Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations [*†]

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Abstract

We prove $C^{1,\alpha}$ regularity of Lipschitz free boundaries in two-phase problems for a class of nonconcave fully nonlinear elliptic equations.

In the papers [2]–[4], L. Caffarelli developed the theory of two-phase free boundary problems for the Laplacian equation. In particular, $C^{1,\alpha}$ regularity of Lipschitz free boundaries was proved in [2]. The results of [2] were extended to the case of nonisotropic problems for linear equations in [10], and to the case of concave fully nonlinear uniformly elliptic equations in P. Wang [15].

In this paper we prove $C^{1,\alpha}$ regularity of Lipschitz free boundaries in two-phase problems for a class of nonconcave fully nonlinear uniformly elliptic equations.

We consider the following free boundary problem in the domain $\Omega \subset \mathbf{R}^n$ with smooth boundary:

$$F(D^2u, Du) = 0 \quad \text{in } \ \Omega^+ = \{x \in \Omega, u(x) > 0\}, \tag{1}$$

$$F(D^2u, Du) = 0 \quad \text{in } \ \Omega^- = \Omega \setminus \overline{\Omega^+}, \tag{2}$$

$$u_\nu^+ = G(u_\nu^-) \quad \text{on } \ \Gamma = \partial\Omega^+ \setminus \partial\Omega, \tag{3}$$

$$u_{|\partial\Omega} = \varphi, \tag{4}$$

where (1) and (2) are fully nonlinear uniformly elliptic equations satisfying standard structural assumptions, see (7) and (10) below. Here u_ν^+ and u_ν^- denote normal derivatives of u^+ and u^- at the points of free boundary Γ, where u^+ and u^- are the positive and negative parts of u. Function $\varphi \in C(\partial\Omega)$ is given. We consider weak solutions of (1) – (4) in the sense of [2], [3]:

Definition 1 *A function $u \in C(\overline{\Omega})$ is a weak solution of the free boundary problem (1)-(4) if:*

The equations (1), (2) are satisfied in the viscosity sense, and (4) is satisfied.

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The free boundary condition (3) is understood in the following sense: If $x_0 \in \Gamma$ and Γ has a tangent ball at x_0 from the Ω^+ side, which means that there exists $B_\rho(y) \subset \Omega^+$ such that $x_0 \in \partial B_\rho(y)$, then

$$u(x) = \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|) \tag{5}$$

where $\nu = \frac{x_0 - y}{|x_0 - y|}$, $\alpha, \beta \in [0, \infty]$, and

$$\alpha = G(\beta). \tag{6}$$

If $x_0 \in \Gamma$ and Γ has a tangent ball at x_0 from the Ω^- side, i.e. $B_\rho(y) \subset \Omega^-$ and $x_0 \in \partial B_\rho(y)$, then (5), (6) hold with $\nu = -\frac{x_0 - y}{|x_0 - y|}$, $\alpha, \beta \in [0, \infty]$.

We assume that F in (1), (2) satisfies the following conditions:
(H1) For each $R, S \in \mathcal{S}(n)$, $p, q \in \mathbf{R}^n$

$$\begin{aligned} \mathcal{P}^-_{\Lambda,\lambda}(R - S) - M|p - q| &\leq F(R, p) - F(S, q) \\ &\leq \mathcal{P}^+_{\Lambda,\lambda}(R - S) + M|p - q| \end{aligned} \tag{7}$$

Here $\mathcal{S}(n)$ denotes the set of symmetric $n \times n$ matrices, $\Lambda \geq \lambda > 0$ and $M \geq 0$ are constants. $\mathcal{P}^+_{\Lambda,\lambda}$ and $\mathcal{P}^-_{\Lambda,\lambda}$ are the extremal Pucci operators defined by

$$\mathcal{P}^-_{\Lambda,\lambda}(R) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \sum_{i,j=1}^n a_{ij} R_{ij}, \quad \mathcal{P}^+_{\Lambda,\lambda} = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \sum_{i,j=1}^n a_{ij} R_{ij}, \tag{8}$$

where $R \in \mathcal{S}(n)$, and $\mathcal{A}_{\lambda,\Lambda}$ denotes the set of all symmetric $n \times n$ matrices $A \equiv \{a_{ij}\}$ which satisfy

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any} \quad \xi \in \mathbf{R}^n. \tag{9}$$

(H2) F is positively homogeneous degree 1, i.e.

$$F(\gamma R, \gamma p) = \gamma F(R, p) \quad \text{for all } \gamma > 0,\ R \in \mathcal{S}(n),\ p \in \mathbf{R}^n. \tag{10}$$

Note that (7) implies that F is Lipschitz continuous, and then from (10)

$$F(0, 0) = 0. \tag{11}$$

The main result of this paper is

Theorem 2 *Let u be a weak solution of (1) - (4) in the sense of Definition 1. Assume that $B_1^{n-1} \times [-L, L] \subset \Omega$, where B_1^{n-1} is the unit ball in $\mathbf{R}^{n-1}$ centered at the origin, and $L \geq 1$. Denote points of Ω by (x', x_n) where $x' = (x_1, \ldots, x_{n-1}) \in B_1^{n-1}$ and $x_n \in [-L, L]$. Assume that u satisfies:*

1. *$\Omega^+ = \{(x', x_n) \in \Omega \,:\, x_n > f(x')\}$, where $f(x')$ is a Lipschitz continuous function on B_1^{n-1} with Lipschitz constant satisfying $Lip[f] \leq L$;*

2. *$0 \in \Gamma$, where Γ is the free boundary, defined in (3).*

Assume that $F(\cdot,\cdot)$ in (1), (2) satisfies (7) and (10). Assume that $G(s)$ is strictly increasing, and for some $N>0$, $s^{-N}G(s)$ is decreasing. Then, on $B^{n-1}_{\frac{1}{8}}$, f is a $C^{1,\alpha}$ function, where α depends only on n, N, L, and constants λ, Λ, M in (7).

Our proof is an extension of the argument of L. Caffarelli [2] to the nonlinear nonconcave case. In [2] the $C^{1,\alpha}$ regularity of Lipschitz free boundaries is proved in the case of two-phase free boundary problems for the Laplacian equation. The approach in [2] is based on the Harnack principle and the following two properties of harmonic functions:

(i) boundary comparison principle ([8]), and monotonicity of positive solutions near Lipschitz boundary ([2, Lemma 5]);

(ii) construction of a subharmonic function by taking supremum of harmonic function over balls of variable radii, i.e., proving that if $u(x)$ is harmonic, and $v(x)=\sup_{y\in B_{g(x)}(x)}u(y)$ where $g(\cdot)$ is a suitable nonconstant nonnegative function and $B_r(x)$ is a ball in $\mathbf{R}^n$, then $v(x)$ is subharmonic ([2, Lemmas 9, 13]).

The results of [2] were extended in [10] and [15].

In [10] we consider a two-phase problem, in which the positive and negative parts of a solution satisfy two different linear elliptic equations with constant coefficients, and regularity of Lipschitz free boundary is proved. In order to use the approach of [2], the following analogue of the property (ii) is proved for linear elliptic equations with constant coefficients ([10, Lemmas 7,8]):

(ii') for any two constants $\Lambda>\lambda>0$ there exists a suitable nonconstant nonnegative function $g(x)$, such that if u is a solution of an elliptic equation with ellipticity constants λ,Λ, then $v(x)=\sup_{y\in B_{g(x)}(x)}u(y)$ is a subsolution of the same equation.

P. Wang [15] considers a two-phase problem for concave elliptic equations, i.e., a positive and negative parts of a solution satisfy a fully nonlinear uniformly elliptic equation

$$F(D^2u)=0, \tag{12}$$

where F is concave, smooth, homogeneous of degree one. For such problems, Wang proves that Lipschitz free boundaries are in fact $C^{1,\alpha}$. In particular, the boundary comparison for positive solutions of (12) in Lipschitz domains is proved in [15, Lemma 2.1]. Note that regularity results for concave elliptic equations imply that interior $C^{2,\alpha}$ regularity holds for solutions of (12). This fact is used in [15] for the proofs of properties of solutions of (12) near a Lipschitz boundary. The property (ii') for equation (12) is obtained from the property (ii') of linear equations by representing the concave function $F(\cdot)$ as the infimum of a family of linear functions $\{L_\alpha(\cdot)\}$.

In this paper we do not assume concavity of F with respect to D^2u in the equations (1), (2). Examples of nonconcave equations with F satisfying (7) and (10) include the following Isaacs-type equations. Let $\mathcal{A},\mathcal{B}\subset\mathbf{R}^1$ and let for each $\alpha\in\mathcal{A}$, $\beta\in\mathcal{B}$ the constants $\{a_{ij}^{\alpha\beta}\}_{i,j=1}^n$ are given and satisfy (9) with λ,Λ independent of α,β. Let $b^{\alpha\beta}:\mathbf{R}^n\to\mathbf{R}^1$ satisfy $\|b^{\alpha\beta}\|_{C^{0,1}}\le C$ with C independent of α,β, and $b^{\alpha\beta}(\cdot)$ is positively homogeneous degree 1. Then the equation

$$F(D^2u,Du)=\sup_{\alpha}\inf_{\beta}\left\{\sum_{i,j=1}^{n}a_{ij}^{\alpha\beta}u_{x_ix_j}-b^{\alpha\beta}(Du)\right\}=0,$$

satisfies the conditions (7) and (10) and is in general not a concave or convex equation (i.e. $F(R,p)$ is not concave or convex with respect to $R \in \mathcal{S}(n)$) if both sets $\mathcal{A}$, $\mathcal{B}$ contain more than one element. One example of such an equation is

$$F(D^2u, Du) = \sup_{\alpha} \inf_{\beta} \left\{ \sum_{i,j=1}^{n} a_{ij}^{\alpha\beta} u_{x_i x_j} \right\} - |Du| = 0.$$

For the proof of Theorem 2 we need mainly to prove the properties (i) and (ii') for the solutions of (1). Regularity of free boundary follows then from the argument of [2].

Unlike the case of concave equations, the property (ii') for solutions of nonconcave elliptic equations does not readily follow from the property (ii') of linear equations. Our argument uses the nonlinear structure of the equation. Since the property (ii') for nonlinear elliptic equations is of independent interest, we prove it for a wider class of equations, than the one we consider in the rest of the paper.

We also prove the boundary comparison principle in Lipschitz domains for positive solutions of (1), and monotonicity of such solutions near Lipschitz free boundary for a cone of directions. New features arise in these proofs since the interior $C^{2,\alpha}$ regularity fails in general for solutions of nonconcave elliptic equations, as was shown by N. Nadirashvili.

Finally, a generalization. Similarly to [10] we can consider a nonisotropic problem, in which the positive and negative part of u satisfy two different nonlinear equations. The proof of Theorem 2 remains unchanged since by Proposition 4 and Lemma 7, if both equations satisfy the structure conditions (7) and (10) then the construction of continuous family of subsolutions can be done for the nonisotropic problem (i.e., the proof of Lemma 13 below does not change). In addition, following [2, Section 7] and [10, Remark 3] we can introduce dependence on x and ν into the free boundary relation. Thus we obtain:

Theorem 3 *Let u be a weak solution of the free boundary problem*

$$\begin{aligned}
&F_1(D^2u, Du) = 0 \quad \text{in } \Omega^+ = \{x \in \Omega, u(x) > 0\}, \\
&F_2(D^2u, Du) = 0 \quad \text{in } \Omega^- = \Omega \setminus \overline{\Omega^+}, \\
&u_\nu^+ = G(x, \nu, u_\nu^-) \quad \text{on } \Gamma = \partial\Omega^+ \setminus \partial\Omega, \\
&u_{|\partial\Omega} = \varphi
\end{aligned}$$

in the sense of Definition 1. Assume that $B_1^{n-1} \times [-L, L] \subset \Omega$, where $L \geq 1$. Assume that u satisfies:

1. *$\Omega^+ = \{(x', x_n) \in \Omega \,:\, x_n > f(x')\}$, where $f(x')$ is a Lipschitz continuous function on B_1^{n-1} with Lipschitz constant satisfying $Lip[f] \leq L$;*
2. *$0 \in \Gamma$.*

Assume that F_1, F_2 satisfy (7) and (10). Assume that $G(x, \nu, s)$ satisfy:

(i) $\log G(x, \nu, s)$ is a Lipschitz function of x, ν with a Lipschitz constant L_1 independent of s;

(ii) $G(x, \nu, s)$ is increasing on s, and $s^{-N} G(x, \nu, s)$ is decreasing on $s \in (0, \infty)$, for some N independent of x, ν.

Then, on $B_{\frac{1}{8}}^{n-1}$, f is a $C^{1,\alpha}$ function, where α depends only on n, N, L_1, L, and constants λ, Λ, M in (7).

The rest of the paper is organized as following. In Section 1 we prove the property (ii') for nonlinear equations. In Section 2 we prove boundary comparison principle and monotonicity for positive solutions in Lipschitz domains. In Section 3 we prove a comparison lemma for solutions of nonlinear free boundary problem, and show that the rest of the proof of Theorem 2 follows the argument of [2].

1 A family of subsolutions

In this section we consider a fully nonlinear elliptic equation

$$F(D^2u, Du, u) = 0, \tag{13}$$

where $u(x)$ is defined on the open set $\Omega \in \mathbf{R}^n$.

A standard structure assumption for the function F, implying that (13) is uniformly elliptic with the constants $\Lambda \geq \lambda > 0$, is the following (see e. g. [6]):

$$\begin{aligned} \mathcal{P}^-_{\Lambda,\lambda}(R-S) - M|p-q| - \omega(|z-w|) &\leq F(R,p,z) - F(S,q,w) \\ &\leq \mathcal{P}^+_{\Lambda,\lambda}(R-S) + M|p-q| + \omega(|z-w|) \end{aligned} \tag{14}$$

for each $R, S \in \mathcal{S}(n)$, $p, q \in \mathbf{R}^n$, $z, w \in \mathbf{R}^1$. Here $\mathcal{P}^-_{\Lambda,\lambda}$ and $\mathcal{P}^+_{\Lambda,\lambda}$ are Pucci extremal operators defined by (8), and ω is a nondecreasing continuous function satisfying $\omega(0) = 0$.

Since condition (14) does not allow to consider elliptic equations of the form

$$\sum_{i,j=1}^n a_{ij}(Du) u_{x_i x_j} = 0,$$

we introduce a relaxed structure condition: we require the inequality

$$\begin{aligned} \mathcal{P}^-_{\Lambda,\lambda}(R-S) &- M\left[|p-q| + (|R|+|S|)\frac{|p-q|}{|p|+|q|+1}\right] - \omega(|z-w|) \\ &\leq F(R,p,z) - F(S,q,w) \\ &\leq \mathcal{P}^+_{\Lambda,\lambda}(R-S) + M\left[|p-q| + (|R|+|S|)\frac{|p-q|}{|p|+|q|+1}\right] + \omega(|z-w|) \end{aligned} \tag{15}$$

for each $R, S \in \mathcal{S}(n)$, $p, q \in \mathbf{R}^n$, $z, w \in \mathbf{R}^1$.

Example of an equation which satisfies (15) is the following. Let $\mathcal{A}, \mathcal{B} \subset \mathbf{R}^1$, and let for each $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ the functions are given $a^{\alpha\beta}_{i,j} : \overline{B}_1 \to \mathbf{R}^1$, where $i, j = 1, \ldots, n$ and B_1 denotes the unit ball in $\mathbf{R}^n$, and $b^{\alpha\beta} : \mathbf{R}^n \to \mathbf{R}^1$. Let $\|a^{\alpha\beta}_{i,j}\|_{C^{0,1}}, \|b^{\alpha\beta}\|_{C^{0,1}} \leq C$ with C independent of α, β. Let in addition for each $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $p \in \overline{B}_1$ the numbers $\{a^{\alpha\beta}_{i,j}(p)\}^n_{i,j=1}$ satisfy (9) with λ, Λ independent of α, β, p. The Isaacs-type equation:

$$F(D^2u, Du) = \sup_\beta \inf_\alpha \left\{ \sum_{i,j=1}^n a^{\alpha\beta}_{ij}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) u_{x_i x_j} - b^{\alpha\beta}(Du) \right\} = 0, \tag{16}$$

satisfies (15).

The main result of this section is

Proposition 4 *Let the function $F : \mathcal{S}(n) \times \mathbf{R}^n \times \mathbf{R}^1 \to \mathbf{R}^1$ satisfy (15) and*

$$F(0,0,z) = 0 \quad \textit{for any} \;\; z. \tag{17}$$

Let $\Omega \subset \mathbf{R}^n$ be an open domain and let $u \in C(\Omega)$ be a viscosity subsolution of (13) in $\Omega^+(u){:=}\{u > 0\}$, and u is differentiable in the interior of $\Omega^+(u)$. Let $g \in C^2(\Omega)$, satisfy the following properties:

$$0 < g < 2, \qquad |Dg| \le \varepsilon, \tag{18}$$

$$\mathcal{P}^-_{\Lambda,\lambda} g \ge C\left(\frac{|Dg|^2}{g} + |Dg|\right), \tag{19}$$

where C, ε depend only on n, Λ, λ, M. Let

$$\Omega_g = \{x \in \mathbf{R}^n \mid \; dist(x,\Omega) < g(x)\}.$$

Define on Ω_g the function

$$v(x) = \sup_{B_{g(x)}(x)\cap\Omega} u(\cdot) \tag{20}$$

Let $\Omega_g^+ = \{v > 0\}$. Then v is a viscosity subsolution of (13) in Ω_g^+.

Proof. Let $\tilde{\phi} \in C^2(R^n)$ and $v - \tilde{\phi}$ attains its maximum at $x_0 \in \Omega_g^+$. Then $v(x_0) > 0$. Shifting coordinates, we can assume that $x_0 = 0$. Then $g(x)$ is defined and satisfies (19) on $\overline{B}_{g(0)}(0)$. We need to show that

$$F(D^2\tilde{\phi}(0), D\tilde{\phi}(0), \tilde{\phi}(0)) \ge 0. \tag{21}$$

Let $y_0 \in \overline{B}_{g(0)}(0)$ be such that $v(0) = u(y_0) = \sup_{B_{g(0)}(0)} u(\cdot)$, i.e., the supremum in the right-hand side of (20) with $x = 0$ is attained at y_0. Then

$$u(y_0) > 0, \quad \text{i.e. } \; y_0 \in \Omega^+. \tag{22}$$

Consider first the possibility that

$$y_0 \in B_{g(0)}(0).$$

Then denoting $\psi(x) = \tilde{\phi}(x - y_0)$, we find that the function $u - \psi$ attains its local maximum at y_0. But u is a subsolution of (13) in Ω^+, and then by (22) the inequality

$$F(D^2\psi(y_0), D\psi(y_0), \psi(y_0)) \ge 0$$

holds. This implies (21).

Thus we can assume that

$$y_0 \in \partial B_{g(0)}(0).$$

By choice of coordinates we can write $y_0 = (0, g(0))$ in (x', x_n) coordinates, where $x' \in \mathbf{R}^{n-1}$, $x_n \in \mathbf{R}^1$. Let $e_1, ..., e_n$ be the coordinate unit vectors,

$$\nu(x) = e_n + \frac{1}{g(0)} \sum_{i=1}^{n-1} \sum_{j=1}^{n} l_{ij} x_j e_i, \tag{23}$$

and

$$y(x) = x + \frac{g(x)}{|\nu(x)|}\nu(x), \tag{24}$$

where the constants l_{ij} will be chosen later. We have

$$\begin{array}{l} v(x) \geq u(y(x)) \quad \text{for all } x \in \Omega_g; \\ v(0) = u(y(0)). \end{array} \tag{25}$$

We will choose the constants l_{ij} below so that the inequality is satisfied

$$|l_{ij}| \leq C|Dg(0)|$$

where C depends only on Λ, λ, n, M. By conditions, $|Dg|$ is bounded by ε. So, if ε is small depending on Λ, λ, n, M, then the map (24) is invertible in a small enough neighborhood of $x = 0$. Let

$$\Psi : B_r(g(0)e_n) \to B_{r_1}(0) \tag{26}$$

be the inverse map. Then Ψ is a C^2 map, and so the function

$$\varphi(y) := \tilde{\phi}(\Psi(y))$$

is defined and C^2 smooth on $B_r(g(0)e_n)$. Then for any $y \in B_r(g(0)e_n)$ we have

$$u(y) \leq v(\Psi(y))$$

and thus

$$u(y) - \varphi(y) \leq v(\Psi(y)) - \tilde{\phi}(\Psi(y)) \leq v(0) - \tilde{\phi}(0) = u(g(0)e_n) - \varphi(g(0)e_n).$$

So the function $u - \varphi$ attains a local maximum at the point $g(0)e_n$. Since u is a viscosity subsolution of (13) in Ω^+ and $g(0)e_n = y_0 \in \Omega^+$, we get

$$F(D^2\varphi(g(0)e_n), D\varphi(g(0)e_n), \varphi(g(0)e_n)) \geq 0. \tag{27}$$

Note that for $x \in B_\rho(0)$, where $\rho > 0$ is small enough, we have

$$\tilde{\phi}(x) = \varphi(y(x)). \tag{28}$$

Introduce the following notation:

$$\varphi_i = \partial_{x_i}\varphi(g(0)e_n), \quad \varphi_{ij} = \partial_{x_i x_j}\varphi(g(0)e_n), \quad \Phi = [\varphi_{ij}]_{i,j=1}^n.$$

Note that Φ is symmetric matrix and that

$$\varphi_i = 0 \quad \text{for } i = 1, ..., n-1, \tag{29}$$

$$\varphi_n = |D\varphi(g(0)e_n)| \geq 0, \tag{30}$$

since $D\varphi(g(0)e_n) = Du(g(0)e_n)$ and maximum of $u(x)$ on $\bar{B}_{g(0)}(0)$ is attained at $x = g(0)e_n$.

In the following computations the function φ and its derivatives are always estimated at $x = g(0)e_n$, and the function g and its derivatives are always estimated at $x = 0$.

Explicit computations with use of (28), (24), (23) show that

$$
\begin{aligned}
\partial_{x_i}\tilde{\phi}(0) &= \varphi_i + g_{x_i}\varphi_n, \\
\partial_{x_i x_j}\tilde{\phi}(0) &= \varphi_{ij} + \sum_{k=1}^{n-1}(l_{ki}\varphi_{kj} + l_{kj}\varphi_{ki}) + (g_{x_i}\varphi_{jn} + g_{x_j}\varphi_{in}) \\
&+ \left(g_{x_i}\sum_{k=1}^{n-1} l_{kj}\varphi_{kn} + g_{x_j}\sum_{k=1}^{n-1} l_{ki}\varphi_{kn}\right) + g_{x_i}g_{x_j}\varphi_{nn} \\
&+ \sum_{k,m=1}^{n-1} l_{mi}l_{kj}\varphi_{km} + g_{x_i x_j}\varphi_{x_n} - \frac{\varphi_n}{2g(0)}\sum_{k=1}^{n-1} l_{ki}l_{kj}.
\end{aligned}
\tag{31}
$$

We use the following notation: if $A = [a_{ij}]$ is a $n \times n$ matrix then

$$
\mathbf{a}_{\cdot,\mathbf{j}} = \begin{pmatrix} a_{1j} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj} \end{pmatrix}, \quad \mathbf{a}_{\mathbf{i},\cdot} = \begin{pmatrix} a_{i1} \\ \cdot \\ \cdot \\ \cdot \\ a_{in} \end{pmatrix}.
$$

Denote by L the $n \times n$ matrix, in which first $n-1$ rows contain numbers l_{ij}, $(i = 1, ..., n-1,\ \ j = 1, ...n)$, and the last row contains all zeroes.

We calculate, using (31)

$$
\begin{aligned}
\mathcal{F} &\equiv F(D^2\tilde{\phi}(0), D\tilde{\phi}(0), \tilde{\phi}(0)) \\
&= F(\Phi + Q(\Phi, L, Dg) + \varphi_{x_n}D^2 g - \frac{\varphi_{x_n}}{2g}L^t L,\ \ D\varphi + \varphi_n Dg,\ \ \varphi),
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
Q(\Phi, L, h) &= \Phi L + \Phi^t L^t + h \otimes \varphi_{\cdot,\mathbf{n}} + \varphi_{\cdot,\mathbf{n}} \otimes h \\
&+ h \otimes (\mathbf{\Phi L})_{\mathbf{n},\cdot} + (\mathbf{\Phi L})_{\mathbf{n},\cdot} \otimes h + L^t \Phi L + \varphi_{nn} h \otimes h
\end{aligned}
\tag{33}
$$

for $h \in \mathbf{R}^n$.

We need to show that, with an appropriate choice of $\{l_{ij}\}$

$$
\mathcal{F} \geq 0.
$$

First we prove

Lemma 5 *Let $F(R, p, z)$ satisfy (15) and (17). Then there exist positive ε, C, $\overline{C}$ depending only on Λ, λ, n, M such that the following is true. For any $h \in \mathbf{R}^n$ satisfying*

$$
|h| < \varepsilon
$$

there exist numbers $\{l_{ij}\}$, γ satisfying

$$
|l_{ij}|, |\gamma| \leq C|h|, \tag{34}
$$

such that the inequality holds

$$
\left| F\Big(\Phi + Q(\Phi, L, h),\ \ D\varphi + \varphi_n h,\ \ \varphi\Big) - (1+\gamma)F\Big(\Phi,\ \ D\varphi,\ \ \varphi\Big)\right| \leq \overline{C}|D\varphi|\,|h|. \tag{35}
$$

Proof. We first prove the lemma assuming the additional condition that F is smooth, and then we remove this condition by an approximation argument.

Step 1. Smooth F. We assume in this step that $F \in C^\infty$ and satisfies (15) and (17). If $\Phi = 0$, then $Q(\Phi, L, h) = 0$, and thus, taking $l_{ij}, \gamma = 0$, we get (35) by applying (15). Thus from now on we assume that $\Phi \neq 0$.

The left-hand side of (35) is smaller than or equal to the expression

$$\left|\left[F(\Phi + Q(\Phi, L, h),\ \ D\varphi + \varphi_n h,\ \ \varphi) - F(\Phi,\ \ D\varphi + \varphi_n h,\ \ \varphi)\right] - \gamma\left[F(\Phi,\ \ D\varphi,\ \ \varphi) - F(0,\ \ D\varphi,\ \ \varphi)\right] + \left[F(\Phi,\ \ D\varphi + \varphi_n h,\ \ \varphi) - F(\Phi,\ \ D\varphi,\ \ \varphi)\right]\right| + \gamma\left|F(0,\ \ D\varphi,\ \ \varphi)\right| \tag{36}$$

From (17), (15) we get

$$|F(0,\ \ D\varphi,\ \ \varphi)| \leq 2M|D\varphi|. \tag{37}$$

By (15),

$$\left|F(\Phi,\ \ D\varphi + \varphi_n h,\ \ \varphi) - F(\Phi,\ \ D\varphi,\ \ \varphi)\right| \leq M|h|\,|D\varphi| + M|\Phi|\frac{|D\varphi||h|}{1 + |D\varphi| + |D\varphi + \varphi_n h|},$$

where $|\Phi| = \sqrt{\sum_{i,j=1}^n \phi_{ij}^2}$. Thus we can find $G_1, G_2 \in \mathbf{R}^1$ such that

$$F(\Phi,\ \ D\varphi + \varphi_n h,\ \ \varphi) - F(\Phi,\ \ D\varphi,\ \ \varphi) = G_1 + G_2,$$

where

$$|G_1| \leq M|h|\,|D\varphi|, \quad\ |G_2| \leq M|\Phi|\frac{|D\varphi||h|}{1 + |D\varphi|}.$$

Since $\Phi \neq 0$ there exists $k, l \in \{1, \ldots, n\}$ such that $\varphi_{kl} \geq \frac{1}{n^2}|\Phi| > 0$. Thus we have $G_2 = -c_{kl}\varphi_{kl}$, where $|c_{kl}| \leq Mn^2|h|$. So, setting $c_{ij} = 0$ for $(i, j) \neq (k, l)$, we have

$$G_2 = -\sum_{i,j=1}^n c_{ij}\varphi_{ij}, \quad \text{ where } \quad |c_{i,j}| \leq C(M, n)|h|. \tag{38}$$

Now using (36), (37) we see that the left-hand side of (35) is bounded from above by

$$\left|\left[F(\Phi + Q(\Phi, L, h),\ \ D\varphi + \varphi_n h,\ \ \varphi) - F(\Phi,\ \ D\varphi + \varphi_n h,\ \ \varphi)\right] - \gamma\left[F(\Phi,\ \ D\varphi,\ \ \varphi) - F(0,\ \ D\varphi,\ \ \varphi)\right] - \textstyle\sum_{i,j=1}^n c_{ij}\varphi_{ij}\right| + C(M, n)|D\varphi|\,|h|,$$

if γ is chosen so that (34) holds. Thus in order to complete the proof of lemma, it is enough to find $\{l_{ij}\}_{i,j=1}^n$, γ satisfying (34) and the following equation

$$\left[F(\Phi + Q(\Phi, L, h),\ \ D\varphi + \varphi_n h,\ \ \varphi) - F(\Phi,\ \ D\varphi + \varphi_n h,\ \ \varphi)\right] - \gamma\left[F(\Phi,\ \ D\varphi,\ \ \varphi) - F(0,\ \ D\varphi,\ \ \varphi)\right] - \textstyle\sum_{i,j=1}^n c_{ij}\varphi_{ij} = 0$$

We can write this equation in the form:

$$
\begin{aligned}
\sum_{i,j=1}^{n}\sum_{k,m=1}^{n-1} a_{ij}\Big[(l_{ki}\varphi_{kj}+l_{kj}\varphi_{ki}) \quad &+ \quad (h_i\varphi_{jn}+h_j\varphi_{in})+ \\
(h_il_{kj}\varphi_{kn}+h_jl_{ki}\varphi_{kn}) \quad &+ \quad l_{mi}l_{kj}\varphi_{km}+h_ih_j\varphi_{nn}\Big] \\
&= \quad \gamma\sum_{i,j=1}^{n} b_{ij}\varphi_{ij}+\sum_{i,j=1}^{n} c_{ij}\varphi_{ij},
\end{aligned}
\tag{39}
$$

where the coefficients $\{a_{ij}\}$, $\{b_{ij}\}$ are defined as following:

$$
\begin{aligned}
a_{ij}=a_{ij}(L,h) \quad &= \quad \int_0^1 \frac{\partial F}{\partial R_{ij}}(\Phi+\theta Q(\Phi,L,h),\ D\varphi+\varphi_n h,\ \varphi)d\theta, \\
b_{ij}=b_{ij}(L,h) \quad &= \quad \int_0^1 \frac{\partial F}{\partial R_{ij}}(\theta\Phi,\ D\varphi,\ \varphi)d\theta.
\end{aligned}
\tag{40}
$$

Denote A, B, C the $n\times n$ matrices $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$. The structure condition (15) implies that

$$
\lambda I \le A, B \le \Lambda I, \tag{41}
$$

where I is the $n\times n$ identity matrix.

The equation (39) can be written in the form

$$
\begin{aligned}
&\sum_{i,j=1}^{n-1}\varphi_{ij}\sum_{k=1}^{n}\left[l_{ik}(a_{kj}+a_{jk})+\sum_{m=1}^{n}a_{km}l_{im}l_{jk}\right]+\sum_{i=1}^{n-1}\varphi_{in}\left\{\sum_{k=1}^{n}[l_{ik}(a_{kn}+a_{nk})+h_k(a_{ik}+a_{ki})]\right. \\
&\left.+\sum_{k,m=1}^{n}[h_m l_{ik}(a_{mk}+a_{km})]\right\}+\varphi_{nn}\left[\sum_{k=1}^{n}h_k(a_{kn}+a_{nk})+\sum_{k,m=1}^{n}h_kh_m\right] \\
&=\sum_{i,j=1}^{n}\varphi_{ij}(\gamma b_{ij}+c_{ij}).
\end{aligned}
\tag{42}
$$

The equation (42) is satisfied if $(\{l_{ij}\},\ \gamma)$ is a solution of the following system of equations:

$$
\left\{
\begin{array}{l}
2A\mathbf{l}_{\mathbf{i},\cdot} \quad + \quad \begin{pmatrix} \sum_{k,m=1}^{n} a_{km}l_{im}l_{1k} \\ \cdots \\ \sum_{k,m=1}^{n} a_{km}l_{im}l_{(n-1)k} \\ 2\sum_{k,m=1}^{n} a_{km}l_{im}h_k \end{pmatrix} = \gamma\mathbf{b}_{\mathbf{i},\cdot}+\mathbf{c}_{\mathbf{i},\cdot}-\begin{pmatrix} 0 \\ \cdots \\ 0 \\ 2\sum_{k=1}^{n} a_{ik}h_k \end{pmatrix} \\
\qquad\qquad \text{for} \quad i=1,\ldots,n-1; \\
\gamma b_{nn} \quad = \quad -c_{nn}+2\sum_{k=1}^{n} h_k a_{kn}+\sum_{k,m=1}^{n} h_kh_m.
\end{array}
\right.
\tag{43}
$$

We solve this nonlinear system by an iteration process. Denote $(\{l_{ij}^N\},\ \gamma^N)$ the iterations for $N=0,1,\ldots$. We set

$$
l_{ij}^0=0, \quad \gamma^0=0.
$$

Suppose that $\{l_{ij}^{N-1}\}$, γ^{N-1} are given and satisfy

$$
|l_{ij}^{N-1}| \le K|h|, \tag{44}
$$

where K will be defined below, depending only on Λ, λ, n, M. We find $(\{l_{ij}^N\},\ \gamma^N)$ as follows. Denote L^{N-1} the matrix with entries l_{ij}^{N-1} for $i = 1, ..., n-1$; $j = 1, ...n$ and n-th row containing all zeroes. Let for $i, j = 1, \ldots, n$

$$\begin{aligned} a_{ij}^{N-1} &= a_{ij}(L^{N-1}, h), \\ b_{ij}^{N-1} &= b_{ij}(L^{N-1}, h), \\ c_{ij}^{N-1} &= c_{ij}(h), \end{aligned}$$

be defined by (40), (38), and let $A^{N-1}, B^{N-1}, C^{N-1}$ be the corresponding matrices. Then A^{N-1}, B^{N-1} satisfy (41). We find γ^N from the last equation of (43) with coefficients $A^{N-1}, B^{N-1}, C^{N-1}$, i.e.,

$$\gamma^N = \frac{1}{b_{nn}^{N-1}} \left[-c_{nn}^{N-1} + 2\sum_{k=1}^{n} h_k a_{kn}^{N-1} + \sum_{k,m=1}^{n} h_k h_m \right].$$

Then, assuming that $|h| < 1$ (i.e. $\varepsilon < 1$), we get by (41), (38)

$$|\gamma^N| \le C(\Lambda, \lambda, n, M)|h|. \tag{45}$$

We find l_{ij}^N for $N \ge 1$ by solving $n-1$ linear systems which correspond the first part of (43). Namely, for $i = 1, ..., n-1$ we consider the system

$$\left(2A^{N-1} + D^{i,N-1}\right) \mathbf{l}_{\mathbf{i},\cdot}^N = \gamma^N \mathbf{b}_{\mathbf{i},\cdot}^{N-1} + \mathbf{c}_{\mathbf{i},\cdot}^{N-1} - \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 2\sum_{k=1}^{n} a_{ik}^{N-1} h_k \end{pmatrix}, \tag{46}$$

where $D^{i,N-1}$ is the matrix with entries

$$\begin{aligned} d_{jm}^{i,N-1} &= \sum_{k=1}^{n} a_{km}^{N-1} l_{jk}^{N-1} \qquad \text{for} \quad j = 1, ..., n-1, \quad m = 1, ..., n; \\ d_{nm}^{i,N-1} &= \sum_{k=1}^{n} a_{km}^{N-1} h_k \qquad \text{for} \quad m = 1, ..., n. \end{aligned}$$

From (44), (41)

$$|d_{ij}^{i,N-1}| \le KC(\Lambda, n)|h|. \tag{47}$$

Choosing $\varepsilon > 0$ small depending on Λ, λ, n, K, we get for $|h| < \varepsilon$

$$2A^{N-1} + D^{i,N-1} \ge \lambda I. \tag{48}$$

Thus there exists a unique solution $\mathbf{l}_{\mathbf{i},\cdot}^N$ of (46), and it satisfies (recall (38), (45))

$$|l_{ij}^N| \le C(\Lambda, \lambda, n, M)|h|. \tag{49}$$

Now we can choose the constant K in (44) to be equal to the constant $C(\Lambda, \lambda, n, M)$ from (49). By (47), (48) this fixes the choice of ε depending only on Λ, λ, n, M.

Thus, assuming the estimate (44) for $\{l_{ij}^{N-1}\} \in \mathbf{R}^{n(n-1)}$, we are getting the same estimate for $\{l_{ij}^N\}$. Recall also that γ^N satisfies (45). The set

$$\mathcal{B} = \left\{ (\{l_{ij}^{N-1}\}, \gamma^{N-1}) \ \mid \ \{l_{ij}^{N-1}\} \text{ and } \gamma^{N-1} \text{ satisfy (44) and (45) resp.} \right\}$$

is a compact convex subset of $\mathbf{R}^{n(n-1)} \times \mathbf{R}^1$. A map T on $\mathcal{B}$ defined by $T(\{l_{ij}^{N-1}\}, \gamma^{N-1}) = (\{l_{ij}^N\}, \gamma^N)$ maps $\mathcal{B}$ into itself as we showed above. Also, T is continuous by (48). Thus by Schauder fixed point theorem, T has a fixed point $(L, \gamma) \in \mathcal{B}$, which is a solution of (43), satisfying (34). Lemma is proved for smooth F.

Step 2. Nonsmooth F. Now we assume only that F satisfies (15) and (17).

The condition (15) implies that F is a continuous function. By identifying $\mathcal{S}(n)$ with $\mathbf{R}^{\frac{n(n+1)}{2}}$ we can consider F as a continuous function on the space $V = \mathbf{R}^{\frac{n(n+1)}{2}} \times \mathbf{R}^n \times \mathbf{R}^1$. Let $\eta \in C^\infty(V)$ satisfy $\operatorname{supp} \eta \subset B_1$, where $B_1 \subset V$ is a unit ball centered at the origin, $\eta \geq 0$ and $\int_V \eta = 1$. Let $\delta > 0$, and $\eta_\delta(X) = \frac{1}{\delta^n}\eta(\frac{X}{\delta})$. Then the function $F_\delta = F * \eta_\delta$ is smooth, and, since F is locally Lipschitz on V, then

$$F_\delta \to F \ \text{ uniformly on compact subsets of } V \text{ as } \delta \to 0+. \tag{50}$$

Define for $\delta > 0$ the functions $G_\delta(R, p, z) = F_\delta(R, p, z) - F_\delta(0, 0, z)$. It follows from (17), (50) that $G_\delta \to F$ uniformly on compact subsets of V as $\delta \to 0+$. Also, we have $G_\delta(0, 0, \cdot) \equiv 0$, i.e. each G_δ satisfies (17).

Each G_δ for $\delta \in (0, 1)$ satisfies (15) with M and $\omega(\cdot)$ replaced by $3M$ and $2\omega(\cdot)$. Indeed, we have $F_\delta(X) = \int_{B_\delta} \eta_\delta(X')F(X - X')\,dX'$. Let $X' = (\hat{R}, \hat{p}, \hat{z}) \in B_\delta$. Estimate the difference $F(R - \hat{R}, p - \hat{p}, z - \hat{z}) - F(S - \hat{R}, q - \hat{p}, w - \hat{w})$ from above and below using (15). Similarly, estimate $F(-\hat{R}, -\hat{p}, w - \hat{z}) - F(-\hat{R}, -\hat{p}, z - \hat{z})$. Add the above estimates. Then, assuming $\delta < 1$ (and thus $|(\hat{R}, \hat{p}, \hat{z})| < 1$) and estimating

$$|p - q| + (|R - \hat{R}| + |S - \hat{R}|)\frac{|p - q|}{|p| + |q| + 1} \leq 3\left(|p - q| + (|R| + |S|)\frac{|p - q|}{|p| + |q| + 1}\right),$$

we obtain

$$\begin{aligned}
\mathcal{P}^-_{\Lambda,\lambda}(R - S) &- 3M\left[|p - q| + (|R| + |S|)\frac{|p - q|}{|p| + |q| + 1}\right] - 2\omega(|z - w|) \\
&\leq \left[F(R - \hat{R}, p - \hat{p}, z - \hat{z}) - F(-\hat{R}, -\hat{p}, z - \hat{z})\right] \\
&\quad -\left[F(S - \hat{R}, q - \hat{p}, w - \hat{z}) - F(-\hat{R}, -\hat{p}, w - \hat{z})\right] \\
&\leq \mathcal{P}^+_{\Lambda,\lambda}(R - S) + 3M\left[|p - q| + (|R| + |S|)\frac{|p - q|}{|p| + |q| + 1}\right] + 2\omega(|z - w|).
\end{aligned}$$

Multiplying this inequality by $\eta_\delta(X')$ where $X' = (\hat{R}, \hat{p}, \hat{z})$, and integrating with respect to X' over B_δ, we obtain the desired estimate for G_δ.

From the above discussion and Step 1 of the proof it follows that Lemma 5 holds for each G_δ for $\delta \in (0, 1)$, with M and $\omega(\cdot)$ replaced by $3M$ and $2\omega(\cdot)$ in (15). Thus, if ε, C are the constants defined in Step 1 by $\Lambda, \lambda, n, 3M$, and if $h \in \mathbf{R}^n$ satisfies $|h| < \varepsilon$, then for each $\delta \in (0, 1)$ there exist $L_\delta = \{l_{ij}^\delta\}$ and γ_δ satisfying

$$|l_{ij}^\delta|, |\gamma_\delta| \leq C|h|, \tag{51}$$

such that (35) holds for $G_\delta, L_\delta, \gamma_\delta$. By (51) we can choose a sequence $\delta_k \to 0$ such that L_{δ_k}, γ_{δ_k} converge. Then the limits $L = \{l_{ij}\}$, γ satisfy (34). Also by (51) we obtain $|\,Q(\Phi, L_{\delta_k}, h)\,| \leq C(h)$. Since G_δ converge to F locally uniformly on V, we can pass to the limit in the inequalities (35) for $G_\delta, L_\delta, \gamma_\delta$, and conclude that Lemma 5 is true for nonsmooth F. $\square$

We continue the proof of Proposition 4. Note that making ε in Lemma 5 smaller if necessary, we get from (34)

$$|\gamma| < 1 \tag{52}$$

for solution (L, γ) of (35). Let $\varepsilon > 0$ in (18) is chosen so small that it satisfies conditions of Lemma 5 and (52) holds, and such that the mapping (26) exists. Let L, γ be the solution of (35) with $h = Dg(0)$. Then we estimate the quantity $\mathcal{F}$ defined in (32) as follows. We use Lemma 5, (33), (34), (52), (27), (30) and assumptions (15), (19), recall also that in our calculations the function φ and its derivatives are always estimated at the point $x = g(0)e_n$, and the function g and its derivatives are always estimated at the point $x = 0$:

$$\begin{aligned}
\mathcal{F} &= F(\Phi + Q(\Phi, L, Dg) + \varphi_{x_n} D^2 g - \frac{\varphi_{x_n}}{2g} L^t L, \;\; D\varphi + \varphi_n Dg, \;\; \varphi) \\
&\geq F(\Phi + Q(\Phi, L, Dg), \;\; D\varphi + \varphi_n Dg, \;\; \varphi) \\
&\quad + |D\varphi| \left[\mathcal{P}^-_{\Lambda,\lambda} g - C(\Lambda, n) \frac{\sum_{i,j} l_{ij}^2}{g} \right] \\
&\geq (1+\gamma) F(\Phi, \;\; D\varphi, \;\; \varphi) + |D\varphi| \left[\mathcal{P}^-_{\Lambda,\lambda} g - C(\Lambda, \lambda, n, M) \left(\frac{|Dg|^2}{g} + |Dg| \right) \right] \\
&\geq 0
\end{aligned}$$

if the constant C in (19) is large enough.

Proposition 4 is proved. $\square$

Corollary 6 *Let F satisfy (14) and (17). Let $\Omega \subset \mathbf{R}^n$ be an open domain and let $u \in C(\Omega)$ be a viscosity solution of (13) in $\Omega^+(u)$. Let $g \in C^2(\Omega)$ be as in Proposition 4. Then assertion of Proposition 4 holds, i.e. the function v defined by (20) is a viscosity subsolution of (13) in Ω^+_g.*

Proof. By [14, Theorem 2.1], $u \in C^{1,\alpha}_{loc}(\Omega^+(u))$. Thus u satisfies all conditions of Proposition 4. $\square$

In the next lemma we construct a family of functions satisfying (19), (18). In fact we show that a slight modification of the construction of [10, Lemma 8], provides such a family.

We use for $x \in \mathbf{R}^n$ the notation $x = (x', x_n)$ where $x' = (x_1, \ldots, x_{n-1})$.

Lemma 7 *Let $0 < \rho < \frac{1}{16}$, $0 < \lambda, \varepsilon < 1$, $\Lambda \geq \lambda$, and $\overline{C} > 0$. There exist numbers $\theta, \mu > 0$ and a continuous family of C^2 functions g_t $(0 \leq t \leq 1)$, defined in $\overline{B}_1 \setminus B_\rho(0, \frac{1}{4})$, such that*

(i) $1 \leq g_t \leq 1 + t\mu$,

(ii) $|Dg_t| \leq \varepsilon$.

(iii) The inequality holds

$$\mathcal{P}^-_{\Lambda,\lambda} g_t \geq \overline{C} \left(\frac{|Dg_t|^2}{g_t} + |Dg_t| \right),$$

where $\overline{C}$ is the constant chosen above, *(iv)* $g_t \equiv 1$ *outside of* $B_{\frac{7}{8}}$,

(v) $g_{|B_{\frac{1}{8}}} \geq 1 + \theta\mu t$.

Proof. By explicit calculation one can check that there exists an integer $M > 0$, depending on $\overline{C}$, ρ and λ, such that in $B_1 \setminus B_\rho$, for every elliptic operator L of the form $Lu = \sum_{i,j=1}^n A_{ij} u_{x_i x_j}$ where A_{ij} satisfy (9), the inequality $Lf \geq \overline{C}|Df|$ holds, where $f(x) = \frac{1}{|x|^{2M}}$.

Let $f_1(x) = \max(\frac{1}{|(0,\frac{1}{4})-x|^{2M}} - 2^{2M}, 0)$. Then $Lf_1 \geq \overline{C}|Df_1|$ on $B_{\frac{1}{2}}(0,\frac{1}{4}) \setminus B_\rho(0,\frac{1}{4})$ and $f_1 \equiv 0$ outside $B_{\frac{1}{2}}(0,\frac{1}{4})$.

Now let $f_2 = \gamma f_1^4$, where γ is chosen such that $0 \leq f_2 \leq 1$ in $B_1 \setminus B_\rho(0,\frac{1}{4})$ (and thus γ depends only on $\overline{C}$, ρ and λ). Then $f_2 \in C^2$. Also, since on the set $\{f_1 > 0\}$

$$Lf_2 = 4\gamma f_1^3 Lf_1 + 12\gamma f_1^2 \sum_{i,j} a_{ij}(f_1)_{x_i}(f_1)_{x_j} \geq 4\gamma f_1^3 Lf_1$$

and

$$|Df_2| = 4\gamma f_1^3 |Df_1|,$$

we have $Lf_2 \geq \overline{C}|Df_2|$ on $B_1 \setminus B_\rho(0,\frac{1}{4})$. Also, $f_2 \equiv 0$ outside $B_{\frac{7}{8}}$, and $f_2 \geq \theta > 0$ on $B_{\frac{1}{8}}$, where θ depends only on $\overline{C}$, ρ and λ.

Now the function $g_t = 1 + t\mu f_2$, for small $\mu > 0$ depending on ε and $\sup_{B_1 \setminus B_\rho(0,\frac{1}{4})} |Df_2|$, satisfies all conditions. Note that from the above construction $\sup_{B_1 \setminus B_\rho(0,\frac{1}{4})} |Df_2|$ depends only on $\overline{C}$, ρ and λ. Thus $\mu = \mu(\varepsilon, \overline{C}, \rho, \lambda)$. $\square$

2 Properties of positive solutions near the Lipschitz boundary

In this and following sections we consider equations of the form

$$F(D^2u, Du) = 0, \tag{53}$$

We will assume that $F(\cdot,\cdot)$ satisfies the structure conditions (7) and (10).

It is well-known (and in particular can be obtained by straightforward modifications of the proofs in Chapters 2-5 of [5]) that comparison principle holds for sub- and super-solutions of (53), and Harnack inequality holds for positive solutions of (53).

We also will use the following

Lemma 8 *Let* $\Omega \subset \mathbf{R}^n$ *be a domain satisfying a uniform exterior cone condition, and* $g \in C(\partial\Omega)$. *Then there exists* $u \in C(\overline{\Omega})$ *satisfying*

$$\begin{cases} F(D^2u, Du) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{54}$$

where the equation is interpreted in the viscosity sense.

Proof. We construct a solution by Perron's method. Indeed, since boundary data are continuous, then, by [6, Corollary 3.10] there exist (classical) solutions $u^+, u^- \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ of the equations

$$\mathcal{P}^-_{\Lambda,\lambda}(D^2u^-) - M|Du^-| = 0,$$
$$\mathcal{P}^+_{\Lambda,\lambda}(D^2u^+) + M|Du^+| = 0$$

respectively, in the domain Ω with boundary data $u^{\pm} = g$ on $\partial\Omega$. By (7), (11) the u^+ is a supersolution and u^- is a subsolution of (54). Now the existence of a solution $u \in C(\overline{\Omega})$ follows from [7, Theorem 4.1]. □

Now we prove a boundary comparison principle.

We consider the following domains. We use coordinates (x', x_n) in $\mathbf{R}^n$, where $x' = (x_1, \ldots, x_{n-1})$. Let $x_n = f(x')$ be a Lipschitz function with Lipschitz constant L, and let $f(0) = 0$. We will assume without loss of generality that $L \geq 1$. For $r > 0$ denote

$$\Omega_r = \{(x', x_n) \mid |x'| < r,\ |x_n| < Lr,\ x_n > f(x')\}. \tag{55}$$

We say that a constant C depends only on data, if it depends at most on on n, the constants λ, Λ, M in condition (7), and the Lipschitz character L of the domain Ω_2. All constants C below are positive and depend only on data, unless otherwise specified.

Proposition 9 *Let $F(R,p)$ satisfies (7) and (10). Let u, v be two nonnegative viscosity solutions of (53) in Ω_2, and let u, v vanish on $\{x_n = f(x')\} \cap \partial\Omega_2$. Suppose that $u \geq \sigma v$ in Ω_2 for some $\sigma \geq 0$. There exists $C > 0$ depending only on data such that in $\Omega_{\frac{1}{2}}$*

$$C\,\frac{u(A) - \sigma v(A)}{v(A)} \leq \frac{u - \sigma v}{v}.$$

where $A{:=}\frac{L}{2}e_n = (0, \frac{L}{2})$.

Proof. We intend to use linearization in the way similar to [15, Lemma 2.1]. However, since solutions of nonconcave equations are not C^2-smooth in general, a straightforward linearization will produce in our case an elliptic equation with discontinuous coefficients. But then uniqueness in the Dirichlet problem for such equations fails even for "good solutions", as was shown by N. Nadirashvili [12], and generalized by M. Safonov [13]. This nonuniqueness creates a difficulty in the proof. Thus we need to do a more careful approximation and linearization of our equation.

Consider

$$w = u - (\sigma + \mu)v \tag{56}$$

in Ω_1, where $\mu \geq 0$ will be defined later. If we prove that $w \geq 0$ in $\Omega_{\frac{1}{2}}$ for $\mu = C\ \frac{u(A)-\sigma v(A)}{v(A)}$ where $C > 0$, then the proposition is proved.

It follows from (10) that $(\sigma + \mu)v$ is a viscosity solution of (53). Then, modifying the proof of Theorem 5.3 of [5] in order to take into account the dependence of F on Du using the structure assumption (7), we find that

$$\mathcal{P}^-_{\Lambda,\lambda}(D^2 w) - M|Dw| \leq 0 \quad \text{in } \Omega_1$$

in the viscosity sense, i.e., w is a viscosity supersolution of the above equation. Let $\hat{w}$ be the unique viscosity solution of the Dirichlet problem

$$\begin{cases} \mathcal{P}^-_{\Lambda,\lambda}(D^2 \hat{w}) - M|D\hat{w}| = 0 & \text{in } \Omega_1; \\ \hat{w} = w & \text{on } \partial\Omega_1. \end{cases} \tag{57}$$

Since $\mathcal{P}^-_{\Lambda,\lambda}$ is a concave uniformly elliptic operator, and Ω_1 is a Lipschitz domain and w is continuous on $\partial\Omega_1$, the problem (57) has a unique solution $\hat{w} \in C^{2,\alpha}(\Omega_1) \cap C(\overline{\Omega}_1)$, see e.g.

[6, Corollary 3.10]. By comparison principle we have $w \geq \hat{w}$ in Ω_1. It remains to prove that $\hat{w} \geq 0$ in $\Omega_{\frac{1}{2}}$ for an appropriate choice of μ.

By mollifying the function $\Phi(R,p) = \mathcal{P}^-_{\Lambda,\lambda}(R) - M|p|$, we find a sequence of functions $\Phi_k(R,p) \in C^\infty$ satisfying (7), (10) concave with respect to R, and converging uniformly on compact sets of $\mathcal{S}(n) \times \mathbf{R}^n$ to Φ as $k \to \infty$ (the construction of the approximations repeats the one in Step 2 of the proof of Lemma 5). Let $\hat{w}_k$ be the (classical) solution of

$$\begin{cases} \Phi_k(D^2\hat{w}_k, D\hat{w}_k) = 0 & \text{in } \Omega_1; \\ \hat{w}_k = w & \text{on } \partial\Omega_1. \end{cases} \tag{58}$$

Then the functions $\hat{w}_k$ for $k = 1, \ldots$ are uniformly bounded and equi-Holder in $\overline{\Omega}_1$. Thus we can find a uniformly converging subsequence. By a standard argument the limit is a viscosity solution of the equation in (57) (see e.g. [5, Proposition 2.9]). Also, the limit clearly satisfies the boundary condition of (57). Thus, by uniqueness of viscosity solution of (57), the limit is in fact the $\hat{w}$. Thus $\hat{w}_k \to \hat{w}$ uniformly in $\overline{\Omega}_1$. It remains to prove that $\hat{w}_k \geq 0$ in $\Omega_{\frac{1}{2}}$ for an appropriate choice of μ independent of k.

Fix k. We have $\hat{w}_k \in C^{2,\alpha}(\Omega_1)$ since $\hat{w}_k$ is a solution of concave elliptic equation (58). Define for $x \in \Omega$

$$a_{ij}(x) = \int_0^1 \frac{\partial \Phi_k}{\partial R_{ij}}\Big(sD^2\hat{w}_k(x), D\hat{w}_k(x)\Big)\,ds,$$
$$b_i(x) = \int_0^1 \frac{\partial \Phi_k}{\partial p_i}\Big(0, sD\hat{w}_k(x)\Big)\,ds.$$

Then

$$A\hat{w}_k \equiv \sum_{i,j=1}^n a_{ij}(x)(\hat{w}_k)_{x_ix_j} + \sum_{i=1}^n b_i(x)(\hat{w}_k)_{x_i} = 0. \tag{59}$$

The linear operator A is (λ, Λ)-elliptic, and $|b_i(x)| \leq C(M)$ in Ω_1. Note that $a_{ij}(\cdot)$, $b_i(\cdot)$ are smooth in Ω_1, but not necessarily in $\overline{\Omega}_1$. For this reason, in order to pass to the limit in the approximations below, we need to use the regularity of $\hat{w}_k$. Specifically, since $\hat{w}_k \in C^{2,\alpha}(\Omega_1)$, it follows from [6, Lemma 2.5 and Theorem 2.10] that $\hat{w}_k$ is a unique L^n-viscosity solution of (59) in Ω_1, satisfying $\hat{w}_k = w$ on $\partial\Omega_1$. Extend $a_{ij}(\cdot)$, $b_i(\cdot)$ to $\mathbf{R}^n$ by setting for $a_{ij}(x) = \delta_{ij}$, $b_i(x) = 0$ for $x \in \mathbf{R}^n \setminus \Omega_1$. Now, mollifying $a_{ij}(\cdot)$, $b_i(\cdot)$ in $\mathbf{R}^n$, we get a sequence of linear operators

$$A_m \equiv \sum_{i,j=1}^n a^m_{ij}(x)\partial_{x_ix_j} + \sum_{i=1}^n b^m_i(x)\partial_{x_i}$$

for $m = 1, 2, \ldots$, with smooth coefficients in $\mathbf{R}^n$, elliptic with constants λ, Λ and satisfying $|b^m_i| \leq M$ in $\mathbf{R}^n$, and such that for all $i,\ j = 1, \ldots, n$ and all $m = 1, \ldots$

$$\begin{aligned} &|a^m_{ij}(x)|,\ |b^m_i(x)| \leq C \quad \text{in } \mathbf{R}^n, \\ &a^m_{ij}(x) = a_{ij}(x), \quad b^m_i(x) = b_i(x) \quad \text{if } \operatorname{dist}(x, \partial\Omega_1) > \tfrac{1}{m}, \end{aligned} \tag{60}$$

where C depends only on λ, Λ, M, n. The solutions $\hat{w}_{km}$ of

$$\begin{cases} A_m(\hat{w}_{km}) = 0 & \text{in } \Omega_1; \\ \hat{w}_{km} = w & \text{on } \partial\Omega_1; \end{cases} \tag{61}$$

are equi-Holder in $\overline{\Omega}_1$. Thus we can extract a subsequence $\hat{w}_{km_j}$ converging uniformly in $\overline{\Omega}_1$. By [6, Theorem 3.8], the limit is a L^n-viscosity solution of (59) in Ω_1 (assumptions of Theorem 3.8 of [6] are satisfied due to the (λ, Λ)-ellipticity of A_m and (60)). Note also that the limit assumes the boundary data w on $\partial\Omega_1$. But, as we pointed out above, $\hat{w}_k$ is a unique L^n-viscosity solution of (59) in Ω_1 with such boundary data. Thus $\hat{w}_{km} \to \hat{w}_k$. Thus it remains to show that $\hat{w}_{km} \geq 0$ in $\Omega_{\frac{1}{2}}$ for an appropriate choice of μ independent of k, m.

Fix k, m. Let W_1 and W_2 be the solutions of the Dirichlet problems $A_m W_l = 0$ in Ω_1 for $l = 1, 2$, with $W_1 = u - \sigma v$ on $\partial\Omega_1$ and $W_2 = v$ on $\partial\Omega_1$. By interior Harnack inequality (applied in the an open domain $\mathcal{D}$ such that $\overline{\mathcal{D}} \subset \Omega_2$, the point $A = (0, \frac{L}{2}) \in \mathcal{D}$ and $\mathcal{D} \cap \partial\Omega_1 \neq \emptyset$) we get

$$C_1^{-1} \leq \frac{W_1(A)}{u(A) - \sigma v(A)}, \ \frac{W_2(A)}{v(A)} \leq C_1,$$

where $C_1 > 1$ depends on data. By [9, Theorem I.2.2]

$$\frac{W_1}{W_2} \geq C_2 \frac{W_1(A)}{W_2(A)} \quad \text{in } \Omega_{\frac{1}{2}},$$

where $C_2 > 0$ depends only on data (note that we have first-order terms in the linear elliptic operator A_m, but it is easy to check that Theorem I.2.2 of [9] is true in this case too, with the constant depending on the ellipticity, Lipschitz character of the domain, dimension, and L^∞ bound of the first-order coefficients).

Thus

$$W_1 \geq C_3 \, \frac{u(A) - \sigma v(A)}{v(A)} W_2 \quad \text{in } \Omega_{\frac{1}{2}}, \tag{62}$$

where $C_3 > 0$ depends only on data.

By uniqueness of solutions for Dirichlet problem (61) (recall, the coefficients of A_m are smooth) we get

$$\hat{w}_{km} = W_1 - \mu W_2 \quad \text{in } \Omega_1. \tag{63}$$

Choose $\mu = C_3 \, \frac{u(A) - \sigma v(A)}{v(A)}$ in (56). Then by (62) we get $\hat{w}_{km} \geq 0$ in $\Omega_{\frac{1}{2}}$.

Thus $w \geq 0$ in $\Omega_{\frac{1}{2}}$ if μ is as above, where w is the function (56). The proposition is proved. □

Now we prove monotonicity of positive solutions of (53) near Lipschitz boundary.

Proposition 10 *Let F and u be as in Proposition 9. Then there exists $\delta > 0$ depending only on data, such that in Ω_δ there holds $D_{x_n} u \geq 0$.*

Proof. First we show that assertions of Lemmas 2-4 of [2] hold true for our equation. We summarize this in the following

Lemma 11 *Let F, u, v be as in Proposition 9. Then*

(a) If in addition $u(\frac{L}{2}e_n) = v(\frac{L}{2}e_n)$ then $\frac{u}{v} \in C^\alpha(\Omega_{\frac{1}{2}})$, where $\alpha > 0$ depends only on data.

(b) There exists $\delta > 0$ depending on data such that for any $x \in \Omega_\delta$,

$$u(x) \leq \frac{1}{2} u(\frac{L}{2} e_n).$$

(c) If we assume in addition that $D_{x_n}u \geq 0$ on Ω_1 then, denoting $d(x)$ the distance from x to the Lipschitz bottom of Ω_2 (i.e. to the graph f), we have on $\Omega_{\frac{1}{2}}$

$$0 < C_1 \leq \frac{d\,D_{x_n}u}{u} \leq C_2,$$

i.e., $D_{x_n}u \simeq \frac{u}{d}$ on $\Omega_{\frac{1}{2}}$.

Proof. Using Proposition 9 the proof of (a) follows directly the argument of [1]. Property (b) can be proved like Lemma 2.3 of [15].

In order to prove (c) we modify the definition of class S of [5, Section 2.2], to take into account dependence of F on Du: we say that a continuous function u belongs to the class $S(\lambda, \Lambda, M)$ if in the viscosity sense

$$\begin{aligned} \mathcal{P}^-_{\Lambda,\lambda}(D^2u) - M|Du| &\leq 0; \\ \mathcal{P}^+_{\Lambda,\lambda}(D^2u) + M|Du| &\geq 0. \end{aligned}$$

The following facts are well known (and in particular can be obtained by straightforward modifications of the proofs in Chapters 2-5 of [5]): Positive functions $u \in S(\lambda, \Lambda, M)$ satisfy Harnack inequality. The class $S(\lambda, \Lambda, M)$ is closed with respect to the uniform on compact subsets limits. If u and v are viscosity solutions of (53) where F satisfies (7), then $u - v \in S(\frac{\lambda}{n}, \Lambda, M)$. Also, $C^{1,\alpha}$ estimates hold for viscosity solutions u of (53), and in particular $\frac{u(x+he_n)-u(x)}{h} \to D_{x_n}u$ uniformly on compact subsets of the domain as $h \to 0$.

We conclude from the above facts that $D_{x_n}u \in S(\frac{\lambda}{n}, \Lambda, M)$, and since by assumptions of part (c) of Lemma we have $D_{x_n}u \geq 0$, the Harnack inequality holds for $D_{x_n}u$. Now the proof of (c) directly repeats the proof of Lemma 4 of [2]. □

The proof of Proposition 10 will consist of two steps. First we follow the proof of Lemma 5 of [2]. However, we are not able to complete the proof of Proposition 10 this way since the following estimate is not available in our case: if u and v are solutions of our equation, then

$$\|Du - Dv\|_{L^\infty(B_{\frac{1}{2}})} \leq C\|u - v\|_{L^\infty(B_1)}.$$

Note that if we had $C^{2,\alpha}$ estimates for u, v, then the above estimate would be true since $u - v$ would satisfy a linear elliptic equation with C^α coefficients. Since in our case all these estimates are not available, we obtain a weaker than monotonicity of u statement, see inequality (67) below. Then we improve (67) to the monotonicity using $C^{1,\alpha}$ estimates for solutions of our equation.

Let $\hat{v} \in C(\partial\Omega_2)$ be such that $\hat{v} = 0$ on the Lipschitz bottom of Ω_2, $\hat{v} = 1$ on the horizontal top of Ω_2, and $\hat{v}$ is monotonic (say, linear) in x_n on the vertical part of $\partial\Omega_2$, i.e.

$$\hat{v}(x', x_n) = \begin{cases} 0 & \text{on} \quad \{\,|x'| < 2,\ x_n = f(x')\,\}, \\ 1 & \text{on} \quad \{\,|x'| < 2,\ x_n = 2L\,\}, \\ \frac{x_n - f(x')}{2L - f(x')} & \text{on} \quad \{\,|x'| = 2,\ x_n \in [f(x'), 2L]\,\}. \end{cases}$$

By Lemma 8 there exists a viscosity solution $v \in C^{1,\alpha}(\Omega_2) \cap C(\overline{\Omega_2})$ of the problem

$$\begin{cases} F(D^2v, Dv) = 0 & \text{in} \quad \Omega_2; \\ v = C\hat{v} & \text{on} \quad \partial\Omega_2, \end{cases}$$

where $C > 0$. Then, comparing v with its translations in e_n-direction we see that $D_{e_n} v \geq 0$ in Ω_2. Adjust C such that $u(\frac{L}{2}e_n) = v(\frac{L}{2}e_n)$. Then by Proposition 9 applied with $\sigma = 0$, we conclude that

$$0 < C_1 \leq \frac{u}{v} \leq C_2 \quad \text{ in } \Omega_{\frac{1}{2}}, \tag{64}$$

and by Lemma 11 (a)

$$\left|\frac{u}{v}(x) - \frac{u}{v}(\overline{x})\right| \leq C|x - \overline{x}|^\alpha \quad \text{ in } \Omega_{\frac{1}{2}}.$$

Remark 12 *For simplicity of notations, we use the same exponent α in the estimate of Lemma 11 (a) and in $C^{1,\alpha}$ estimates for u. We can do this without loss of generality by choosing smaller of two exponents.*

Let x be on the distance d from the graph of f, and $\overline{x} = x + he_n$ where $h \in (0, \frac{d}{2})$, then, denoting $K := \frac{u}{v}(\overline{x})$, we obtain

$$|u(x) - Kv(x)| \leq Cv(x)h^\alpha.$$

Recalling definition of K we get $u(x) - Kv(x) = [u(x) - u(\overline{x})] - K[v(x) - v(\overline{x})]$, and so

$$\left|[u(\overline{x}) - u(x)] - K[v(\overline{x}) - v(x)]\right| \leq \tilde{C}v(x)h^\alpha. \tag{65}$$

Now, by Lemma 11 (c), we get $D_{e_n} v \simeq \frac{v}{d}$ in $B_{\frac{d}{2}}(x)$, and so, using Harnack inequality we get

$$v(\overline{x}) - v(x) = \int_0^h D_{e_n} v(x + se_n)ds \geq C\frac{h}{d}v(x). \tag{66}$$

By (64) we obtain $K \geq C_1 > 0$. Thus we have by (65), (66)

$$u(\overline{x}) - u(x) \geq (C_1 - C_2\frac{d}{h^{1-\alpha}})[v(\overline{x}) - v(x)]. \tag{67}$$

Let $h = d^{1+\alpha}$ and let d be so small that $d^{\alpha^2} \leq \frac{C_1}{2C_2}$. Then we have

$$C_1 - C_2\frac{d}{h^{1-\alpha}} \geq \frac{1}{2}C_1,$$

and since $v(\overline{x}) - v(x) \geq 0$ it follows from (66), (67) and (64) that

$$u(\overline{x}) - u(x) \geq C\frac{h}{d}u(x) \geq C\frac{h}{d} \sup_{B_{\frac{d}{2}}(x)} u, \tag{68}$$

where we used Harnack in the last inequality. Let d be so small that $h = d^{1+\alpha} < \frac{d}{16}$. By the properly scaled $C^{1,\alpha}$ estimates for solutions of (53), we have for any $\eta \in [0, h)$

$$|Du(x + \eta e_n) - Du(x)| \leq \frac{C}{d^{1+\alpha}}\left(\sup_{B_{\frac{d}{2}}(x)} u\right)|(x + \eta e_n) - x|^\alpha \leq C\frac{h^\alpha}{d^{1+\alpha}} \sup_{B_{\frac{d}{2}}(x)} u$$

and, using (68) we get

$$|Du(x + \eta e_n) - Du(x)| \leq C\frac{h^{\alpha-1}}{d^\alpha}[u(\overline{x}) - u(x)].$$

Now we estimate:

$$u(\overline{x}) - u(x) = \int_0^h D_{e_n} u(x + \eta e_n)\, d\eta \le \int_0^h |D_{e_n} u(x + \eta e_n) - D_{e_n} u(x)|\, d\eta + h D_{e_n} u(x)$$

$$\le hC \frac{h^{\alpha-1}}{d^\alpha}[u(\overline{x}) - u(x)] + h D_{e_n} u(x),$$

thus

$$u(\overline{x}) - u(x) \le C \frac{h^\alpha}{d^\alpha}[u(\overline{x}) - u(x)] + h D_{e_n} u(x). \tag{69}$$

Let N be so large that $\frac{C}{N^\alpha} < \frac{1}{2}$, where C is from the right-hand side of (69). Let d be so small that $\frac{h}{d} = d^\alpha < \frac{1}{N}$. Then, since by (68) there holds $u(\overline{x}) - u(x) \ge 0$, we get from (69)

$$D_{e_n} u(x) \ge \frac{1}{2h}[u(\overline{x}) - u(x)] \ge 0.$$

Proposition 10 is proved. □

3 Regularity of Lipschitz free boundaries in two-phase problems

In this section we discuss the proof of Theorem 2.

In fact, since Harnack principle holds for nonnegative solutions of (1), (2), the argument of [2], Sections 4–6 can be applied, with the use of Corollary 6 (with Lemma 7) and Propositions 9 and 10 on the place of the corresponding linear results. Some new features appear only in the proof of a nonlinear version of Lemma 14 of [2] (the Harnack principle for solutions of the free boundary problem), and we give the proof below.

Lemma 13 *Let $u_1 \ne u_2$ be two weak solutions of the problem (1) - (3) in $B_1 \subset \mathbf{R}^n$, in the sense of Definition 1. Let $\Gamma_i{:=}\Gamma(u_i)$ be a free boundary of u_i, $i = 1, 2$. We assume that $0 \in \Gamma_2$, that Γ_1 lies in $\frac{1}{16}$-neighborhood of Γ_2, and that Γ_1 is Lipschitz, i.e., $\Omega^+(u_1) = \{x_n > f(x')\}$ with Lipschitz f in some coordinate system (x', x_n), with Lipschitz constant $L \ge 1$. Assume further that*

$$v_\varepsilon(x) := \sup_{B_\varepsilon(x)} u_1 \le u_2(x)$$

in $B_{1-\varepsilon}$, that

$$v_\varepsilon(0, \frac{1}{4}) \le (1 - \sigma\varepsilon) u_2(0, \frac{1}{4}),$$

where we use the coordinates (x', x_n) introduced above. Then for ε small enough, there exists a $\delta > 0$, depending only on data and σ, such that on $B_{\frac{1}{8}}$

$$v_{(1+\delta\sigma)\varepsilon} \le u_2(x).$$

Proof. First we note that Lemmas 10, 12 of [2] hold in our case. Indeed, the proof of Lemma 10 of [2] does not use the equation, and thus the lemma holds for our case. In order to obtain Lemma 12 of [2] we note that Harnack inequality and interior gradient bounds hold for solutions of (53) – now the proof given in [2] works in our case.

Note also that the following version of Hopf's boundary point lemma holds for viscosity solutions u, v of (53) in a domain Ω: if $u > v$ in Ω and $u(x_0) = v(x_0)$ for $x_0 \in \partial\Omega$, and $\partial\Omega$ satisfies an interior sphere condition at x_0, i.e. $x_0 \in \partial B_r(y)$ where $r > 0$ and $B_r(y) \subset \Omega$, then $\frac{\partial(u-v)}{\partial n}(x_0) > 0$ where $n = \frac{x-y}{|x-y|}$. This is true since by (7) the $w := v - u$ is a negative subsolution of $\mathcal{P}^-_{\Lambda,\lambda}(D^2 w) - M|Dw| \geq 0$ in Ω and $w(x_0) = 0$, and we can follow the proof of [11, Lemma 3.4].

Now we prove Lemma 13. We will construct a continuous family $\overline{v}_t$ of functions such that $\overline{v}_0 = u_1$, $\overline{v}_t \leq u_2$ for $t \in [0,1]$, and

$$\overline{v}_{1|\, B_{\frac{1}{8}}} \geq v_{(1+\delta\sigma)\varepsilon}. \tag{70}$$

This will imply Lemma 13.

Let $\rho = \frac{1}{64\sqrt{1+L^2}}$. From assumptions of the lemma

$$B_{4\rho}(0, \frac{1}{4}) \subset \Omega^+(u_1).$$

By [2, Lemma 12] we have

$$v_{(1+\mu\sigma)\varepsilon}(x) \leq u_2(x) - \gamma\sigma\varepsilon u_2(0, \frac{1}{4}) \quad \text{ on } B_\rho(0, \tfrac{1}{4}), \tag{71}$$

where $\gamma > 0$, $\mu > 0$ depend only on data. Let

$$\mathcal{D} {:=} B_{\frac{9}{10}} \setminus \overline{B}_\rho(0, \frac{1}{4}).$$

Let g_t be the family of functions constructed in Lemma 7 with the following choice of constants: ρ as chosen above, λ, Λ from (7), and $\varepsilon > 0$ so small and $\overline{C}$ so large that conditions of Proposition 4 are satisfied for F from (53) and g_t, and μ is from (71).

Fix $t \in [0,1]$. Let $\widetilde{v}_t \in C(\overline{\mathcal{D}})$ be defined by $\widetilde{v}_t(x) {:=} \sup_{B_{\varepsilon g_{\sigma t}}(x)} u_1$. Denote $\Omega^+(\widetilde{v}_t) = \{x \in \mathcal{D} \mid \widetilde{v}_t(x) > 0\}$. By Proposition 4 and Corollary 6, $\widetilde{v}_t$ is a subsolution of the equation (53) in both $\Omega^+(\widetilde{v}_t)$ and $\mathcal{D} \setminus \Omega^+(\widetilde{v}_t)$. From the assumptions of the lemma $\widetilde{v}_0 \leq u_2$ in $\mathcal{D}$. By (71)

$$\widetilde{v}_0 < u_2 \text{ on } \partial B_\rho(0, \tfrac{1}{4}). \tag{72}$$

Denote $\Omega^+(\widetilde{v}_t) = \{x \in \mathcal{D} \mid \widetilde{v}_t(x) > 0\}$.

Let w_t be a viscosity solution in the domain $\Omega^+(\widetilde{v}_t)$ of the problem

$$\begin{array}{ll} F(D^2 w_t, Dw_t) = 0 & \text{in } \Omega^+(\widetilde{v}_t); \\ w_t = \widetilde{v}_t + \frac{\gamma}{2}\sigma\varepsilon u_2(0, \frac{1}{4}) & \text{on } \partial B_\rho(0, \frac{1}{4}); \\ w_t = \widetilde{v}_t & \text{on } \partial\Omega^+(\widetilde{v}_t) \setminus \partial B_\rho(0, \frac{1}{4}). \end{array} \tag{73}$$

Existence of solution follows by Lemma 8.

We define the family $\overline{v}_t$ of continuous functions on $\mathcal{D}$ for $t \in [0,1]$ by

$$\overline{v}_t = \begin{cases} w_t & \text{on } \Omega^+(\widetilde{v}_t); \\ \widetilde{v}_t & \text{on } \mathcal{D} \setminus \Omega^+(\widetilde{v}_t). \end{cases}$$

We need to show that $\overline{v}_t \leq u_2$ for all t, and that $\overline{v}_1$ satisfies (70) – then the lemma is proved.

Since in $\overline{v}_t$ and $\widetilde{v}_t$ are a solution and a subsolution of (53) respectively in $\Omega^+(\widetilde{v}_t)$, and $\overline{v}_t \geq \widetilde{v}_t$ on $\partial\Omega^+(\widetilde{v}_t)$, it follows that $\overline{v}_t \geq \widetilde{v}_t$ in $\Omega^+(\widetilde{v}_t)$. Thus

$$\overline{v}_t \geq \widetilde{v}_t = \sup_{B_{\varepsilon g_{\sigma t}}} u_1 \quad \text{ in } \mathcal{D}. \tag{74}$$

In particular, properties of g_t (specifically, assertion (v) of Lemma 7) imply that (70) holds.

By construction $\overline{v}_t \leq u_2$ on $\partial\mathcal{D}$ and in particular $\overline{v}_t < u_2$ on $\partial B_\rho(0, \frac{1}{4})$ for all $t \in [0,1]$. Since we have $\widetilde{v}_0 \equiv u_1 \leq u_2$, it follows that $\Omega^+(\widetilde{v}_0) \subseteq \Omega^+(u_2)$ and thus by maximum principle $\overline{v}_0 < u_2$ in $\Omega^+(\widetilde{v}_t)$. Thus $\overline{v}_0 \leq u_2$ in $\mathcal{D}$.

Consider the set $\mathcal{A} \subseteq [0,1]$ of all t's for which $\overline{v}_t \leq u_2$ in $\mathcal{D}$. As we showed $\mathcal{A}$ is nonempty since it contains $t = 0$. It is easy to see that the set $\mathcal{A}$ is closed. It remains to show that $\mathcal{A}$ is open – then $\mathcal{A} = [0,1]$.

Let $t \in \mathcal{A}$ and $t \neq 1$. Then $\Omega^+(\widetilde{v}_t) \subseteq \Omega^+(u_2)$. Thus, from the above discussion, if

$$\Gamma(u_2) \cap \Gamma(\widetilde{v}_t) \cap \mathcal{D} = \emptyset, \tag{75}$$

then $\overline{v}_t < u_2$ in $\mathcal{D}$. Since $g_t \equiv 1$ outside $B_{\frac{7}{8}}$ and $u_2 - \overline{v}_t$ satisfies Harnack inequality in $B_{\frac{7}{8}} \cap \mathcal{D}$, it follows that t is an interior point of A. So is enough to show that (75) holds for all $t \in A \setminus \{1\}$.

Let, on contrary, there exists $x_0 \in \Gamma(u_2) \cap \Gamma(\widetilde{v}_t) \cap \mathcal{D}$. Since $g_t \equiv 1$ outside $B_{\frac{7}{8}}$, then $x_0 \in B_{\frac{7}{8}}$. By estimate (ii) of Lemma 7 and [2, Lemma 10] we get that $\Gamma(\widetilde{v}_t)$ has at x_0 a one-sided tangent ball with center y from $\Omega^+(\widetilde{v}_t)$ side, and

$$\widetilde{v}_t(x) \geq \alpha\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

where $\nu = \frac{x_0 - y}{|x_0 - y|}$ and

$$\frac{\alpha}{1 - \varepsilon\nabla g_{\sigma t}} \geq G\left(\frac{\beta}{1 + \varepsilon\nabla g_{\sigma t}}\right) \tag{76}$$

Since $t \in \mathcal{A}$, then $\Omega^+(\overline{v}_t) \subseteq \Omega^+(u_2)$, and thus $\Gamma(u_2)$ has the same tangent ball at x_0. As u_2 is a solution of the free boundary problem,

$$u_2(x) = \alpha_1\langle x - x_0, \nu\rangle^+ - \beta_1\langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

and

$$\alpha_1 = G(\beta_1) \tag{77}$$

By Lemma 7(ii) and [2, Lemma 10] it follows that (after possibly making the parameter ε smaller) the free boundaries $\Gamma(\overline{v}_t)$ for $t \in [0,1]$ are equiLipschitz with $\mathrm{Lip}[\Gamma(\overline{v}_t)] \leq L + C\varepsilon$, where C depends on data. Also, $\Omega^+(u_1) \subset \Omega^+(\overline{v}_t) \subset \Omega^+(u_2)$ for all $t \in \mathcal{A}$, and thus the $\Gamma(\overline{v}_t)$ lies in $\frac{1}{16}$-neighborhood of $\Gamma(u_2)$. In particular, for each t there exists $(0, x_n(t)) \in \Gamma(\overline{v}_t)$ where $x_n(t) \in [0, \frac{1}{16}]$, and thus if $\varepsilon > 0$ is small then all domains $\Omega^+(\overline{v}_t)$ contain the ball $B_{3\rho}(0, \frac{1}{4})$.

Then by Proposition 9 there exists $C > 0$ independent on $t \in \mathcal{A}$ such that for all $t \in \mathcal{A}$

$$\frac{\overline{v}_t - \widetilde{v}_t}{\widetilde{v}_t} \geq C\frac{\overline{v}_t(A) - \widetilde{v}_t(A)}{\widetilde{v}_t(A)}$$

in $B_{\frac{7}{8}} \cap \Omega^+(\overline{v}_t)$, where $A = (0, \frac{1}{4} - 2\rho)$.

Using definition of $\overline{v}_t$ and Harnack inequality for u_2, on $\partial B_\rho(0,\frac{1}{4})$ we have

$$\overline{v}_t - \widetilde{v}_t = \frac{\gamma}{2}\sigma\varepsilon u_2(0,\frac{1}{4}) \geq C\gamma\sigma\varepsilon u_2 \geq C\gamma\sigma\varepsilon\overline{v}_t$$

where $C > 0$ depends only on ρ and data. Since $\overline{v}_t - \widetilde{v}_t$ is a positive supersolution of F, and thus a supersolution of $\mathcal{P}^-_{\Lambda,\lambda}(D^2 w) - M|Dw| \leq 0$ in $\Omega^+(\overline{v}_t)$ it follows that $\overline{v}_t(A) - \widetilde{v}_t(A) \geq \frac{\gamma}{2}\sigma\varepsilon u_2(0,\frac{1}{4})W(A)$ where $W(A)$ is a solution of

$$\begin{aligned} \mathcal{P}^-_{\Lambda,\lambda}(D^2 W) - M|DW| = 0, &\quad \text{in} \quad B_{3\rho}(0,\frac{1}{4}) \setminus \overline{B}_\rho(0,\frac{1}{4}) \\ W = 1 &\quad \text{on} \quad \partial B_\rho(0,\frac{1}{4}) \\ W = 0 &\quad \text{on} \quad \partial B_{3\rho}(0,\frac{1}{4}) \end{aligned}$$

Thus $\frac{\overline{v}_t(A) - \widetilde{v}_t(A)}{\widetilde{v}_t(A)} \geq C\gamma\sigma\varepsilon\frac{u_2(0,\frac{1}{4})}{\widetilde{v}_t(A)} \geq \gamma\sigma\varepsilon\tilde{C} > 0$, where we used Harnack inequality for u_2 and the fact that $u_2 \geq \widetilde{v}_t$ in $\mathcal{D}$. Now we conclude that for all $t \in \mathcal{A}$ we have

$$\frac{\overline{v}_t - \widetilde{v}_t}{\widetilde{v}_t} \geq \gamma\sigma\varepsilon C > 0 \quad \text{in } B_{\frac{7}{8}} \cap \Omega^+(\overline{v}_t),$$

where C depends only on data.

Since (76) holds, we obtain

$$\overline{v}_t(x) \geq \overline{\alpha}\langle x - x_0, \nu\rangle^+ - \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|),$$

where $\overline{\alpha} \geq (1 + \gamma\sigma\varepsilon C)\alpha$

Now we estimate for small enough $\varepsilon > 0$ using the fact that $s^{-N}G(s)$ is decreasing and the inequality (76):

$$\begin{aligned} G(\beta) &\leq \beta^N \left(\frac{\beta}{1+\varepsilon|Dg_{\sigma t}|}\right)^{-N} G\left(\frac{\beta}{1+\varepsilon|Dg_{\sigma t}|}\right) \\ &\leq \frac{(1+\varepsilon|Dg_{\sigma t}|)^N}{1-\varepsilon|Dg_{\sigma t}|}\alpha \\ &\leq \frac{1+2N\varepsilon|Dg_{\sigma t}|}{1-\varepsilon|Dg_{\sigma t}|}\frac{\overline{\alpha}}{1+\gamma\sigma\varepsilon C}, \end{aligned}$$

where we used that $|Dg_{\sigma t}| \leq \varepsilon$ in the last inequality. Using again the estimate $|Dg_{\sigma t}| \leq \varepsilon$ we conclude that if ε is small enough depending on data and σ then

$$\overline{\alpha} \geq G(\beta). \tag{78}$$

Then since $\overline{v}_t \leq u_2$ and G is increasing, we compare (77) and (78) and conclude that $\overline{\alpha} = \alpha_1$ and $\beta = \beta_1$. Since $\Gamma(\widetilde{v}_t)$ has at x_0 a one-sided tangent ball from $\Omega^+(\widetilde{v}_t)$ side, and u_2, $\overline{v}_t$ are solutions of (53) in $\Omega^+(\overline{v}_t)$ and $u_2 > \overline{v}_t$ in $\Omega^+(\overline{v}_t)$, the equality $\overline{\alpha} = \alpha_1$ contradicts the Hopf's lemma. This shows that $\Gamma(u_2) \cap \Gamma(\widetilde{v}_t) \cap \mathcal{D} = \emptyset$ for all $t \in \mathcal{A} \setminus \{1\}$, which proves Lemma 13. □

The remaining part of the proof of Theorem 2 follows Sections 5, 6 of [2]. In particular, Lemma 13 is applied with $u_2(x) = u(x)$, $u_1(x) = u(x - \tau)$, where u is a solution of the free boundary problem, and τ is a small vector from the cone of monotonicity of u near the free boundary. Thus $\Gamma(u_1)$ and $\Gamma(u_2)$ are Lipschitz, and $\Gamma(u_1)$ lies in a small neighborhood of $\Gamma(u_2)$. So these conditions of Lemma 13 are satisfied. Note also that the structure conditions (7) and (10) are preserved under the blowup in the following sense: if u is a viscosity solution of (53) then for any $r \in (0,1)$ the function u_r defined by $u_r(x) = \frac{1}{r}u(rx)$ is a solution of $F_r(D^2u, Du) = 0$ where F_r is defined by $F_r(R,p) = F(R, rp)$, and clearly F_r satisfies (7) with same as F constants, and (10). Now the rest of the argument follows [2].

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