducing r_*^i as needed for the choice of r.

It remains to prove (78). For any open bounded convex set Ω we consider the function $\mathcal{E}_\Omega : \partial\Omega \to [0, \infty)$ defined for $y \in \partial\Omega$ by

$$\begin{aligned} \mathcal{E}_\Omega(y) = \inf \quad \{ \quad & |y - x| \ : \ x \ \text{ is an extreme point of } \ \partial\Omega, \\ & \lambda x + (1-\lambda)y \in \partial\Omega \ \text{ for any } \lambda \in [0,1]\}. \end{aligned}$$

Then obviously $0 \leq r_*^i \leq \mathcal{E}_{\Omega_{t^i}}(y^i)$. Also, $\mathcal{E}_{\Omega_{t_0}}(y_1) = 0$, since y_1 is an extreme point of Ω_{t_0}. Thus (78) follows from

Lemma 30 *Let $\{\Omega_t\}$ be a Lipschitz family of open bounded convex sets in $\mathbf{R}^n$, $n \geq 2$. Let $y^* \in \partial\Omega_{t^*}$ and let $\mathcal{E}_{\Omega_{t^*}}(y^*) = 0$. Then the function $\bigcup_t (\partial\Omega_t \times \{t\}) \to \mathbf{R}^1$, defined by $(y,t) \to \mathcal{E}_{\Omega_t}(y)$, is uppersemicontinuous at (y^*, t^*).*

Proof. First let us derive another expression for the function $\mathcal{E}_\Omega(y)$, where $\Omega \in \mathbf{R}^n$ is a bounded convex set. For $y \in \partial\Omega$ denote by $\mathcal{F}_\Omega(y)$ the face of $\overline{\Omega}$ of minimal dimension which contains y (see [R72, Section 18]) for definitions). Then

$$\mathcal{E}_\Omega(y) = \text{dist } (y, \partial_r \mathcal{F}_\Omega(y)), \tag{79}$$

where $\partial_r \mathcal{F}$ is the relative boundary of $\mathcal{F}$ ([R72, Section 6]). If $k = \dim \mathcal{F}_\Omega(y)$, then $0 \le k \le n-1$ since $y \in \partial\Omega$. If $k = 0$, then y is an extreme point and $\mathcal{E}_\Omega(y) = 0$. If $k > 0$ then $\mathcal{E}_\Omega(y) > 0$ and y lies in a relative interior of $\mathcal{F}_\Omega(y)$.

Let $y_k \in \partial\Omega_{t_k}$ for $k = 1, 2, \ldots$, and $(y_k, t_k) \to (y^*, t^*)$ as $k \to \infty$, and $\mathcal{E}_{\Omega_{t^*}}(y^*) = 0$. We have to show that $\lim\limits_{k\to\infty} \mathcal{E}_{\Omega_{t_k}}(y_k) = 0$. It is enough to show that $\limsup\ \mathcal{E}_{\Omega_{t_k}}(y_k) \le 0$. Without loss of generality we can assume that $\mathcal{E}_{\Omega_{t_k}}(y_k) > 0$ for each k, and thus $\dim \mathcal{F}_{\Omega_{t_k}}(y_k) > 0$ for each k.

Fix $\varepsilon > 0$. There exists $K(\varepsilon)$ such that for each $k > K(\varepsilon)$:

$$x_k \in B_\varepsilon(y^*) \cap \partial\Omega_{t_k}. \tag{80}$$

From now on we consider only such k. In particular, for each such k we have $B_\varepsilon(y^*) \cap \mathcal{F}_{\Omega_{t_k}}(y_k) \ne \emptyset$. Let us show that

$$B_\varepsilon(x_*) \cap \partial_r \mathcal{F}_{\Omega_{t_k}}(y_k) \ne \emptyset \text{ for sufficiently large } k. \tag{81}$$

Suppose (81) is false. Then there exists a subsequence $k_j \to \infty$ such that

$$B_\varepsilon(y^*) \cap \partial\Omega_{t_{k_j}} = B_\varepsilon(y^*) \cap \mathcal{F}_{\Omega_{t_{k_j}}}(y_{k_j}) \ne \emptyset.$$

Then, denoting by H_j the affine subspace of $\mathbf{R}^n$ such that $\dim H_j = \dim \mathcal{F}_{\Omega_{t_{k_j}}}(y_{k_j})$ and $\mathcal{F}_{\Omega_{t_{k_j}}}(y_{k_j}) \subset H_j$, we get

$$B_\varepsilon(y^*) \cap \partial\Omega_{t_{k_j}} = B_\varepsilon(y^*) \cap H_j \ne \emptyset.$$

By continuity of the family of sets $\{\Omega_t\}$, there exist $z_j \in \mathcal{F}_{\Omega_{t_{k_j}}}(y_{k_j})$ such that $z_j \to y^*$ as $j \to \infty$. Then for large j there exist $\hat{z}_j \in H_j \cap \partial B_{\varepsilon/2}(y^*)$. Then $\hat{z}_j \in \mathcal{F}_{\Omega_{t_{k_j}}}(y_{k_j}) \cap \partial B_{\varepsilon/2}(y^*)$, and for a subsequence we get $\hat{z}_{j_m} \to \hat{z}^*$, which implies $z^* \in \partial\Omega_{t^*} \cap \partial B_{\varepsilon/2}(y^*)$ by Lipschitz continuity of the family $\{\Omega_t\}$. We have $l_j \cap B_\varepsilon(y^*) \subset H_j \cap B_\varepsilon(y^*) \subset \partial\Omega_{t_{k_j}}$, where $l_j {:=} \{\alpha z_j + (1-\alpha)\hat{z}_j \ : \ \alpha \in \mathbf{R}^1\}$ is the line through z_j and $\hat{z}_j$. It follows that $\partial\Omega_{t^*} \cap B_\varepsilon(y^*)$ contains $l \cap B_\varepsilon(y^*)$, where $l {:=} \{\alpha y^* + (1-\alpha) z^* \ : \ \alpha \in \mathbf{R}^1\}$ is the line through y^* and z^*. This contradicts the fact that $\mathcal{E}_{\Omega_{t^*}}(y^*) = 0$. Thus (81) is proved.

Now (79) – (81) imply

$$\limsup\ \mathcal{E}_{\Omega_{t_k}}(y_k) \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, Lemma is proved. □

Thus Case 2 is proved, and the proof of Proposition 25 is complete. □

Construction 20 and Propositions 18, 22, 24, 25 imply Theorem 3.

4 Large time behaviour

In this section we will show that under suitable conditions, convex solutions of Eq. (1) become circular in shape as they expand to infinite size. We will prove this for a class of speed functions F which includes the two main examples. The extra conditions we will impose are the following:

(H5) For any κ, γ, and t and any $\lambda > 0$ we have

$$F(\kappa/\lambda, \lambda\gamma, t) = \lambda F(\kappa, \gamma, t);$$

(H6) $\kappa F(\kappa, 1, t) < F(1, 1, t)$ for $0 \le \kappa < 1$ and $t \ge 0$.

These conditions are both satisfied in the two motivating examples (4) and (5). For ease of exposition we will only discuss those cases where F does not explicitly depend on t. This holds in example (5), and also in example (4) if we introduce a new time variable $\tau = \log(t + a)$.

Theorem 31 *Let F be independent of t and satisfy conditions (H1)-(H6), and suppose $\{\Omega_t\}_{t\geq 0}$ is a convex viscosity solution of Eq. (1). Then the rescaled family of convex sets $\{\tilde{\Omega}_t\}_{t\geq 0}$ defined by*

$$\tilde{\Omega}_t = e^{-F(1,1)t}\Omega_t$$

converges in Hausdorff distance as $t \to \infty$ to a circle with centre at the origin.

Proof. We will derive differential inequalities for the circumradius R and the inradius r of the evolving convex sets Ω_t, by constructing comparison families $\{\Lambda_t\}$ and applying the inequality (27) from the definition of viscosity solutions.

A useful differential inequality for the inradius can be deduced very easily: Let $t_0 \geq 0$, and let $B_{r_0}(x_0)$ be an incircle of Ω_{t_0}. For any $\varepsilon > 0$, consider the family of sets $\{\Lambda_t^\varepsilon\}$ defined by

$$\Lambda_t^\varepsilon = \begin{cases} B_{(1-\varepsilon)r_0\exp((1-\varepsilon)(t-t_0)F(1,1))}(x_0) & \text{for } t \geq t_0; \\ B_{(1-\varepsilon)r_0}(x_0) \cap \Omega_t & \text{for } 0 \leq t \leq t_0. \end{cases}$$

Then we have $\Lambda_t^\varepsilon \subseteq \Omega_t$ for $0 \leq t \leq t_0$, and $d(\Lambda_{t_0}^\varepsilon, \partial\Omega_{t_0}) \geq \varepsilon r_0 > 0$. Suppose that it is not true that $\Lambda_t^\varepsilon \subseteq \Omega_t$ for all $t \geq 0$. There there is a first time $t_1 > t_0$ beyond which the inclusion fails. Then $\{\Lambda_t^\varepsilon\}_{0\leq t\leq t_1}$ satisfies conditions (a) and (b) in Definition 15 for the supersolution $\{\Omega_t\}$, and so we have at some point (using (H5))

$$(1-\varepsilon)^2 r_0 F(1,1)e^{(1-\varepsilon)F(1,1)(t-t_0)} = V_{\Lambda_{t_1}^\varepsilon} \geq F(\kappa_{\Lambda_{t_1}^\varepsilon}, \gamma_{\Lambda_{t_1}^\varepsilon}) = (1-\varepsilon)r_0 F(1,1)e^{(1-\varepsilon)F(1,1)(t-t_0)}.$$

This is a contradiction. Therefore $\Lambda_t^\varepsilon \subseteq \Omega_t$ for all $t \geq t_0$ and all $\varepsilon > 0$. Letting $\varepsilon \to 0+$ we find that Ω_t encloses a ball of radius $r_0 \exp(F(1,1)(t-t_0))$ for each $t > t_0$, and therefore

$$r(\Omega_t) \geq r(\Omega_{t_0}) \exp(F(1,1)(t-t_0)),$$

and

$$\frac{d}{dt} r(\Omega_t)\Big|_{t=t_0} := \liminf_{\tau\to 0+} \frac{(r(\Omega_{t_0+\tau}) - r(\Omega_{t_0}))}{\tau} \geq F(1,1)r(\Omega_{t_0}). \tag{82}$$

A similar argument gives an inequality for the circumradius:

$$\frac{d}{dt} R(\Omega_t)\Big|_{t=t_0} \leq F\left(\frac{1}{R(\Omega_t)}, R(\Omega_t)\right). \tag{83}$$

However, we will deduce a more useful differential inequality for the $R(\Omega_t)$ by working a little harder.

Choose the origin of $\mathbf{R}^2$ to be at the centre of a circumcircle of Ω_0. Fix any $T > 0$, and let L be the Lipschitz constant of $\{\Omega_t\}$ on $[0,T]$. Let $t_0 \in [0,T)$, and define R_t^ε for any $t \in (t_0, T]$ and $\varepsilon > 0$ by $R_{t_0}^\varepsilon = (1+\varepsilon)R(\Omega_{t_0})$, and

$$\frac{d}{dt} R_t^\varepsilon = (1+\varepsilon)F\left(\frac{1}{R_t^\varepsilon}, r(\Omega_{t_0}) + L(t-t_0)\right).$$

Let x_0 be the centre of the circumcircle of Ω_{t_0}, and define for any $\sigma > 0$

$$\Lambda_t^{\varepsilon,\sigma} = \begin{cases} B_{R_t^\varepsilon}(x_0) \cap (1+\sigma)\Omega_t & \text{for } t \in [t_0, T]; \\ B_{R_{t_0}^\varepsilon}(x_0) \cap (1+\sigma)\Omega_t & \text{for } 0 \leq t \leq t_0. \end{cases}$$

Suppose it is not true that $\Omega_t \subseteq B_{R_t^\varepsilon}(x_0)$ for all $t \in [t_0, T]$. Then there is a first time $t_1 > t_0$ beyond which this inclusion fails, and a point $y_1 \in \partial\Omega_{t_1} \cap \partial B_{R_t^\varepsilon}(x_0)$. It follows that for any $\sigma > 0$, the family $\{\Lambda_t^{\varepsilon,\sigma}\}$ satisfies the conditions (a) and (b) in Definition 15 for the subsolution $\{\Omega_t\}$, and therefore by (27)

$$\begin{aligned} (1+\varepsilon)F\left(\frac{1}{R_{t_1}^\varepsilon}, r(\Omega_{t_0}) + L(t_1-t_0)\right) &= V_{\{\Lambda^{\varepsilon,\sigma}\}}(y_1, t_1) \\ &\leq F(\kappa_{\Lambda_{t_1}^{\varepsilon,\sigma}}(y_1), \gamma_{\Lambda_{t_1}^{\varepsilon,\sigma}}(y_1)) \end{aligned}$$

$$= \quad F(1/R^\varepsilon_{t_1}, \gamma_{\Lambda^{\varepsilon,\sigma}_{t_1}}(y_1)).$$

Now by Lemma 14 (applied to $(1+\sigma)^{-1}\Lambda^{\varepsilon,\sigma}_{t_1} \subseteq \Omega_{t_1}$), for any $\delta > 0$ there exists $\sigma_0 > 0$ such that $\gamma_{\Lambda^{\varepsilon,\sigma}_{t_1}}(y_1) \le \gamma_{\Omega_{t_1}}(y_1) + \delta$ for $\sigma \le \sigma_0$. Therefore we have (taking $\sigma \to 0$)

$$(1+\varepsilon)F\left(\frac{1}{R^\varepsilon_{t_1}}, r(\Omega_{t_0}) + L(t_1 - t_0)\right) \le F(1/R^\varepsilon_{t_1}, \gamma_{\Omega_{t_1}}(y_1)) \le F(1/R^\varepsilon_{t_1}, r(\Omega_{t_1})),$$

since for convex Ω, $\sup_{\partial\Omega} \gamma = r(\Omega)$. But our choice of L gives $r(\Omega_{t_1}) \le r(\Omega_{t_0}) + L(t_1 - t_0)$, so we have a contradiction. Therefore $\Omega_t \subseteq B_{R^\varepsilon_t}(x_0)$ for all $t \in [t_0, T]$, and in particular (after taking $\varepsilon \to 0$)

$$\frac{d}{dt}R(\Omega_t)\big|_{t=t_0} := \limsup_{\tau\to 0+} \frac{R(\Omega_{t_0+\tau}) - R(\Omega_{t_0})}{\tau} \le F\left(\frac{1}{R(\Omega_{t_0})}, r(\Omega_{t_0})\right). \tag{84}$$

We get more information by combining the inequalities (82) and (84): We compute using (H5)

$$\frac{d}{dt}\left(\frac{R(\Omega_t)}{r(\Omega_t)}\right) \le \frac{R}{r}\left(\frac{r}{R}F\left(\frac{r}{R}, 1\right) - F\left(1, 1\right)\right) \le 0 \tag{85}$$

where we used (H6) in the last inequality. Furthermore, the inequality is strict unless $R = r$.

Consider the function $\psi(t) := \frac{R(\Omega_t)}{r(\Omega_t)}$ on $t \in [0,\infty)$. Clearly $\psi(t) \ge 1$ for all t, and, by (85), $\psi(\cdot)$ is nonincreasing on $[0,\infty)$. Thus there exists $\alpha := \lim_{t\to\infty}\psi(t)$, and $\psi(t) \ge \alpha \ge 1$ for all $t \ge 0$. Suppose $\alpha > 1$. Then by (85) and (H6)

$$\frac{d}{dt}\left(\frac{R(\Omega_t)}{r(\Omega_t)}\right) \le \frac{1}{\alpha}F\left(\frac{1}{\alpha}, 1\right) - F(1,1) < 0,$$

and therefore $\lim_{t\to\infty} R(\Omega_t)/r(\Omega_t) = -\infty$, which is impossible. Therefore $R(\Omega_t)/r(\Omega_t) \to 1$ as $t \to \infty$

Finally we show that Ω_t converges to a circle centred at the origin: From now on we write f for $F(1,1)$ for brevity. The proofs of the estimates (82) and (83) show that if $B_r(x_0) \subseteq \Omega_0 \subseteq B_R(x_0)$ then $B_{re^{ft}}(x_0) \subseteq \Omega_t \subseteq B_{Re^{ft}}(x_0)$. Since $R/r \to 1$, for any $\varepsilon > 0$ there exists $T > 0$, $r = r_\varepsilon \in [r(\Omega_0), R(\Omega_0)]$ and $z \in \mathbf{R}^2$ such that

$$B_{re^{fT}}(z) \subseteq \Omega_T \subseteq B_{(1+\varepsilon)re^{fT}}(z).$$

It follows from (82) and (83) that

$$B_{re^{ft}}(z) \subseteq \Omega_t \subseteq B_{(1+\varepsilon)re^{ft}}(z)$$

for $t > T$, and hence $t > T^*$ where T^* is large enough so that $|z|e^{-f(T^*-T)} \le \varepsilon r$,

$$B_{(1-\varepsilon)r}(0) \subseteq B_r(ze^{-f(t-T)}) \subseteq \tilde{\Omega}_t \subseteq B_{(1+\varepsilon)r}(ze^{-f(t-T)}) \subseteq B_{(1+2\varepsilon)r}(0)$$

for $t > T^*$. It follows that $d(\tilde{\Omega}_t, B_{r_\varepsilon}(0)) \le 3\varepsilon R(\Omega_0)$ for $t > T^*(\varepsilon)$. Now it follows that r_ε converges to some limit $r^* \in [r(\Omega_0), R(\Omega_0)]$ as $\varepsilon \to 0+$, and that $\tilde{\Omega}_t$ converges to $B_{r^*}(0)$ as $t \to \infty$. This completes the proof. □

5 Applications

1. Limits as $p \to \infty$ of solutions of p-Laplacian evolution problems (collapsing sandpiles model).

Consider p-Laplacian evolution problems

$$\begin{cases} \partial_t u_p - \operatorname{div}(|\, Du_p\,|^{p-2}\, Du_p) = 0, & \text{in } \mathbf{R}^n \times (0,\infty), \\ u_p = g, & \text{on } \mathbf{R}^n \times \{0\}, \end{cases} \tag{86}$$

in the "infinitely fast/slow diffusion limit" $p \to \infty$. Here $u = u(x,t)$, $x \in \mathbf{R}^2$, and Du_p is the gradient of u_p with respect to the spatial variables x_1, x_2. The operator

$$\Delta_p(u) = \operatorname{div}(\mid Du_p \mid^{p-2} Du_p)$$

is p-Laplacian. The initial data g is a nonnegative, Lipschitz continuous function with compact support. We assume that

$$\mathrm{Lip}[g] = L > 1 \tag{87}$$

where $\mathrm{Lip}[g]$ is the Lipschitz constant of g.

The initial data g satisfying (87) is unstable in the following sense. It is shown in [EFG97] that $u_p \to u$ uniformly on $\mathbf{R}^n \times (0, \infty)$, and that

$$\mid Du \mid \leq 1 \quad \text{a.e. in} \quad \mathbf{R}^n \times (0, \infty). \tag{88}$$

This limit u does not depend on t for $t > 0$, i.e. $u = u(x)$. It follows from (87) and (88) that $u \neq g$. Thus the transformation $g \to u$ takes place at $t = 0$.

The described problem can be interpreted as a model of a collapsing sandpile on a flat surface. The function $u(x,t)$ is the height of the pile at the location x, time t. The main physical assumption is that a sandpile is stable if its slope does not exceed one, i.e. if $\mathrm{Lip}[u(\cdot,t)] \leq 1$. The condition (87) implies that the initial profile g is unstable. Thus according to the model the initially unstable sandpile collapses instantaneously to a stable one, with the height function u.

Combining Theorem 3 and Proposition 17 with [F99, Theorem 1] and [EFG97, Section 3c] we obtain the following:

Theorem 32 *Let $\Omega_0 \subset \mathbf{R}^2$ be an open, bounded, convex set. Let $p > 2$. Let u_p be a solution of (86) with*

$$g = L\, d_{\Omega_0}(\cdot),$$

where $d_{\Omega_0}(\cdot)$ is the interior distance function of Ω_0, and $L > 1$. Then for each $t > 0$

$$\lim_{p \to \infty} u_p(\cdot, t) = d_{\Omega_1}(\cdot) \qquad \textit{uniformly in } \mathbf{R}^2,$$

where Ω_1 is a bounded open convex set defined as follows: Let the Lipschitz family of convex sets $\{\Lambda_t\}$ be a viscosity solution of the equation (1) with $F(\kappa, \gamma, t) = \frac{\gamma}{3} \cdot \frac{3 - 2\kappa\gamma}{2 - \kappa\gamma}$ on the time interval $[0, \log L]$, with the initial condition $\Lambda_0 = \Omega_0$ at time $t = 0$. Then $\Omega_1 = \Lambda_{\log L}$.

2. Compression molding model.

Compression molding is the process of deformation of an incompressible plastic material between two horizontal plates. A simplified mathematical model of the process was derived by G. Aronsson [Ar95] based on Hele-Show model for non-Newtonian fluid. Suppose that the distance between the horizontal plates is small. Then we can assume that the region occupied by plastic at each time t has the form $U_t = \Omega_t \times [0, h_t]$ where $\Omega_t \subset \mathbf{R}^2$, and that pressure in plastic does not depend on the vertical coordinate, i.e., pressure is the function $u(x,t)$ where $x \in \Omega_t$. Evolution of rescaled Ω_t and $u(x,t)$ is described by the following free boundary problem. Given an open set $\Omega_0 \in \mathbf{R}^2$ find an expanding family of open sets $\Omega_t \in \mathbf{R}^2, t \geq 0$, and a function $u(x,t)$ defined on $\cup_t(\Omega_t \times \{t\})$ such that

$$-\operatorname{div}(|Du|^{p-2} Du) = 1 \qquad \text{in } \Omega_t, \tag{89}$$

$$u = 0 \qquad \text{on } \Gamma_t, \tag{90}$$

$$V = |Du|^{p-2} \qquad \text{on } \Gamma_t, \tag{91}$$

where $\Gamma_t = \partial\Omega_t$, V denotes the outer normal velocity of Γ_t. Condition (91) means that the free boundary Γ_t moves with the velocity of the flow of material.

G. Aronsson and L. C. Evans [AE] considered the formal asymptotic limit as $p \to \infty$ in the problem (89) - (91). This limit corresponds to the case of highly non-Newtonian fluid. The following evolution problem was obtained in [AE] as the limit of (89) - (91) as $p \to \infty$: given an initial set Ω_0 find a continuous family of sets $\{\Omega_t\}_{t\geq 0}$ such that

$$w - \partial_t w \in \partial I_\infty[u]. \tag{92}$$

Here

$$\begin{aligned} w(\cdot,t) &= \chi_{\Omega_t}(\cdot), \\ u(\cdot,t) &= d_{\Omega_t}(\cdot), \end{aligned} \tag{93}$$

where $\chi_{\Omega_t}(\cdot)$ is the indicator function of Ω_t that equals 1 on Ω_t and 0 outside Ω_t. $\partial I_\infty[\cdot]$ is the subdifferential of the convex functional $I_\infty : L^2(\mathbf{R}^2) \to \mathbf{R}^1 \cup \{\infty\}$ defined by

$$I_\infty[v] = \begin{cases} 0 & \text{if } v \in L^2(\mathbf{R}^2), \quad | Dv | \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Interpretation of this model in the sense of Monge-Kantorovich mass transfer theory is given in [AE], [F99].

Heuristic calculations in [AE] imply that smooth solutions $\{\Omega_t\}$ of (92), (93) satisfy equation (1) with right-hand side (5). It was proved in [F99] that convex (weak) solutions of (1), (5) satisfy (92), (93).

Combining Theorem 3 and Proposition 17 with [F99, Theorem 32] we obtain the following

Theorem 33 *Let $\Lambda \subset \mathbf{R}^2$ be a convex open bounded set. Then there exists a solution $\{\Omega_t\}$, $t \geq 0$, of (92), (93) with the initial data $\Omega_0 = \Lambda$, satisfying in addition the following: $\{\Omega_t\}$ is an expanding Lipschitz family of convex open bounded sets, and $\{\Omega_t\}$ is a viscosity solution of the equation (1) with $F(\kappa, \gamma, t)$ defined by (5).*

A Appendix. Proof of Lemma 26

We change notations from $U(x,t)$, $V(x,t)$ to $u(x,t)$, $v(x,t)$.

We always assume below that $(x,t) \in I_{2r_0} \times (-2r_0, 0]$.

Let $r > 2r_*$ and $\varepsilon > 0$ be fixed.

By (52), (53) there exists $T_0 < 0$ such that

$$u(0,t) < \varepsilon|t| \quad \text{ for } \quad t \in (T_0, 0). \tag{94}$$

Since $u(x,0)$ is concave and $u(0,0) = 0$, it follows from (51) that

$$u(\cdot,0) \text{ is strictly decreasing on } (r_*, \infty). \tag{95}$$

In addition, for any $\rho > r_*$ there exists $\tau_\rho < 0$ such that

$$u(x,t) < 0 \quad \text{ for } \quad x > \rho, \quad t \in [\tau_\rho, 0]. \tag{96}$$

Indeed, let $\rho > r_*$ and (96) is false. Then there exists a sequence $\theta_i \to 0$ such that $\theta_i < 0$ for all i, and for each i there exist $x_i > \rho$ such that $u(x_i, \theta_i) \geq 0$. But by (95) we get $u(x_i, 0) < u(\rho, 0) < 0$, and so $|u(x_i,\theta_i) - u(x_i,0)| > |u(\rho,0)|$. A contradiction with (48) for $|\theta_i| < \frac{|u(\rho,0)|}{M}$. Thus there exists $\tau_\rho < 0$ satisfying (96).

Consider two cases.

Case 1. There exists a sequence $t_i \to 0$ such that

$$\begin{aligned} &T_0 < t_1 < t_2 < \ldots < 0; \\ &\sup_{x>0} u(x, t_{2i}) \leq \varepsilon |t_{2i}|, \qquad \text{where} \quad i = 1, 2, \ldots ; \\ &\sup_{x>0} u(x, t_{2i+1}) > \varepsilon |t_{2i+1}|, \qquad \text{where} \quad i = 0, 1, 2, \ldots . \end{aligned}$$

Proof of Lemma 26 in the Case 1. Let $\rho = \frac{r}{2}$, then $\rho > r_*$. Choose the corresponding $\tau \equiv \tau_{\frac{r}{2}} < 0$, where τ_ρ is defined by (96). Let i be such that $|t_{2i}| < \min(\tau_{\frac{r}{2}}, \frac{r}{2})$. Let

$$\mathcal{B}_{\frac{r}{2}} = \{x \in [0, \frac{r}{2}],\ \ t \in [t_{2i}, t_{2i+2}]\}.$$

Then define

$$v(x,t) = \begin{cases} \min(\,\varepsilon|t|,\ u(x,t)\,) & \text{if} \quad (x,t) \in \mathcal{B}_{\frac{r}{2}}; \\ u(x,t) & \text{otherwise.} \end{cases}$$

We have $u(x,t) \le \varepsilon|t|$ on $\{x > 0, \quad t = t_{2i}\}$ and on $\{x > 0, \quad t = t_{2i+2}\}$ by the assumptions of Case 1. The same inequality is true on $\{x = 0, \ \ t \in [t_{2i}, t_{2i+2}]\}$ by (94). In addition $u(x,t) < 0$ on $\{x > \frac{r}{2}, \ \ t \in [t_{2i}, t_{2i+2}]\}$ by (96) and the choice of i. Thus $u \equiv v$ on the boundary of the set $\mathcal{B}_{\frac{r}{2}}$. It follows that $v(x,t)$ is continuous. It also follows that $v(\cdot,t)$ is concave for every t. Indeed, from (94) we have $u \equiv v$ on $\{(x,t) \mid x < \delta, \ t \in [t_{2i}, t_{2i+2}]\}$ for some $\delta > 0$. On $\{(x,t) \mid x > 0, \ t \in [t_{2i}, t_{2i+2}]\}$ we have $v(x,t) = \min(\,\varepsilon|t|, \ u(x,t)\,)$, and $u(\cdot,t)$ is concave. Thus $v(\cdot,t)$ is concave.

Consider $v(\cdot, t_{2i+1})$. We have

$$u(0, t_{2i+1}) < \varepsilon|t_{2i+1}|, \quad \sup_{x>0} u(x, t_{2i+1}) > \varepsilon|t_{2i+1}|, \quad \sup_{x>\frac{r}{2}} u(x, t_{2i+1}) \le 0.$$

Thus there exists $\hat{x} \in (0, \frac{r}{2})$ such that

$$u(\hat{x}, t_{2i+1}) > \varepsilon|t_{2i+1}| = v(\hat{x}, t_{2i+1}),$$

i.e. (56) is true. Let now $(\tilde{x}, \tilde{t})$ be any point such that

$$u(\tilde{x}, \tilde{t}) > v(\tilde{x}, \tilde{t}).$$

Then

$$(\tilde{x}, \tilde{t}) \in \mathcal{B}_{\frac{r}{2}}, \qquad \text{and} \qquad v(x,t) = \varepsilon|t| = -\varepsilon t \quad \text{in a neighborhood of} \quad (\tilde{x}, \tilde{t}).$$

Thus (57) holds with $\varphi(x,t) = -\varepsilon t$ for some $\rho > 0$. Lemma 26 is proved in Case 1.

Case 2. Conditions of Case 1 fail. In this case there exists $T_1 < 0$ such that

$$\text{either} \qquad \sup_{x>0} u(x,t) \le \varepsilon|t| \quad \text{for all} \quad t \in [T_1, 0), \tag{97}$$

$$\text{or} \qquad \sup_{x>0} u(x,t) > \varepsilon|t| \quad \text{for all} \quad t \in [T_1, 0). \tag{98}$$

Proof of Lemma 26 in the Case 2. We can assume that $|T_1| \le |T_0|$ and thus (94) holds on $[T_1, 0]$. We can also assume that $|T_1| \le |\tau_\rho|$ where $\rho = \frac{r}{2} > r_*$, and thus (96) holds on $[T_1, 0]$, i.e.

$$u(x,t) < 0 \quad \text{for} \quad x > \frac{r}{2}, \ \ t \in [T_1, 0]. \tag{99}$$

We claim that there exists a constant $A \ge 0$ such that

$$u(x,t) < (\varepsilon + Ax)\,|t| \quad \text{for} \quad (x,t) \in \{\,x \ge 0, \ t \in [T_1, 0]\,\} \setminus \{\,(0,0)\,\}. \tag{100}$$

Consider 3 cases.

Case 2a. Suppose that (97) holds. Then from (94) it follows that we can choose any $A > 0$ and (100) holds true.

Case 2b. Suppose that (98) holds, and $u(x,0) < 0$ for all $x < -r'$ where $0 < r' < \sqrt{r_*}$. Then we can replace $u(x,t)$ by the function $u^*(x,t) = u(-x,t)$. The function $u^*(x,t)$ satisfies the conditions of Lemma 26 with r_* replaced by $\sqrt{r_*}$, and the conclusions of the Lemma 26 for $u^*(x,t)$ imply the same for $u(x,t)$. Thus it is enough to prove the Lemma for the function $u^*(x,t)$. But concavity of $u(\cdot,t)$, (94) and (98) imply that $u^*(x,t)$ satisfies (97). Thus we choose any $A > 0$ for $u^*(x,t)$.

Case 2c. It remains to consider the case when (98) holds and

$$u(x,0) = 0 \quad \text{for all} \quad x \in [-\sqrt{r_*}, 0]. \tag{101}$$

Since for every $t \in [T_1, 0]$ the function $u(\cdot, t)$ is concave and (94), (99) hold, we can choose a number L_t for each $t \in [T_1, 0]$ such that

$$u(x,t) \le \varepsilon|t| + L_t x \quad \text{for} \quad x \ge 0. \tag{102}$$

For $t \in [T_1, 0]$ define

$$\psi(t) = \inf\{L_t \in \mathbf{R} \ : \ (102) \text{ holds}\}.$$

Note that by (98)

$$\psi(t) \ge 0 \quad \text{for} \quad t \in [T_1, 0]. \tag{103}$$

It follows from the definition of ψ, (99) and (98) that for each $t \in [T_1, 0]$ there exists $\hat{x}_t \in \overline{I_{2r_0}}$ such that

$$u(\hat{x}_t, t) = \varepsilon|t| + \psi(t)\hat{x}_t.$$

Then using concavity of $u(\cdot, t)$ and (94) we get

$$u(x,t) \le \varepsilon|t| + \psi(t)x \quad \text{for all} \quad x \in I_{2r_0}, \ \ t \in [T_1, 0]. \tag{104}$$

Let us show that

$$\psi(t) \le \frac{M+\varepsilon}{\sqrt{r_*}}|t| \quad \text{for} \quad t \in [T_1, 0]. \tag{105}$$

Suppose (105) is false. Then there exists $t_1 \in [T_1, 0)$ such that

$$\psi(t_1) > \frac{M+\varepsilon}{\sqrt{r_*}}|t_1|. \tag{106}$$

Using (104), (106) we get

$$u(-\sqrt{r_*}, t_1) \le \varepsilon|t_1| + \psi(t_1)(-\sqrt{r_*}) < (\varepsilon - \frac{M+\varepsilon}{\sqrt{r_*}}\sqrt{r_*})|t_1| = -M|t_1|.$$

But $u(-\sqrt{r_*}, 0) = 0$, and thus

$$|u(-\sqrt{r_*}, t_1) - u(-\sqrt{r_*}, 0)| > M|t_1|.$$

A contradiction to (48). Thus (105) is proved.

From (104), (105) we conclude that if (98), (101) is true then the inequality (100) holds with $A = \frac{M+1}{\sqrt{r_*}}$.

Thus, summarizing Cases 2a – 2c, we conclude that in Case 2, after possibly replacing $u(x,t)$ by $u(-x,t)$, we have:

Inequality (51) is replaced by the following: there exists $s_0 \in [r_*, \sqrt{r_*}]$ such that

$$u(x,0) < 0 \quad \textit{for} \quad x > s_0, \tag{107}$$

and the estimate (100) holds with

$$s_0 A \le (M+1)\sqrt{r_*}. \tag{108}$$

Indeed, in Cases 2a, 2b we can choose $A = 1$. In Case 2c we have $A = \frac{M+1}{\sqrt{r_*}}$ and $s_0 = r_*$.

Define a function h on $I_{2r_0} \times (T_1, 0]$ by

$$h(x,t) = (\varepsilon + Ax)|t|. \tag{109}$$

where A is the constant from (100). The function $h(x,t)$ is continuous, and by (100)

$$h(x,t) > u(x,t) \quad \text{on} \quad \{\, x \ge 0, \ t \in (T_1, 0] \,\} \setminus \ (0,0)\,.$$

It follows from (49), (50), (107) and the concavity of $u(\cdot,0)$ that for any $\rho > s_0$ there exists $\delta_\rho > 0$ such that that

$$-2\delta_\rho x > u(x,0) \quad \text{for} \quad x \geq \rho. \tag{110}$$

Now we claim that there exists $\tau^*_\rho \in (0,T_1]$ such that

$$-\delta_\rho x > u(x,t) \quad \text{for} \quad t \in (-\tau^*_\rho, 0], \ \ x \geq \rho. \tag{111}$$

Indeed, if (111) is false, then there are sequences $\theta_k \to 0$ and $x_k \geq \rho$ with $u(x_k,\theta_k) \geq -\delta_\rho x_k$. By (110) we have $|u(x_k,\theta_k) - u(x_k,0)| \geq \delta_\rho x_k \geq \delta_\rho \rho$. This contradicts (48) for $|\theta_k| < \frac{\delta_\rho \rho}{M}$.

Fix $s \in (2s_0, r)$, where s_0 is from (107). Let $\delta > 0$ satisfy $\delta \leq \min(\delta_s, \ (\tau^*_s)^2)$. Here δ_s, τ^*_s are the constants from (110), (111) with $\rho = s$. We claim that

$$\begin{aligned} &h(x,t) + (t^2-\delta)x > u(x,t), \\ &\text{for any } \ (x,t) \in ([0,2r_0] \times (T_1,0]) \setminus \Big([0,s) \times (-\sqrt{\delta}, 0]\Big). \end{aligned} \tag{112}$$

Indeed, $h(x,t) \geq 0$ and thus for $t \in (-\sqrt{\delta}, \ 0]$ we have using (111)

$$h(x,t) + (t^2-\delta)x \geq -\delta x > u(x,t) \quad \text{for} \quad x \in [s, 2r_0].$$

For $t \in [T_1, -\sqrt{\delta}]$ we have

$$h(x,t) + (t^2-\delta)x \geq h(x,t) > u(x,t) \quad \text{for} \quad x \in [0, 2r_0].$$

Thus (112) is proved.

Next we want to show that the inequality (112) holds for some negative x. First note that (52), (53) imply that there exists $a_0 > 0$ such that

$$\delta|x| + \frac{\varepsilon}{10}|t| > \hat{f}(x,t) \geq u(x,t) \quad \text{on} \quad Q^-_{a_0}(0,0) \setminus \{\,(0,0)\,\}.$$

Let $a \in (0, a_0)$ be so small that $a^2 + Aa < \varepsilon/2$. Then we have for $(x,t) \in ([-a,0] \times [-a,0]) \setminus (0,0)$

$$h(x,t) + (t^2-\delta)x \geq \varepsilon|t| - A|xt| - t^2|x| + \delta|x| \geq \frac{\varepsilon}{2}|t| + \delta|x| > u(x,t). \tag{113}$$

Now we choose δ in (112) so that it satisfies $\sqrt{\delta} < a/4$ in addition to the conditions stated above. Then, combining (112) and (113) we conclude that

$$h(x,t) + (t^2-\delta)x > u(x,t)$$

for any

$$(x,t) \in ([-a, 2r_0] \times [-a, 0]) \setminus ([0,s) \times (-a/4, 0]).$$

By continuity there exists $\delta_1 > 0$ such that

$$h(x,t) + (t^2-\delta)x - \delta_1 > u(x,t) \quad \text{in} \quad \mathcal{P}, \tag{114}$$

where

$$\mathcal{P} = ([-a, 2r_0] \times [-a, 0]) \setminus ((-a/4, s) \times (-a/2, 0]) \subset Q^-_{r_0}. \tag{115}$$

Let

$$\varphi(x,t) = h(x,t) + (t^2-\delta)x - \delta_1 = -(\varepsilon + Ax)t + (t^2-\delta)x - \delta_1. \tag{116}$$

and

$$v(x,t) = \begin{cases} \min(\,\varphi(x,t), \ u(x,t)\,) & \text{if} \quad (x,t) \in [-a,s] \times [-a,0]; \\ u(x,t) & \text{in} \ \ Q^-_{2r_0} \setminus ([-a,s] \times [-a,0]). \end{cases}$$

It follows from (114), (115) that $v(x,t)$ is continuous. $\varphi(\cdot,t)$ is linear for every t, and as in Case 1 we conclude that $v(\cdot,t)$ is concave for every t.

We have

$$\partial_t \varphi(x,t) = -\varepsilon - A|x| - 2|tx| \geq -\varepsilon - As - 2as \quad \text{on } [-a,s]\times[-a,0],$$

and, using $\sqrt{\delta} \leq a/4$,

$$|\partial_x \varphi(x,t)| \leq A|t| + |t^2 - \delta| \leq Aa + a^2 \quad \text{on } [-a,s]\times[-a,0].$$

We can make s, a smaller but satisfying $s > s_0$ and $a > 0$ and obtain, using $s_0 \leq \sqrt{r}_*$ and (108):

$$|\partial_x \varphi| \leq \varepsilon, \qquad \partial_t \varphi \geq -2\varepsilon - (M+1)\sqrt{r}_* \qquad \text{on} \quad [-a,s]\times[-a,0].$$

Now it follows that $v(x,t)$ satisfies all conclusions of Lemma 26. Lemma 26 is proved.

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