A GEOMETRIC LEVEL-SET FORMULATION OF A PLASMA-SHEATH INTERFACE

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Abstract. In this paper, we present a new geometric level-set formulation of a plasma-sheath interface arising in the plasma physics. We formally derive the explicit dynamics of the interface from the Euler-Poisson equations and study the local-time evolution of the interface and sheath in some special cases.

Contents

1. Introduction

The purpose of this paper is to continue the study begun in [37] of the dynamics of a plasma-sheath interface (in short "sheath interface") arising from the plasma sheath problem [2, 15, 35, 38, 40, 48, 56]. The issues can be easily understood by the examination of the Euler-Poisson system **(E-P)**. Consider a plasma consisting of cold ions and hot electrons confined to a domain $\Omega = \mathbb{R}^3 - \Omega_0$ which is exterior to a target $\Omega_0 \subset \mathbb{R}^3$. Both ions and electrons have constant temperature, the temperature of the ions being absolute zero Kelvin. The density of ions is denoted by n, the density of electrons is $e^{-\phi}$ (Boltzmann relation [40]), $-\phi$ is the potential field and $\mathbf{u}$ is the velocity of the ions. In this case, **(E-P)** reads

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, \infty), \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \phi, \\ \varepsilon^2 \Delta \phi = n - e^{-\phi}, \end{cases} \tag{1.1}$$

subject to initial and boundary conditions:

$$\begin{aligned} &(n, \mathbf{u}, \phi)(\mathbf{x}, 0) = (n_0, \mathbf{u}_0, \phi_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ &\nabla \phi \cdot \boldsymbol{\nu}_0 = \frac{g(\mathbf{x}, t)}{\varepsilon}, \quad (\mathbf{x}, t) \in \partial\Omega_0 \times [0, \infty). \end{aligned} \tag{1.2}$$

Here ε is proportional to the Debye length λ_D [40] and $\boldsymbol{\nu}_0$ is the exterior normal at the target boundary $\partial\Omega_0$. Typically away from the boundary $\partial\Omega_0$, the formal $\varepsilon \to 0$ limit in **(E-P)** can be used to yield the quasi-neutral relation $n = e^{-\phi}$. However near the boundary $\partial\Omega_0$, this quasi-neutrality breaks down (see Section 2) and a *sheath* boundary layer of width ε forms.

In [37], Ha and Slemrod gave a description of sheath dynamics for the case of planar, cylindrically and spherically symmetric motion, generalizing earlier work of Daube and Riemann [48]. In this paper, we make no restriction as to symmetry and formulate the dynamics of the sheath interface in terms of a geometric level-set, where the dynamics of the sheath interface is based on a step-sheath model. In the step sheath model, the spatial-time domain is separated by a propagating sheath interface into distinct quasi-neutral and sheath regions. Particularly interesting in our approach is a set of equations describing the evolution of the sheath interface as a curvature driven flow. Specifically, we show that the sheath interface evolution is governed by the equations:

$$\boxed{\frac{\delta \psi}{\delta t} = 0, \quad \frac{\delta n}{\delta t} = n \nabla \cdot \boldsymbol{\nu}, \quad (V + 1) + \frac{\mathbf{h} \cdot \boldsymbol{\nu}}{n} = -\frac{1}{n} \nabla_s \cdot (V \nabla_s \ln n),}$$

where

(i) the level set $\mathcal{S}(t) = \{(\mathbf{x}, t) : \psi(\mathbf{x}, t) = 0\}$ is the sheath interface;
(ii) $\frac{\delta}{\delta t} = \partial_t + V \boldsymbol{\nu} \cdot \nabla$ is the normal time derivative following $\mathcal{S}(t)$ and ∇_s is the surface gradient on $\mathcal{S}(t)$;
(iii) $\boldsymbol{\nu}$ is the exterior unit normal to $\mathcal{S}(t)$. Since $\nabla \cdot \boldsymbol{\nu}$ is twice the mean curvature of $\mathcal{S}(t)$, motion is curvature driven;
(iv) $\mathbf{h}$ is the ion current and n is the ion density on the sheath interface.

Usefulness of such models is seen in studying material processing [40] and in particular the plasma source ion implantation (PSII) technique invented by Conrad and his collaborators [16]. Other applications may be found in the related problems for the modelling of electron beam where again loss of quasi-neutrality is a crucial issue (see [7, 8, 9, 10, 21, 22, 23, 24]).

We note that in this paper we have taken the normal component of the electric field to be prescribed on the boundary $\partial\Omega_0$. This boundary condition was used by Cipolla and Silevitch [15] in their study of plasma-sheath evolution and considerably simplifies the proof of the existence and uniqueness theorems presented in Section 7. On the other hand the derivation of the evolution equations for the sheath interface is independent of the boundary conditions. (An existence and uniqueness theorem for the boundary conditions used in [37, 48] $\mathbf{u} = \mathbf{u}_w, \phi = \phi_w$ on $\partial\Omega_0$ remains an open problem).

The rest of this paper is organized as follows.

In Section 2, we present a formulation of the plasma-sheath problem for general targets under suitable physical assumptions. We decompose the domain Ω into two sub-domains (a quasi-neutral region and a sheath region) and a common boundary (the sheath interface). On each sub-domain, we simplify the Euler-Poisson system according to suitable physical relations (the equations of isothermal gas dynamics in the quasi-neutral region and the zero electron density limit in the sheath region). In Section 3, we review the definitions of normal-time derivative and surface gradient for the fields defined on a sheath interface. In Section 4, we derive the explicit kinematics of the plasma-sheath interface from the Euler-Poisson system. In Section 5, we simplify the sheath-interface dynamics by considering orthogonal flow (the tangent component of ion velocity is zero) on the sheath interface. In Section 6, we summarize the systems governing dynamics of sheath, sheath-interface and quasi-neutral plasma and in Section 7 we study the local dynamics of sheath and sheath interface when the sheath interface is represented by the graph of a smooth function in $\mathbb{R}^2$. The proof of local existence for the full initial-boundary value problem describing sheath, quasi-neutral and interface regions is modelled on free boundary studies of Chen and Feldman [13] and Canic, Keyfitz and Lieberman [12]. We also borrow many ideas from the fundamental paper of Nouri [43] where our sheath system was considered in the absence of boundary conditions and the interface region. In Section 8, we present a new "bulk interface" level-set formulation of the initial-boundary value problem. Appendix A provides a detailed proof of the local existence theorem of Section 7 while Appendices B and C provide other technical lemmas.

2. Level-set formulation of the plasma-sheath interface

In this section, we present a level-set formulation of the sheath interface for general three-dimensional targets. This formulation was partly employed in [37] in the case of planar, cylindrical and spherical targets.

First we give a rather elementary description of the plasma sheath. Since the Debye length ε is a small parameter in (1.1), the Poisson equation suggests that the quasi-neutral relation $n = e^{-\phi}$ should pervade in our problem. Substitution of this relation into (1.1) yields the quasi-neutral system:

$$\begin{cases} \partial_t n + \nabla\cdot(n\mathbf{u}) = 0, \quad (\mathbf{x},t)\in\Omega\times[0,\infty), \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(\ln n) = \mathbf{0}, \end{cases} \tag{2.3}$$

with prescribed initial data for n and $\mathbf{u}$ at $t = 0$ and boundary data $\nabla \ln n \cdot \boldsymbol{\nu}_0 = -g$ on $\partial\Omega_0$. In general, the initial-boundary value problem for (2.3) is not well-posed. For example, consider the symmetric cases of planar, cylindrical and spherical targets. In these cases, the Euler-Poisson system (1.1) becomes

$$\begin{cases} \partial_t\rho + \partial_r(\rho u) = 0, \quad r_0 \le r < \infty, \quad t > 0, \\ \partial_t u + \partial_r\left(\frac{u^2}{2}\right) = \partial_r\phi, \\ \varepsilon^2\partial_r(r^\nu\partial_r\phi) = \rho - \rho_e, \quad \rho = nr^\nu, \quad \rho_e = e^{-\phi}r^\nu, \end{cases} \tag{2.4}$$

where $\nu = 0, 1, 2$ correspond to the planar, cylindrical and spherical cases respectively.

With this one-dimensional symmetry, (2.3) possesses two distinct characteristic curves:

$$\frac{d\chi_1}{dt} = u - 1, \qquad \frac{d\chi_2}{dt} = u + 1,$$

which carry the prescribed data into the domain $(r_0, \infty) \times \mathbb{R}_+$. Notice that when u decreases below the critical value $u = -1$, both characteristics χ_1 and χ_2 run into the boundary $r = r_0$, thus making the initial-boundary value problem for (2.3) overdetermined and hence unsolvable in the class $C^1((r_0,\infty)\times(0,T)) \cap C^0([r_0,\infty)\times[0,T))$, for some positive constant T. Hence near the "Bohm velocity" $u = -1$, quasi-neutrality breaks down and a sheath boundary layer forms. Since the Poisson equation reads

$$\varepsilon^2 \partial_r (r^\nu \partial_r \phi) = r^\nu (n - e^{-\phi}),$$

the quasi-neutrality relation is violated when the left hand side becomes non-negligible. This has been quantified by Franklin and Ockendon for steady problems [30], where a matched asymptotic expansion yields $\partial_r \phi \approx \varepsilon^{-\beta}, \quad 0 < \beta < 1$, so that the electric potential develops a large gradient near the sheath edge (see also [47]). Since ϕ has rapidly increased as the ions entered the "*sheath*" boundary layer, we formally set the electron density $\rho_e = 0$ of (2.4) in the boundary layer to define the "*step sheath*" model which we now describe in more detail.

Specifically we return to the Euler-Poisson system. Since the sheath width is order of ε, we use fast variables $(\bar{\mathbf{x}}, \bar{t})$:

$$\bar{\mathbf{x}} = \frac{\mathbf{x}}{\varepsilon}, \qquad \bar{t} = \frac{t}{\varepsilon},$$

to get a rescaled system:

$$\begin{cases} \partial_{\bar{t}} n + \nabla_{\bar{\mathbf{x}}} \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega \times (0,\infty), \\ \partial_{\bar{t}} \mathbf{u} + (\mathbf{u} \cdot \nabla_{\bar{\mathbf{x}}})\mathbf{u} = \nabla_{\bar{\mathbf{x}}} \phi, \\ \Delta_{\bar{\mathbf{x}}} \phi = n - e^{-\phi}, \end{cases} \tag{2.5}$$

and rescaled initial and boundary data

$$\begin{cases} (n, \mathbf{u}, \phi)(\bar{\mathbf{x}}, 0) = (n_0, \mathbf{u}_0, \phi_0)(\bar{\mathbf{x}}), \quad \bar{\mathbf{x}} \in \Omega, \\ \nabla_{\bar{\mathbf{x}}} \phi \cdot \boldsymbol{\nu}_0 = g(\bar{\mathbf{x}}, t), \quad (\bar{\mathbf{x}}, \bar{t}) \in \partial\Omega_0 \times [0,\infty). \end{cases}$$

where now the gradient $\nabla_{\bar{x}}$ and the Laplacian $\Delta_{\bar{x}}$ are taken in terms of rescaled variables $\bar{\mathbf{x}}$.

As mentioned above in the sheath region, we formally set the electron density to be zero to get the rescaled sheath system **(S)**:

$$\begin{cases} \partial_{\bar{t}} n + \nabla_{\bar{\mathbf{x}}} \cdot (n\mathbf{u}) = 0, \quad (\bar{\mathbf{x}}, t) \in \Omega \times (0,\infty), \\ \partial_{\bar{t}} \mathbf{u} + (\mathbf{u} \cdot \nabla_{\bar{\mathbf{x}}})\mathbf{u} = \nabla_{\bar{\mathbf{x}}} \phi, \\ \Delta_{\bar{\mathbf{x}}} \phi = n. \end{cases} \tag{2.6}$$

In contrast, in the quasi-neutral region, we use the rescaled quasi-neutral system **(Q)**:

$$\begin{cases} \partial_{\bar{t}} n + \nabla_{\bar{\mathbf{x}}} \cdot (n\mathbf{u}) = 0, \quad (\bar{\mathbf{x}}, t) \in \Omega \times (0,\infty), \\ \partial_{\bar{t}} \mathbf{u} + (\mathbf{u} \cdot \nabla_{\bar{\mathbf{x}}})\mathbf{u} + \nabla_{\bar{\mathbf{x}}}(\ln n) = \mathbf{0}. \end{cases} \tag{2.7}$$

Of course it is readily noted that (2.7) is just the system of compressible inviscid isothermal gas dynamics.

The boundary surface $\partial\Omega_0$ is described by an implicit relation:

$$b(\mathbf{x},\varepsilon)=0, \quad \text{for a smooth function } b:\mathbb{R}^3\times\mathbb{R}_+\to\mathbb{R}.$$

Furthermore we assume b satisfies the scaling relation:

$$\text{If} \quad \bar{\mathbf{x}}=\frac{\mathbf{x}}{\varepsilon}, \text{ then } b(\varepsilon\bar{\mathbf{x}},\varepsilon)=c(\varepsilon)\bar{b}(\bar{\mathbf{x}}), \text{ for some smooth functions } c, \text{ and } \bar{b}.$$

For these rescaled independent variables $\bar{\mathbf{x}}$, the boundary surface $\partial\Omega_0$ of the target Ω_0 can be represented as

$$\bar{b}(\bar{\mathbf{x}})=0. \tag{2.8}$$

Example 1. (Perturbation of a planar surface $x_1=0$). Consider a high frequency small perturbation of our planar surface such that

$$x_1=\varepsilon b_1\Big(\frac{x_2}{\varepsilon},\frac{x_3}{\varepsilon}\Big), \text{ where } b_1:\mathbb{R}^2\to\mathbb{R};$$
$$\bar{x}_1-b_1(\bar{x}_2,\bar{x}_3)=0, \text{ and } \bar{b}(\bar{x}_1,\bar{x}_2,\bar{x}_3)=\bar{x}_1-b_1(\bar{x}_2,\bar{x}_3).$$

Example 2. (Perturbation of a circle in the $\mathbb{R}^2$: $r=\varepsilon r_0$). Consider a perturbation of a circle such that

$$r=\varepsilon r_0 b_2(\theta) \quad \text{where } b_2:\mathbb{R}\to\mathbb{R}: \text{ smooth and } 2\pi\text{-periodic;}$$

Set

$$\bar{x}_i=\frac{x_i}{\varepsilon}, i=1,2, \text{ and } \bar{r}=\frac{r}{\varepsilon}, \bar{r}=r_0b_2(\theta).$$

and

$$\bar{x}_1^2+\bar{x}_2^2-r_0^2b^2\tan^{-1}\left(\frac{\bar{x}_2}{\bar{x}_1}\right)=0,$$

describes the curve in the x_1-x_2 plane.

Example 3. (Perturbation of a sphere $r=\varepsilon r_0$). Consider the sphere in $\mathbb{R}^3$ described by $r=\varepsilon r_0$ and let θ,ϕ denote the standard polar and azimuthal angles such that

$$x_1=r\sin\phi\cos\theta, \qquad x_2=r\sin\phi\sin\theta, \qquad x_3=r\cos\phi.$$

As a perturbation of our sphere, consider

$$r=\varepsilon r_0 b_3(\theta,\phi)$$

where $b_3:\mathbb{R}^3\to\mathbb{R}$ is smooth and 2π periodic in θ,ϕ. Then again setting

$$\bar{x}_i=\frac{x}{\varepsilon}, \qquad \bar{r}=\frac{r}{\varepsilon},$$

we see

$$\bar{r}=r_0b_3(\theta,\phi),$$

and

$$\bar{x}_1^2+\bar{x}_2^2+\bar{x}_3^2-r_0^2b_3^2\Big(\tan^{-1}\left(\frac{\bar{x}_2}{\bar{x}_1}\right),\cos^{-1}\left(\frac{\bar{x}_3}{\sqrt{\sum_{i=1}^3\bar{x}_i^2}}\right)\Big)=0,$$

describes the surface.

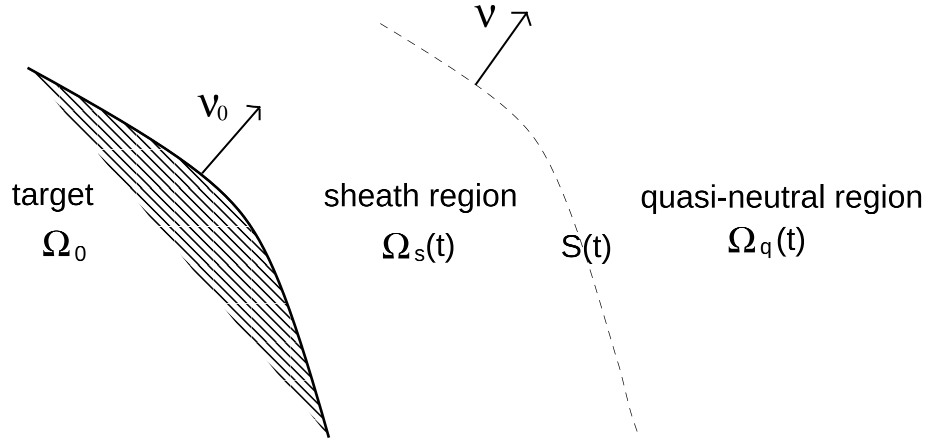

FIGURE 1. Schematic diagram of a physical domain at time t

In this paper, for fixed time t, we decompose the domain Ω into the sheath region, the quasi-neutral region and their interface, i.e.,

$$\Omega = \Omega_s(t) \cup \mathcal{S}(t) \cup \Omega_q(t), \quad \Omega_s(t): \text{ the sheath region,}$$
$$\Omega_q(t): \text{ the quasi-neutral region,} \quad \mathcal{S}(t): \text{ plasma-sheath interface.}$$

Now we return to the issue of the sheath edge. As noted above for the steady motion with planar, cylindrical and spherical symmetry, a matched asymptotic expansion [30] yields $\partial_r\phi = \varepsilon^{-\beta},\ 0 < \beta < 1$. Hence in the formal quasi-neutral limit ($\varepsilon \to 0+$), we obtain the sheath edge relation

$$\partial_{\bar{r}}\phi = \varepsilon\partial_r\phi \approx \varepsilon^{1-\beta} \to 0, \qquad \text{as } \varepsilon \to 0+,$$

so that the normal component of $\nabla_{\mathbf{x}}\phi$ on the interface becomes zero, i.e.,

$$\nabla\phi \cdot \boldsymbol{\nu} = 0 \quad \text{on } \mathcal{S}(t). \tag{2.9}$$

We will incorporate this relation in defining the sheath interface below.

First drop the over bars in (2.6) and (2.7) for notational simplicity and set n_e to be the electron density so that our governing equations become

$$\begin{aligned}
&\partial_t n + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega_s(t) \times (0, \infty),\\
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \nabla\phi,\\
&\Delta_{\mathbf{x}}\phi = n,\ n_e = 0.
\end{aligned} \tag{2.10}$$

in the sheath region and

$$\begin{aligned}
&\partial_t n + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega_q \times (0, \infty),\\
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}}(\ln n) = \mathbf{0}\\
&n = n_e = e^{-\phi} \text{ (quasi-neutrality and the Boltzmann relation)}\ ,
\end{aligned} \tag{2.11}$$

in the quasi-neutral region.

Note that with overbars deleted the target boundary surface (2.8) can be rewritten as

$$b(\mathbf{x}) = 0.$$

We combine (2.9) with the "Bohm-relation" $\mathbf{u}\cdot\boldsymbol{\nu} = -1$ (where $\boldsymbol{\nu}$ is the unit exterior normal to the plasma-sheath interface) to give a definition of the sheath interface for (2.5). Specifically the definition of the plasma-sheath interface $\mathcal{S}(t)$ is defined by the level set of the normal component of the electric field and ion-velocity fields:

Definition 2.1. *A plasma-sheath interface $\mathcal{S}(t)$ separating a quasi-neutral region and an ion-sheath region is the level set of the normal component of the ion velocity and electric fields,*

$$\mathcal{S}(t) \equiv \{\mathbf{x}\in\mathbb{R}^3 : (\mathbf{u}\cdot\boldsymbol{\nu})(\mathbf{x},t) = -1, \quad (\nabla\phi\cdot\boldsymbol{\nu})(\mathbf{x},t) = 0\}, \qquad t\geq 0,$$

where $\boldsymbol{\nu}$ is the exterior normal to the interface.

Notice our definition is motivated by the observation that in the symmetric case [37] only the normal component of fluid velocity $\mathbf{u}$ and electric field $\nabla\phi$ affect the sheath location.

We list main assumptions (M) employed in this paper.

- (M1) The sheath interface is non-characteristic for the exterior quasi-neutral system (2.11) and whose (signed) normal speed satisfies
$$V \neq 0, -1.$$
- (M2) Continuity relation: $n, \mathbf{u}, \phi, \nabla n, \nabla\mathbf{u}, \nabla\phi$ and h are continuous across the sheath interface.
- (M3) Continuity of surface Laplacian of ϕ, i.e., $\Delta_s\phi$ across the interface.
- (M4) The current density $\mathbf{h}$ decays to zero at ∞, i.e., for each $t\geq 0$,
$$\lim_{|\mathbf{x}|\to\infty}\mathbf{h}(\mathbf{x},t) = \mathbf{0}.$$
- (M5) The target boundary is C^2-regular, i.e., the boundary can be represented by the graph of a C^2-function locally.

3. Preliminaries

In this section, we review the concept of a normal-time derivative and some basic lemmas which will be used in Section 4. In the sequel, we use the Einstein summation convention for repeated indices and assume that the sheath interface $\mathcal{S}(t)$ is represented by the zero-level set of a scalar valued function ψ, i.e.,

$$\mathcal{S}(t) = \{\mathbf{x}\in\mathbb{R}^3 \quad | \quad \psi(\mathbf{x},t) = 0\} \quad \text{for} \quad t\geq 0.$$

We let $\{\mathcal{S}(t)\}$ be a C^2-*regular* sheath interface in $\mathbb{R}^3$ such that tangent planes, normal lines and mean curvature are well defined. Since $\mathcal{S}(t)$ is regular, we have

$$|\nabla_{\mathbf{x}}\psi(\mathbf{x},t)| \neq 0.$$

Let $\boldsymbol{\nu}(\mathbf{x},t) = (\nu_1,\nu_2,\nu_3)(\mathbf{x},t)$ be the exterior unit normal vector of the sheath region at the interface $\mathcal{S}(t)$. Then it follows from e.g. [33, 36] that we have

$$\nu_i = \frac{\partial_{x_i}\psi}{|\nabla\psi|}, \quad i = 1,2,3 \quad \text{and} \quad \nabla\cdot\boldsymbol{\nu} = 2\kappa_m,$$

where κ_m is the mean curvature. Throughout this paper, we follow the terminology of Gurtin [36], and we use "*bulk*" field to denote the fields (e.g., scalar, vector and tensor fields) defined on $\mathbb{R}^3 - \Omega_0$ and in contrast, we use "*superficial*" field to represent fields only defined on the interface $\mathcal{S}(t)$.

For given $t \geq 0$, let $\boldsymbol{\nu}(\mathcal{S}(t))$ be the normal bundle of $\mathcal{S}(t)$ which is a subbundle of a tangent bundle of $\mathbb{R}^3$ restricted on $\mathcal{S}(t)$, i.e., $T(\mathbb{R}^3)|_{\mathcal{S}(t)}$, moreover, we have the following orthogonal decomposition of $T(\mathbb{R}^3) = \mathbb{R}^3$:

$$T(\mathbb{R}^3) = T(\mathcal{S}(t)) \oplus \boldsymbol{\nu}(\mathcal{S}(t)).$$

The fiber of any bundle at a point $p \in \mathcal{S}(t)$ will be denoted using a subscript p, for example,

$$\text{if} \quad p \in \mathcal{S}(t), \qquad T_p(\mathbb{R}^3) = T_p(\mathcal{S}(t)) \oplus \boldsymbol{\nu}_p(\mathcal{S}(t)).$$

We define the projection operator $\mathbb{P}$ onto the tangent bundle of $\mathcal{S}(t)$:

$$\mathbb{P} : T(\mathbb{R}^3) \to T(\mathcal{S}(t)); \qquad \mathbb{P}(\boldsymbol{\omega}) \equiv \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nu})\boldsymbol{\nu}.$$

For simplicity of presentation, we use the following notation: $\boldsymbol{\omega} \in T(\mathbb{R}^3)$,

$$\boldsymbol{\omega}^{\top} = \mathbb{P}(\boldsymbol{\omega}), \quad \boldsymbol{\omega}^{\perp} = (\boldsymbol{\omega} \cdot \boldsymbol{\nu})\boldsymbol{\nu}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}^{\perp} + \boldsymbol{\omega}^{\top}.$$

Definition 3.1. [36] *The surface gradients $\nabla_s f$ and $\nabla_s \mathbf{F}$ of f and $\mathbf{F}$ are defined as projection of bulk gradients, i.e.,*

$$\nabla_s f = \mathbb{P}\nabla f \quad \text{and} \quad \nabla_s \mathbf{F} = \nabla_s \mathbf{F}\mathbb{P}.$$

Suppose the interface moves in the direction $-\boldsymbol{\nu}$. Consider two surfaces $\mathcal{S}(t)$ and $\mathcal{S}(t)(t+\Delta t)$ and choose the unit interior normal $-\boldsymbol{\nu}$ to $\mathcal{S}(t)$ at a position p_0 where the normal intersects the surface $\mathcal{S}(t)(t+\Delta t)$ at the point p_1.

Definition 3.2. [36] *If the velocity field $\mathbf{v}$ of the interface $\mathcal{S}(t)$ is a normal field $\mathbf{v} = V\boldsymbol{\nu}$ and f is a superficial scalar field on the $\{\mathcal{S}(t)\}$, the normal-time derivative of f is defined as:*

$$\frac{\delta f(p_0,t)}{\delta t} \equiv \lim_{\Delta t \to 0} \frac{f(p_1, t+\Delta t) - f(p_0,t)}{\Delta t}.$$

Remark 3.1. *If f is a C^1-bulk field, then the normal-time derivative of f can be defined as*

$$\frac{\delta f}{\delta t} = \partial_t f + V\boldsymbol{\nu} \cdot \nabla f,$$

moreover, the normal-time derivative for the C^1-bulk vector field $\mathbf{F}$ can be defined similarly, i.e.,

$$\frac{\delta \mathbf{F}}{\delta t} = \partial_t \mathbf{F} + \nabla \mathbf{F}(V\boldsymbol{\nu}).$$

Here the sign of (scalar) normal velocity V is positive or negative depending on expansion or contraction of $\mathcal{S}(t)$ respectively.

Recall that $\boldsymbol{\nu}$ is the outward normal at the interface of the sheath region (see Figure 1).

Lemma 3.1. [36] *The normal-time derivative of an exterior normal $\boldsymbol{\nu}$ is the negation of surface gradient of the scalar normal velocity V of the interface, i.e.,*

$$\frac{\delta \boldsymbol{\nu}}{\delta t} = -\nabla_s V.$$

Proof. We consider the following five identities:

$$V = -\frac{\partial_t\psi}{|\nabla\psi|}, \quad \partial_t|\nabla\psi| = (\nabla\partial_t\psi)\cdot\boldsymbol{\nu}, \quad \nabla|\nabla\psi| = (\nabla\otimes\nabla\psi)\boldsymbol{\nu},$$
$$|\nabla\psi|\nabla\boldsymbol{\nu} = \nabla\otimes\nabla\psi - \boldsymbol{\nu}(\nabla\otimes\nabla\psi)\boldsymbol{\nu}, \quad |\nabla\psi|\partial_t\boldsymbol{\nu} = \mathbb{P}(\nabla\partial_t\psi).$$

The first three identities can be obtained easily, so we only present the calculation for the last two identities. First we observe

$$\begin{aligned} |\nabla\psi|\nabla\boldsymbol{\nu} &= |\nabla\psi|\Big(\frac{(\nabla\otimes\nabla\psi)|\nabla\psi| - \nabla\psi(\nabla\otimes\nabla\psi)\boldsymbol{\nu}}{|\nabla\psi|^2}\Big) \\ &= \nabla\otimes\nabla\psi - \boldsymbol{\nu}(\nabla\otimes\nabla\psi)\boldsymbol{\nu}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |\nabla\psi|\partial_t\boldsymbol{\nu} &= |\nabla\psi|\partial_t\Big(\frac{\nabla\psi}{|\nabla\psi|}\Big) = |\nabla\psi|\Big(\frac{|\nabla\psi|(\nabla\partial_t\psi) - (\nabla\partial_t\psi\cdot\boldsymbol{\nu})\nabla\psi}{|\nabla\psi|^2}\Big) \\ &= \nabla\partial_t\psi - (\nabla\partial_t\psi\cdot\boldsymbol{\nu})\boldsymbol{\nu} = \mathbb{P}(\nabla\partial_t\psi). \end{aligned}$$

Based on the above identities, we note that

$$\begin{aligned} \frac{\nabla\partial_t\psi}{|\nabla\psi|} &= \nabla\Big(\frac{\partial_t\psi}{|\nabla\psi|}\Big) - \partial_t\psi\nabla\Big(\frac{1}{|\nabla\psi|}\Big) = \nabla\Big(\frac{\partial_t\psi}{|\nabla\psi|}\Big) + \frac{\partial_t\psi}{|\nabla\psi|^2}\nabla|\nabla\psi| \\ &= -\nabla V - \frac{V}{|\nabla\psi|}\nabla|\nabla\psi|. \end{aligned} \tag{3.12}$$

By definition of the normal-time derivative for the vector field, we have

$$\begin{aligned} \frac{\delta\boldsymbol{\nu}}{\delta t} &= \partial_t\boldsymbol{\nu} + \nabla\boldsymbol{\nu}(V\boldsymbol{\nu}) = \mathbb{P}\Big(\frac{\nabla\partial_t\psi}{|\nabla\psi|}\Big) + \nabla\boldsymbol{\nu}(V\boldsymbol{\nu}) \\ &= -\nabla V - \mathbb{P}\Big(\frac{V}{|\nabla\psi|}\nabla|\nabla\psi|\Big) + \nabla\boldsymbol{\nu}(V\boldsymbol{\nu}). \end{aligned}$$

Here we used the identity (3.12). Now we claim:

$$\mathcal{I} =: -\mathbb{P}\Big(\frac{V}{|\nabla\psi|}\nabla|\nabla\psi|\Big) + \nabla\boldsymbol{\nu}(V\boldsymbol{\nu}) = 0.$$

Proof of the claim:

$$\begin{aligned} \mathcal{I} &= -\Big[\frac{V}{|\nabla\psi|}\nabla|\nabla\psi| - \Big(\frac{V}{|\nabla\psi|}\nabla|\nabla\psi|\cdot\boldsymbol{\nu}\Big)\boldsymbol{\nu}\Big] + \frac{V}{|\nabla\psi|}\nabla\boldsymbol{\nu}(|\nabla\psi|\boldsymbol{\nu}) \\ &= -\frac{V}{|\nabla\psi|}\Big[\nabla|\nabla\psi| - \Big(\nabla|\nabla\psi|\cdot\boldsymbol{\nu}\Big)\boldsymbol{\nu} - |\nabla\psi|(\nabla\boldsymbol{\nu})\boldsymbol{\nu}\Big] \\ &= -\frac{V}{|\nabla\psi|}\Big[(\nabla\otimes\nabla\psi)\boldsymbol{\nu} - \Big((\nabla\otimes\nabla\psi)(\boldsymbol{\nu},\boldsymbol{\nu})\Big)\boldsymbol{\nu} - \Big(\nabla\otimes\nabla\psi - (\nabla\otimes\nabla\psi)(\boldsymbol{\nu},\boldsymbol{\nu})\Big)\boldsymbol{\nu}\Big] \\ &= 0. \end{aligned}$$

□

Lemma 3.2. *Let υ be any bulk scalar field such that $\nabla\upsilon\cdot\boldsymbol{\nu} = 0$ on the sheath interface $\mathcal{S}(t)$. Then we have*

$$\Delta\upsilon = \Delta_s\upsilon + (\nabla\otimes\nabla\upsilon)(\boldsymbol{\nu},\boldsymbol{\nu}) \quad \text{on} \;\; \mathcal{S}(t),$$

where $(\nabla\otimes\nabla\upsilon)(\boldsymbol{\nu},\boldsymbol{\nu}) = \boldsymbol{\nu}(\nabla\otimes\nabla\upsilon)\boldsymbol{\nu}$.

Proof. Let υ denote a bulk scalar field. Recall that

$$\nabla_s \upsilon = \mathbb{P}\nabla \upsilon = \nabla \upsilon - (\nabla \upsilon \cdot \boldsymbol{\nu})\boldsymbol{\nu},$$

which then yields

$$(\nabla_s \upsilon)_i = \partial_{x_i}\upsilon - (\nu_j \partial_{x_j}\upsilon)\nu_i, \text{ and } \nabla_s(\nabla_s \upsilon) = \partial_{x_i}(\nabla \upsilon)_i - (\nu_l \partial_{x_l}(\nabla \upsilon)_i)\nu_i.$$

By definition of the surface Laplacian $\Delta_s \upsilon$, we have

$$\begin{aligned} \Delta_s \upsilon &= \nabla_s \cdot (\nabla_s \upsilon) \\ &= \partial_{x_i}\Big(\partial_{x_i}\upsilon - (\nu_j \partial_{x_j})\nu_i\Big) - \nu_l \partial_{x_l}\Big(\partial_{x_i}\phi - (\nu_j \partial_{x_j}\phi)\nu_i\Big)\nu_i \\ &= \partial^2_{x_i}\upsilon - \nu_i(\partial_{x_i}\nu_j)(\partial_{x_j}\upsilon) - \nu_i\nu_j(\partial_{x_i}\partial_{x_j}\upsilon) - \nu_j(\partial_{x_j}\upsilon)(\partial_{x_i}\nu_i) \\ &- \nu_l\nu_i(\partial_{x_i}\partial_{x_l}\upsilon) + \nu_l\nu_i^2(\partial_{x_l}\nu_j)(\partial_{x_j}\upsilon) + \nu_l\nu_j\nu_i^2(\partial_{x_j}\partial_{x_l}\upsilon) + \nu_l\nu_j\nu_i(\partial_{x_j}\upsilon)(\partial_{x_l}\nu_l) \\ &= \partial^2_{x_i}\upsilon - \nu_i\nu_l\partial_{x_i}\partial_{x_l}\upsilon \\ &= \Delta\upsilon - (\nabla \otimes \nabla \upsilon)(\boldsymbol{\nu}, \boldsymbol{\nu}). \end{aligned}$$

Here we used

$$\nu_j \partial_{x_j}\upsilon = 0, \quad \nu_i\nu_i = 1.$$

□

Remark 3.2. *We have*

$$\begin{aligned} \Delta\phi &= \Delta_s\phi + (\nabla \otimes \nabla\phi)(\boldsymbol{\nu}, \boldsymbol{\nu}), \\ \Delta n &= \Delta_s n + (\nabla \otimes \nabla n)(\boldsymbol{\nu}, \boldsymbol{\nu}) \quad \text{on } \mathcal{S}(t). \end{aligned}$$

Lemma 3.3. *A surface gradient of a scalar superficial quantity lies in the tangent plane of the surface, i.e., for any scalar superficial field w,*

$$\nabla_s w \cdot \boldsymbol{\nu} = 0.$$

Proof. Let w be an arbitrary superficial field. By definition of surface gradient,

$$\nabla_s w = \nabla w - [(\boldsymbol{\nu} \cdot \nabla)w]\boldsymbol{\nu},$$

and hence

$$\nabla_s w \cdot \boldsymbol{\nu} = \nabla w \cdot \boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \nabla)w = 0.$$

□

Remark 3.3. *Since $\boldsymbol{\nu} = \dfrac{\nabla\psi}{|\nabla\psi|}$, it can be regarded as a bulk quantity and hence via Remark 3.2, $\Delta_s\phi, \Delta_s n$ can be regarded as bulk quantities as well.*

4. Derivation of kinematics of the sheath interface

In this section, we derive the explicit dynamics of the plasma-sheath interface $\mathcal{S}(t)$ which is implicit in the Euler-Poisson equation and defining equations (see Definition 2.1). Crucial to our computations is the following assumption.

> **Assumption:** Bulk quantities $\mathbf{u}, n$ and ϕ are C^1 functions in space for each fixed time t in both the quasi-neutral and sheath regions, and hence these functions and their spatial gradients on the sheath interface can be computed as limits from the sheath or quasi-neutral regions. Furthermore

second derivatives of $\phi, n, \mathbf{u}$ in directions *tangential* to $\mathcal{S}(t)$ exist and are continuous across $\mathcal{S}(t)$, e.g.,

$$\Delta_s \phi = -\Delta_s \ln n \quad \text{on } \mathcal{S}(t), \tag{4.13}$$

since $\Delta_s \phi$ can be determined as a limit from the quasi-neutral region (see Remark 3.3).

Remark 4.1. *Notice since $n_e = 0$ in the sheath region and $n_e = e^{-\phi}$ in the quasi-neutral region, the electron density n_e suffers a jump discontinuity across $\mathcal{S}(t)$.*

As noted earlier we have assumed the outward normal from the sheath region to the quasi-neutral region on $\mathcal{S}(t)$:

$$\boldsymbol{\nu} = \frac{\nabla \psi}{|\nabla \psi|} \quad \text{for a bulk quantity} \;\; \psi,$$

and henceforth we treat $\boldsymbol{\nu}$ as a bulk quantity defined by this relationship. Thus the decomposition:

$$\mathbf{u} = -\boldsymbol{\nu} + \mathbf{u}^\top \quad \text{on } \mathcal{S}(t),$$

which decomposes the velocity $\mathbf{u}^\top$ into its normal and tangent component on $\mathcal{S}(t)$ is meaningful as a bulk relation as well, i.e.,

$$\mathbf{u}^\top \equiv \mathbf{u} + \frac{\nabla \psi}{|\nabla \psi|}.$$

However we continue to use the slightly simpler notation for the *bulk* quantity $\mathbf{u}^\top$:

$$\mathbf{u}^\top \equiv \mathbf{u} + \boldsymbol{\nu}.$$

The explicit dynamics for the plasma-sheath interface which we will obtain are as follows: On $\mathcal{S}(t)$,

$$\frac{\delta \psi}{\delta t} = \partial_t \psi + V|\nabla \psi| = 0, \tag{4.14}$$

$$\frac{\delta n}{\delta t} = n \nabla \cdot \boldsymbol{\nu} - \nabla_s \cdot (n\mathbf{u}^\top) - \nabla(n\mathbf{u}^\top)(\boldsymbol{\nu}, \boldsymbol{\nu}), \tag{4.15}$$

$$\begin{aligned} \frac{\delta \mathbf{u}^\top}{\delta t} &= \Big[(V\boldsymbol{\nu} - \mathbf{u}) \cdot \nabla\Big]\mathbf{u} - \nabla_s V - \nabla_s \ln n, \\ &= -\Big[(V\boldsymbol{\nu} - \mathbf{u}) \cdot \nabla\Big]\boldsymbol{\nu} + \Big[(V\boldsymbol{\nu} - \mathbf{u}) \cdot \nabla\Big]\mathbf{u}^\top - \nabla_s V - \nabla_s \ln n, \end{aligned} \tag{4.16}$$

with the implicit "constitutive equation" for V

$$(V+1) + \frac{\mathbf{h} \cdot \boldsymbol{\nu}}{n} = -\frac{1}{n} \nabla_s \cdot (V \nabla_s \ln n). \tag{4.17}$$

Here the current density $\mathbf{h}$ is given as:

$$\mathbf{h} = \partial_t \nabla \phi + n\mathbf{u} \quad \text{on } \mathcal{S}(t).$$

We derive the above dynamics in the following subsections separately and all normal-time derivatives will be calculated from the quasi-neutral region unless otherwise noted.

• **Derivation of (4.14)**. Since the sheath interface is the zero-level set of ψ, we obtain a Hamilton-Jacobi equation:

$$\begin{aligned} 0 = \frac{\delta \psi}{\delta t} &= \partial_t \psi + V\boldsymbol{\nu} \cdot \nabla \psi \\ &= \partial_t \psi + V \frac{\nabla \psi}{|\nabla \psi|} \cdot \nabla \psi \\ &= \partial_t \psi + V|\nabla \psi|. \end{aligned}$$

• **Derivation of (4.15)**. We calculate the normal-time derivative as a limit from the quasi-neutral region to get

$$\begin{aligned} \frac{\delta n}{\delta t} &= \partial_t n + V\boldsymbol{\nu} \cdot \nabla n = -\nabla \cdot (n\mathbf{u}) + V\boldsymbol{\nu} \cdot \nabla n \\ &= -\nabla n \cdot \mathbf{u} - n\nabla \cdot \mathbf{u} + (V\boldsymbol{\nu}) \cdot \nabla n \\ &= -\nabla n \cdot (-\boldsymbol{\nu} + \mathbf{u}^\top) - n\nabla \cdot (-\boldsymbol{\nu} + \mathbf{u}^\top) + (V\boldsymbol{\nu}) \cdot \nabla n \\ &= \nabla n \cdot \boldsymbol{\nu} - \nabla n \cdot \mathbf{u}^\top + n\nabla \cdot \boldsymbol{\nu} - n\nabla \cdot \mathbf{u}^\top + (V\boldsymbol{\nu}) \cdot \nabla n \\ &= -\nabla \cdot (n\mathbf{u}^\top) + n\nabla \cdot \boldsymbol{\nu} + (V+1)\boldsymbol{\nu} \cdot \nabla n \\ &= -\nabla_s \cdot (n\mathbf{u}^\top) - \nabla(n\mathbf{u}^\top)(\boldsymbol{\nu}, \boldsymbol{\nu}) + n\nabla \cdot \boldsymbol{\nu} + (V+1)\boldsymbol{\nu} \cdot \nabla n \\ &= -\nabla_s \cdot (n\mathbf{u}^\top) - \nabla(n\mathbf{u}^\top)(\boldsymbol{\nu}, \boldsymbol{\nu}) + n\nabla \cdot \boldsymbol{\nu} \qquad \text{on } \mathcal{S}(t). \end{aligned}$$

• **Derivation of (4.16)** By definition of $\mathbf{u}^\top$ and momentum equation in (2.11), we have

$$\begin{aligned} \frac{\delta \mathbf{u}^\top}{\delta t} &= \frac{\delta(\mathbf{u} + \boldsymbol{\nu})}{\delta t} \\ &= \partial_t \mathbf{u} + (V\boldsymbol{\nu} \cdot \nabla)\mathbf{u} - \nabla_s V \\ &= -(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla_s \ln n + (V\boldsymbol{\nu} \cdot \nabla)\mathbf{u} - \nabla_s V \\ &= \Big[(V\boldsymbol{\nu} - \mathbf{u}) \cdot \nabla\Big]\mathbf{u} - \nabla_s V - \nabla_s \ln n \\ &= -\Big[(V\boldsymbol{\nu} - \mathbf{u}) \cdot \nabla\Big]\boldsymbol{\nu} + \Big[(V\boldsymbol{\nu} - \mathbf{u}) \cdot \nabla\Big]\mathbf{u}^\top - \nabla_s V - \nabla_s \ln n, \end{aligned}$$

where we used the quasi-neutrality relation $\phi = -\ln n$ and $\mathbf{u} = -\boldsymbol{\nu}$ on $\mathcal{S}(t)$ consistent with our **Assumption** (4.13).

• **Derivation of (4.17)**. For $\mathbf{x} \in \Omega_s(t)$, we differentiate the Poisson equation in (2.10) with respect to t to get

$$0 = \partial_t(\Delta\phi - n) = \nabla \cdot \Big(\nabla \partial_t \phi + n\mathbf{u}\Big). \tag{4.18}$$

We define the current density $\mathbf{h}$ in the sheath region Ω_s as

$$\nabla \partial_t \phi + n\mathbf{u} = \mathbf{h} \quad \text{and hence} \quad \nabla \cdot \mathbf{h} = 0. \tag{4.19}$$

Now we claim:

$$\lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Omega_s(t)}} \Big(\nabla \partial_t \phi \cdot \boldsymbol{\nu}\Big)(\mathbf{y}, t) = \Big[\nabla_s\Big(V\nabla_s \phi\Big) - Vn\Big](\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathcal{S}(t), \tag{4.20}$$

Proof of the claim: Recall that

$$\nabla\phi \cdot \boldsymbol{\nu} = 0 \quad \text{on } \mathcal{S}(t).$$

We take the normal time-derivative of the above equation to get

$$\begin{aligned} 0 &= \frac{\delta(\nabla\phi\cdot\boldsymbol{\nu})}{\delta t} = \frac{\delta\nabla\phi}{\delta t}\cdot\boldsymbol{\nu} + \nabla\phi\cdot\frac{\delta\boldsymbol{\nu}}{\delta t} \\ &= \Big[\partial_t\nabla\phi + (V\boldsymbol{\nu}\cdot\nabla)\nabla\phi\Big]\cdot\boldsymbol{\nu} - \nabla\phi\cdot\nabla_s V \\ &= \nabla\partial_t\phi\cdot\boldsymbol{\nu} + (V\nabla\otimes\nabla\phi)(\boldsymbol{\nu},\boldsymbol{\nu}) - \nabla_s\phi\cdot\nabla_s V \\ &= \nabla\partial_t\phi\cdot\boldsymbol{\nu} - \nabla_s\cdot(V\nabla_s\phi) + V\Big[\Big(\nabla\otimes\nabla\phi\Big)(\boldsymbol{\nu},\boldsymbol{\nu}) + \Delta_s\phi\Big] \quad \text{on } \mathcal{S}(t), \end{aligned} \tag{4.21}$$

where the total quantity on the R.H.S. is well defined and continuous with limits from Ω_s and Ω_q. Of course the individual terms $\nabla\partial_t\phi\cdot\boldsymbol{\nu}$ and $V(\nabla\otimes\nabla\phi)(\boldsymbol{\nu},\boldsymbol{\nu})$ will be discontinuous across the interface. It follows from Lemma 3.2 and (M3) of Section 2 that

$$\Delta_s\phi = \Delta\phi - (\nabla\otimes\nabla\phi)(\nabla\otimes\nabla\phi)(\boldsymbol{\nu},\boldsymbol{\nu}),$$

and again the R.H.S. is continuous across $\mathcal{S}(t)$ even though the individual terms are discontinuous. Hence

$$\begin{aligned} \Delta_s\phi(\mathbf{x},t) &= \lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\Omega_s(t)}} \Big(\Delta\phi - (\nabla\otimes\nabla\phi)(\boldsymbol{\nu},\boldsymbol{\nu})\Big)(\mathbf{y},t) \\ &= n - \lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\Omega_s(t)}} (\nabla\otimes\nabla\phi)(\boldsymbol{\nu},\boldsymbol{\nu})(\mathbf{y},t), \quad (\mathbf{x},t)\in\mathcal{S}(t). \end{aligned} \tag{4.22}$$

Take the limit $\mathbf{y}\to\mathbf{x},\ \mathbf{y}\in\Omega_s$ in (4.21) and use (4.22) to get

$$\begin{aligned} &\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\Omega_s(t)}} \Big(\nabla\partial_t\phi\cdot\boldsymbol{\nu}\Big)(\mathbf{y},t) \\ &= \nabla_s\cdot(V\nabla_s\phi)(\mathbf{x},t) - V\Big[\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\Omega_s(t)}} \Big(\nabla\otimes\nabla\phi(\boldsymbol{\nu},\boldsymbol{\nu})\Big)(\mathbf{y},t) + \Delta_s\phi(\mathbf{x},t)\Big] \\ &= \Big[\nabla_s\Big(V\nabla_s\phi\Big) - Vn\Big](\mathbf{x},t). \end{aligned}$$

This completes the proof of the claim.

Finally use the $\mathbf{u}\cdot\boldsymbol{\nu} = -1$ on $\mathcal{S}(t)$ and the definition of $\mathbf{h}$ in (4.19) to see

$$\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in\Omega_s(t)}} \Big(\nabla\partial_t\phi\cdot\boldsymbol{\nu}\Big)(\mathbf{y},t) = \mathbf{h}\cdot\boldsymbol{\nu} + n, \quad \mathbf{x}\in\mathcal{S}(t). \tag{4.23}$$

We combine (4.23) and (4.20) to get the "constitutive equation" for V

$$\Big(\nabla_s(V\nabla_s\phi)\Big) - (V+1)n = \mathbf{h}\cdot\boldsymbol{\nu}.$$

Remark 4.2. *Since on the level set $\psi = 0$, n satisfies (4.15)*

$$\frac{\delta n}{\delta t} = n\nabla\cdot\boldsymbol{\nu} - \nabla_s\cdot(n\mathbf{u}^\top) - \nabla(n\mathbf{u}^\top)(\boldsymbol{\nu},\boldsymbol{\nu}),$$

on $\mathcal{S}(t)$ and $V \neq -1$, then

$$\nabla n\cdot\boldsymbol{\nu} = 0 \quad \text{on } \mathcal{S}(t).$$

Proof: From above subsection 4.2, we have the relation

$$\frac{\delta n}{\delta t} = -\nabla_s\cdot(n\mathbf{u}^\top) - \nabla(n\mathbf{u}^\top)(\boldsymbol{\nu},\boldsymbol{\nu}) + n\nabla\cdot\boldsymbol{\nu} + (V+1)\boldsymbol{\nu}\cdot\nabla n$$

which combined with (4.15) yields the result.

5. **Orthogonal flow at the interface**

In this section, we consider **orthogonal** ion-flow, i.e., flow which is normal to the interface so that:

$$\mathbf{u}^\top = 0 \quad \text{on } \mathcal{S}(t).$$

For the geometrical motion of the interface, orthogonal flow is crucial for simplifying (4.14) - (4.16). Of course in general $\mathbf{u}^\top$ need not be zero; non-orthogonal flow on the sheath interface may be induced by the presence of a magnetic field $\mathbf{B}$. In this case, the term $\mathbf{u} \times \mathbf{B}$ appears at the right hand side of (1.1 b) and, as shown in [46], non-orthogonal flow occurs on the sheath interface. But notice in the rescaled variables $(\bar{\mathbf{x}}, \bar{t})$, this term is now $\mathcal{O}(\varepsilon)$ in (2.4 b) and hence negligible in our theory. Thus if $\mathbf{u}^\top$ initially zero on the sheath interface, it is reasonable to assume $\mathbf{u}^\top = 0$ on the interface for all time since magnetic field perturbations are omitted in our rescaled theory. On the other hand, recall (4.15):

$$\frac{\delta n}{\delta t} = n\nabla \cdot \boldsymbol{\nu} - \nabla_s \cdot (n\mathbf{u}^\top) - \nabla(n\mathbf{u}^\top)(\boldsymbol{\nu}, \boldsymbol{\nu}).$$

For an orthogonal flow ($\mathbf{u}^\top = 0$), the last two terms in R.H.S. of the above equation are zero, because

- $\mathbf{u}^\top = 0$ on $\mathcal{S}(t)$ implies $\nabla_s \cdot (n\mathbf{u}^\top) = 0$ on $\mathcal{S}(t)$;
- $\mathbf{u}^\top = 0$ on $\mathcal{S}(t)$ yields $\frac{\delta \mathbf{u}^\top}{\delta t} = 0$ on $\mathcal{S}(t)$ and hence the L.H.S. of (4.16) is zero and (4.16) becomes

  $$\Big((V+1)\boldsymbol{\nu} \cdot \nabla\Big)\mathbf{u}^\top = \Big((V+1)\boldsymbol{\nu} \cdot \nabla\Big)\boldsymbol{\nu} + \nabla_s(V + \ln n) \quad \text{on } \mathcal{S}(t).$$

  Here we used $\mathbf{u} = -\boldsymbol{\nu}$. Now take the inner product of the above equation with $\boldsymbol{\nu}$ and use the relations

  $$\nabla\boldsymbol{\nu}(\boldsymbol{\nu}, \boldsymbol{\nu}) = 0, \quad \nabla_s n \cdot \boldsymbol{\nu} = \nabla n \cdot \boldsymbol{\nu} = 0 \quad \text{on } \mathcal{S}(t),$$

  to get

  $$(V+1)\nabla\mathbf{u}^\top(\boldsymbol{\nu}, \boldsymbol{\nu}) = n\nabla_s \ln(V+1) \cdot \boldsymbol{\nu} = 0 \quad \text{by Lemma 3.3.}$$

  Since $V \neq -1$, we have

  $$\nabla\mathbf{u}^\top(\boldsymbol{\nu}, \boldsymbol{\nu}) = 0 \quad \text{on } \mathcal{S}(t).$$

  Hence the term $\nabla(n\mathbf{u}^\top)(\boldsymbol{\nu}, \boldsymbol{\nu})$ becomes zero:

  $$\nabla(n\mathbf{u}^\top)(\boldsymbol{\nu}, \boldsymbol{\nu}) = (\nabla n \otimes \mathbf{u}^\top)(\boldsymbol{\nu}, \boldsymbol{\nu}) + n\nabla\mathbf{u}^\top(\boldsymbol{\nu}, \boldsymbol{\nu}) = 0.$$

In summary, for orthogonal flow at the interface, the dynamics of the sheath interface $\mathcal{S}(t)$ is described by a pair of scalar evolution equations and an implicit "constitutive equation" for V:

$$\boxed{\frac{\delta \psi}{\delta t} = 0, \quad \frac{\delta n}{\delta t} = n\nabla \cdot \boldsymbol{\nu}, \quad (V+1) + \frac{\mathbf{h} \cdot \boldsymbol{\nu}}{n} = -\frac{1}{n}\nabla_s \cdot (V\nabla_s \ln n).} \tag{5.24}$$

Next we recover the dynamics of planar, cylindrical and spherical interfaces from (4.14) - (4.17) to show consistency with the earlier paper [37]. In the case of symmetric motion, surface derivative terms in the constitutive relation for V will be zero, and so V satisfies

$$V = -1 - \Big(\frac{\mathbf{h}\cdot\boldsymbol{\nu}}{n}\Big).$$

5.1. **Dynamics of symmetric targets.** We first consider a planar target. We take the ansatz for the level set function ψ:

$$\psi(\mathbf{x},t) = x_1 - s(t), \quad \mathbf{x} = (x_1, x_2, x_3),$$

then the plasma-sheath interface is given by the zero-level set of ψ, i.e.,

$$\mathcal{S}(t) \equiv \{(\mathbf{x},t) : \psi(\mathbf{x},t) = 0\}.$$

Moreover, we have

$$\boldsymbol{\nu} = (1,0,0), \quad \mathbf{u}^\top = 0, \quad n = 1, \quad \nabla_s \ln n = 0, \quad \Delta_s \ln n = 0 \ \text{ and } \ \nabla_s n = 0 \ \text{ on } \mathcal{S}(t).$$

Hence the dynamics (4.14) - (4.17) reduce to

$$\dot{s}(t) = -1 - h(t),$$

which is the relation obtained in [37].

Next we consider a spherical target. Since all fields are assumed to depend only on the radial variable r and t, the terms involving the tangential derivatives ∇_s and $\mathbf{u}^\top$ in (4.14) - (4.17) vanish, and we get the simplified dynamics:

$$\partial_t\psi - \Big(1 + \frac{\mathbf{h}\cdot\boldsymbol{\nu}}{n}\Big)|\nabla\psi| = 0, \quad \partial_t n = n\nabla\cdot\boldsymbol{\nu}. \tag{5.25}$$

We take the ansatz for the level set function ψ, weighted density ρ and current h:

$$\psi(r,t) = r - s(t), \quad \rho(r,t) = r^2 n(r,t) \quad \text{and} \quad h(r,t) = s(t)^2 h_r(t),$$

where h_r is the radial component of the current $\mathbf{h}$ and only depends on the time t by the divergence free condition. Then (4.24) becomes

$$\dot{s}(t) = -\Big(1 + \frac{h}{\rho_s}\Big) \quad \text{and} \quad \partial_t n = \frac{2n}{r} \quad \text{on } \mathcal{S}(t). \tag{5.26}$$

Here $\rho_s(t) = \rho(s(t),t)$. On the other hand, we have

$$\begin{aligned}
\dot{\rho}_s(t) &= \frac{d}{dt}\rho(s(t),t) = \partial_t\rho(s(t),t) + \dot{s}(t)\partial_r\rho(s(t),t) \\
&= 2s(t)n(s(t),t) + 2s(t)\dot{s}(t)n(s(t),t) \\
&= 2s(t)n(s(t),t)[1+\dot{s}(t)] = -\frac{2h}{s(t)}.
\end{aligned}$$

Hence the interface is governed by

$$\dot{s}(t) = -\Big(1 + \frac{h}{\rho_s}\Big), \qquad \dot{\rho}_s(t) = -\frac{2h}{s(t)},$$

which is again the same dynamics derived in [37]. The cylindrically symmetric case is done analogously to obtain

$$\dot{s}(t) = -\Big(1 + \frac{h}{\rho_s}\Big), \qquad \dot{\rho}_s(t) = -\frac{h}{s(t)},$$

as in [37].

6. Recapitulation: The dynamics of an orthogonal flow

In this section, we summarize the governing systems for the sheath, quasi-neutral plasmas and sheath-interface for an orthogonal flow.

The sheath system **(S)**

$$\boxed{\begin{cases} \partial_t n + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega_s \times (0, \infty), \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \nabla\phi, \\ \Delta_{\mathbf{x}}\phi = n, \end{cases}}$$

subject to initial data and boundary data

$$\begin{aligned} &(n, \mathbf{u}, \phi)(\mathbf{x}, 0) = (n_{s0}, \mathbf{u}_{s0}, \phi_{s0})(\mathbf{x}), \qquad \mathbf{x} \in \Omega_s(0), \\ &\nabla\phi \cdot \boldsymbol{\nu}_0 = g \quad \text{on} \;\; \partial\Omega_0, \end{aligned}$$

in the sheath region;

the quasi-neutral system **(Q)**:

$$\boxed{\begin{cases} \partial_t n + \nabla_{\mathbf{x}} \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega_q \times (0, \infty), \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}}(\ln n) = \mathbf{0}. \end{cases}}$$

subject to initial and boundary data

$$\begin{aligned} &(n, \mathbf{u})(\mathbf{x}, 0) = (n_{q0}, \mathbf{u}_{q0})(\mathbf{x}), \qquad \mathbf{x} \in \Omega^q(0), \\ &(n, \mathbf{u})(\mathbf{x}, t) = (n_s, \mathbf{u}_s)(\mathbf{x}, t) \qquad \text{on} \;\; \mathcal{S}(t), \end{aligned}$$

in the quasi-neutral region;

the sheath interface system **(SI)**:

$$\boxed{\frac{\delta\psi}{\delta t} = 0, \quad \frac{\delta n}{\delta t} = n\nabla \cdot \boldsymbol{\nu}, \quad (V+1) + \frac{\mathbf{h} \cdot \boldsymbol{\nu}}{n} = -\frac{1}{n}\nabla_s \cdot (V\nabla_s \ln n),}$$

subject to initial data:

$$(\psi, n)(\mathbf{x}, 0) = (\psi_0, n_0)(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{S}(0).$$

where $\mathbf{h} := \nabla\partial_t\phi + n\mathbf{u}$.

7. Planar motion and local existence theorems

In this section we present a simplification of the general theory in Section 6 to planar flow. In addition we give local existence theorems for the interface system, the sheath system under a "*small gradient*" assumption for relation for V in (5.24) and the quasi-neutral system.

7.1. **Local existence for the interface system.** In this subsection, we first consider the local existence of the interface system:

$$\boxed{\frac{\delta\psi}{\delta t} = 0, \quad \frac{\delta n}{\delta t} = n\nabla\cdot\boldsymbol{\nu}, \quad (V+1) + \frac{\mathbf{h}\cdot\boldsymbol{\nu}}{n} = -\frac{1}{n}\nabla_s\cdot(V\nabla_s \ln n),} \tag{7.27}$$

in the case that the interface is a curve evolving in the $x_1 - x_2$ plane.

Below, we first explain the procedure to calculate $\mathbf{h}\cdot\boldsymbol{\nu}$ appearing at (7.27). As in [6], for a H^1-vector field $\mathbf{v} = (v_1, v_2)$ and a scalar function v, we define

$$\nabla\cdot\mathbf{v} := \partial_{x_1}v_1 + \partial_{x_2}v_2, \quad \nabla\times\mathbf{v} := \partial_{x_1}v_2 - \partial_{x_2}v_1 \qquad \text{and} \qquad \nabla^{\perp}v := (\partial_{x_2}v, -\partial_{x_1}v).$$

Then it is easy to see

$$\nabla\times\nabla^{\perp}v = \Delta v \quad \text{and} \quad \nabla\cdot\nabla^{\perp}v = 0.$$

We next express the current $\mathbf{h}$ as a direct sum of three vector fields via the planar Hodge-Weyl decomposition (Theorem 6 in [6]):

$$\mathbf{h} = \nabla\zeta + \nabla^{\perp}v + \mathbf{m} \qquad \text{for} \ \ (\mathbf{x}, t) \in \ \Omega_s(t)\times[0,\infty), \tag{7.28}$$

where ζ and v are scalar valued functions, and all components m_i of $\mathbf{m} = (m_1, m_2)$ are harmonic functions.

We now take a $\nabla\times$ (7.28) to find

$$\begin{aligned} \nabla\times\mathbf{h} &= \nabla\times\nabla^{\perp}v + \nabla\times\mathbf{m} \\ &= \Delta v + \nabla\times\mathbf{m}. \end{aligned} \tag{7.29}$$

In contrast, in the sheath region $\Omega_s(t)$

$$\mathbf{h} = \nabla\partial_t\phi + n\mathbf{u} \qquad \text{and} \qquad \nabla\times\mathbf{h} = \nabla\times(n\mathbf{u}). \tag{7.30}$$

We combine (7.29) and (7.30) to get

$$\Delta v = \nabla\times(n\mathbf{u}) - \nabla\times\mathbf{m} \qquad \text{for} \ \ (\mathbf{x}, t)\in\Omega_s(t)\times[0,\infty). \tag{7.31}$$

According to equation (20) of Theorem 3 in [6], we can set Neumann boundary conditions for ζ and Dirichlet boundary condition for v:

$$v \equiv 0 \quad \text{on} \ \ \partial\Omega_0\cup\mathcal{S}(t). \tag{7.32}$$

Let $\boldsymbol{\tau}_0$ and $\boldsymbol{\tau}$ be unit tangent vectors on $\partial\Omega_0$ and the sheath interface respectively. Then zero Dirichlet boundary condition (7.32) yields

$$\nabla v\cdot\boldsymbol{\tau}_0 = 0 \quad \text{on } \partial\Omega_0 \quad \text{and} \quad \nabla v\cdot\boldsymbol{\tau} = 0 \quad \text{on } \mathcal{S}(t). \tag{7.33}$$

On the other hand (7.32) is equivalent to

$$\nabla^{\perp}v = 0 \quad \text{on } \partial\Omega_0\cup\mathcal{S}(t). \tag{7.34}$$

Again by the equation (20) of Theorem 3 in [6], $\mathbf{m}$ satisfies

$$\mathbf{m}\cdot\boldsymbol{\nu}_0 = 0 \quad \text{on } \partial\Omega_0 \quad \text{and} \quad \mathbf{m}\cdot\boldsymbol{\nu} = 0 \quad \text{on } \mathcal{S}(t). \tag{7.35}$$

Furthermore, the proof of Theorem 11 shows: for some scalar valued function p,

$$\mathbf{m} = \nabla^{\perp}p \qquad (\mathbf{x}, t)\in\Omega_s(t)\times[0,\infty). \tag{7.36}$$

In particular, it follows from (7.28), (7.34) and (7.35) that

$$\nabla\zeta \cdot \boldsymbol{\nu}_0 = \mathbf{h} \cdot \boldsymbol{\nu}_0 \quad \text{on } \partial\Omega_0 \qquad \text{and} \tag{7.37}$$

$$\nabla\zeta \cdot \boldsymbol{\nu} = \mathbf{h} \cdot \boldsymbol{\nu} \quad \text{on } \mathcal{S}(t). \tag{7.38}$$

Finally since $\nabla \cdot \mathbf{h} = 0$ and (7.36) in $\Omega_s(t)$, we see from (7.28)

$$\Delta\zeta = 0 \qquad \text{in } \Omega_s(t), \tag{7.39}$$

with Neumann boundary data for ζ given by (7.37) and (7.38). Note again that the function $\mathbf{h}$ is not known a priori and must be computed from the given boundary data g and initial conditions. In fact, however, to produce a solution ζ of (7.37) - (7.39), we solve the exterior problem:

$$\Delta\zeta = 0 \qquad \text{on } \mathbb{R}^2 - \bar{\Omega}_0, \tag{7.40}$$

with the Neumann boundary condition (7.37) on $\partial\Omega_0$ and

$$\lim_{|\mathbf{x}|\to\infty} \nabla\zeta = \mathbf{0}. \tag{7.41}$$

Solution of (7.40) and (7.41) in the full exterior domain reflects the fact that the decomposition (7.28) must be done in the whole exterior domain $\mathbb{R}^2 - \bar{\Omega}_0$ and $\nabla\cdot\mathbf{h} = 0$ in $\mathbb{R}^2 - \bar{\Omega}_0$, where the current is appropriately defined in both the sheath and quasi-neutral regions. More abstractly, if we write the direct sum decomposition (7.28) for $\mathbf{h}(\cdot, t) \in L^2(\mathbb{R}^2 - \Omega_s(t))$, then the projection $\nabla\zeta$ must be in $L^2(\mathbb{R}^2 - \Omega_s(t))$ as well. Notice that the exact computation of v and $\mathbf{m}$ is irrelevant and would be done a posteriori to solve the full evolution in the sheath region from (7.31).

In summary, the procedure for computing the quantity $\mathbf{h} \cdot \boldsymbol{\nu}$ on the sheath interface is as follows. In order to evolve (7.27), we solve the exterior Neumann problem:

$$\begin{cases} \Delta\zeta = 0 \quad \text{in } \mathbb{R}^2 - \bar{\Omega}_0, \\ \nabla\zeta \cdot \boldsymbol{\nu}_0 = \mathbf{h} \cdot \boldsymbol{\nu}_0 \quad \text{on } \partial\Omega_0, \end{cases}$$

and then employ (7.38)

$$\mathbf{h} \cdot \boldsymbol{\nu} = \nabla\zeta \cdot \boldsymbol{\nu} \quad \text{on } \mathcal{S}(t).$$

This provides the mechanism of the transfer of information from the boundary to the sheath interface, enabling the sheath interface to evolve according to (7.27).

7.1.1. **Evolving curve which is a graph of a function**. We consider the situation where the target is a perturbation of a plane so that the interface is given by the graph of a function. Hence we assume that the interfacial quantities and profile are functions of x_1, x_2, t and moreover the profile is given by the graph of some function f, i.e.,

$$x_2 = f(x_1, t).$$

As in the Figure 2, we denote the angle between the exterior unit normal $\boldsymbol{\nu}$ and x_1-axis at (x_1, t) by $\theta(x_1, t)$, hence

$$\boldsymbol{\nu}(x_1, t) = (\cos\theta(x_1, t), \sin\theta(x_1, t)), \qquad \theta \in (0, \pi),$$

and also set

$$\bar{V}(x_1, t) = V(x_1, f(x_1, t), t) \qquad \text{and} \qquad \bar{n}(x_1, t) \equiv n(x_1, f(x_1, t), t).$$

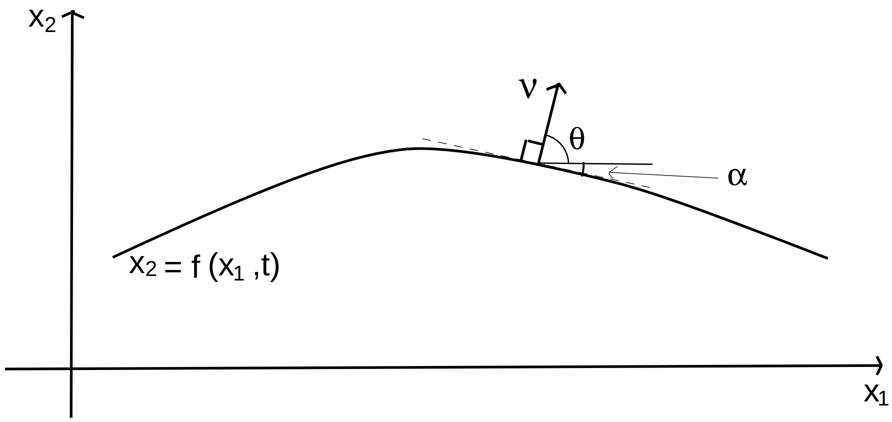

FIGURE 2. Schematic diagram of a sheath interface on the plane

Lemma 7.1. *The time and spatial derivatives of f are given by:*

$$\partial_{x_1} f = -\cot\theta, \qquad \partial_t f = \frac{\bar{V}}{\sin\theta}.$$

Proof. In Figure 2, we notice that

$$\alpha + \theta = \frac{\pi}{2}, \qquad \text{and} \qquad \tan\alpha = -\partial_{x_1} f.$$

These imply

$$\cot\theta = -\partial_{x_1} f.$$

It follows from the above figure 2 that

$$\frac{\bar{V}\Delta t}{f(x_1, t+\Delta t) - f(x_1, t)} = \sin\theta + \mathcal{O}((\Delta t)^2).$$

As $\Delta t \to 0$, we have

$$\frac{\bar{V}}{\partial_t f} = \sin\theta, \qquad \text{or} \qquad \partial_t f = \frac{\bar{V}}{\sin\theta}.$$

□

In the following lemma, we obtain evolution equations for the interface, which are equivalent to (7.27).

Lemma 7.2. *In the case when the sheath interface is a curve in the plane, (7.27) is equivalent to the following system for $\theta, \bar{n}$ and $\bar{V}$:*

$$\begin{pmatrix} \partial_t\theta \\ \partial_t\bar{n} \\ \partial_t f \end{pmatrix} + \begin{pmatrix} \bar{V}\cos\theta & 0 & 0 \\ 2\bar{n}\sin\theta & \bar{V}\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{x_1}\theta \\ \partial_{x_1}\bar{n} \\ \partial_{x_1} f \end{pmatrix} = \begin{pmatrix} \sin\theta\partial_{x_1}\bar{V} \\ 0 \\ \frac{\bar{V}}{\sin\theta} \end{pmatrix}$$

with the constitutive relation for V:

$$(V+1) + \frac{\mathbf{h}\cdot(\cos\theta, \sin\theta)}{n} = -\frac{1}{n}\nabla_s\cdot(V\nabla_s \ln n),$$

which can be expressed as a ODE for $\bar{V}$:

$$\frac{(\partial_{x_1}\bar{V})(\partial_{x_1}\bar{n})\sin^2\theta(1+\cos^2\theta)}{\bar{n}^2}+\bar{V}\Big(1+\frac{\sin\theta}{\bar{n}}\partial_{x_1}\Big(\frac{\sin\theta\partial_{x_1}\bar{n}}{\bar{n}}\Big)\Big)+1+\frac{\mathbf{h}\cdot(\cos\theta,\sin\theta)}{\bar{n}}=0.$$

Proof. (i) $\dfrac{\delta\psi}{\delta t}=0$: So the statement equivalent to $\dfrac{\delta\psi}{\delta t}=0$ is just the equality of cross partial derivatives $\partial_t(\partial_{x_1}f)=\partial_{x_1}(\partial_t f)$ and we have

$$\partial_t(-\cot\theta)=\partial_{x_1}\Big(\frac{\bar{V}}{\sin\theta}\Big).$$

This yields

$$\partial_t\theta+\bar{V}\cos\theta\partial_{x_1}\theta=\sin\theta\partial_{x_1}\bar{V}. \tag{7.42}$$

(ii) $\frac{\delta n}{\delta t}=n\mathrm{div}\boldsymbol{\nu}$: Recall the 2nd equation can be rewritten as

$$\partial_t n+\bar{V}\boldsymbol{\nu}\cdot\nabla n=n\mathrm{div}\boldsymbol{\nu}. \tag{7.43}$$

Here all quantities and their derivatives are evaluated on the interface. Next we claim:

$$\partial_t n=\partial_t\bar{n}-\Big(\frac{\bar{V}}{\sin\theta}\Big)\partial_{x_2}n,\quad \nabla\cdot\boldsymbol{\nu}=-2\sin\theta\partial_{x_1}\theta,$$
$$\bar{V}\cos\theta\partial_{x_1}n=\bar{V}\cos\theta\partial_{x_1}\bar{n}+\bar{V}\Big(\frac{\cos^2\theta}{\sin\theta}\Big)\partial_{x_2}n.$$

Proof of the claim: By direct calculation and Lemma 7.1, we have

$$\partial_t\bar{n}=\partial_t n+\partial_{x_2}n\partial_t f=\partial_t n+\partial_{x_2}n\Big(\frac{\bar{V}}{\sin\theta}\Big).$$

Similarly we obtain

$$\begin{aligned}\bar{V}\cos\theta\partial_{x_1}\bar{n} &= \bar{V}\cos\theta\partial_{x_1}n+\bar{V}\cos\theta\partial_{x_2}n\partial_{x_1}f\\ &= \bar{V}\cos\theta\partial_{x_1}n-\Big(\frac{\bar{V}\cos^2\theta}{\sin\theta}\Big)\partial_{x_2}n.\end{aligned}$$

Next note

$$\nabla\cdot\boldsymbol{\nu}=-2\sin\theta\partial_{x_1}\theta.$$

In (7.43), use the above claim to get

$$\begin{aligned}&\partial_t n+\bar{V}\boldsymbol{\nu}\cdot\nabla n\\ &=\partial_t n+\bar{V}\cos\theta\partial_{x_1}n+\bar{V}\sin\theta\partial_{x_2}n\\ &=\partial_t\bar{n}-\Big(\frac{\bar{V}}{\sin\theta}\Big)\partial_{x_2}n+\bar{V}\cos\theta\partial_{x_1}\bar{n}+\bar{V}\Big(\frac{\cos^2\theta}{\sin\theta}\Big)\partial_{x_2}n+\bar{V}\sin\theta\partial_{x_2}n\\ &=\partial_t\bar{n}+\bar{V}\cos\theta\partial_{x_1}\bar{n}+\frac{\bar{V}\partial_{x_2}n}{\sin\theta}\Big(-1+\cos^2\theta+\sin^2\theta\Big)\\ &=\partial_t\bar{n}+\bar{V}\cos\theta\partial_{x_1}\bar{n}.\end{aligned}$$

Therefore our scaled evolution equation is

$$\partial_t\bar{n}+\bar{V}\cos\theta\partial_{x_1}\bar{n}+2\bar{n}\sin\theta\partial_{x_1}\theta=0. \tag{7.44}$$

Now combine (7.42) and (7.44) to get the equations for the motion:

$$\begin{pmatrix} \partial_t \theta \\ \partial_t \bar{n} \\ \partial_t f \end{pmatrix} + \begin{pmatrix} \bar{V}\cos\theta & 0 & 0 \\ 2\bar{n}\sin\theta & \bar{V}\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{x_1}\theta \\ \partial_{x_1}\bar{n} \\ \partial_{x_1} f \end{pmatrix} = \begin{pmatrix} \sin\theta\partial_{x_1}\bar{V} \\ 0 \\ \frac{\bar{V}}{\sin\theta} \end{pmatrix}. \tag{7.45}$$

Finally we derive ODE satisfied by $\bar{V}$. Recall the interfacial quanities

$$\bar{V}(x_1,t) = V(x_1, f(x_1,t), t) \qquad \text{and} \qquad \bar{n}(x_1,t) \equiv n(x_1, f(x_1,t), t),$$

and use the fact that the unit normal $\boldsymbol{\nu}$ and the unit tangent $\boldsymbol{t}$ to the sheath interface are given by

$$\boldsymbol{\nu} = (\cos\theta, \sin\theta) \qquad \text{and} \qquad \boldsymbol{t} = (\sin\theta, -\cos\theta)$$

to see

$$\begin{aligned} \nabla_s V &= \nabla V - (\nabla V \cdot \boldsymbol{\nu})\boldsymbol{\nu} = (\nabla V \cdot \boldsymbol{t})\boldsymbol{t} \\ &= (\sin\theta\partial_{x_1}V - \cos\theta\partial_{x_2}V)(\sin\theta, -\cos\theta). \end{aligned} \tag{7.46}$$

Furthermore via the chain rule and the relation $\partial_{x_1} f = -\cot\theta$, we find

$$\begin{aligned} \partial_{x_1}\bar{V} &= \partial_{x_1}V + \partial_{x_2}V\partial_{x_1}f = \nabla V \cdot (1, \partial_{x_1}f) \\ &= \nabla V \cdot (1, -\cot\theta) = \frac{\nabla V \cdot \boldsymbol{t}}{\sin\theta}. \end{aligned} \tag{7.47}$$

Thus (7.46) and (7.47) imply

$$\nabla_s V = \partial_{x_1}\bar{V}(\sin^2\theta, -\sin\theta\cos\theta). \tag{7.48}$$

Similarly we find for $\bar{n}(x_1,t)$ that

$$\nabla_s \ln n = \partial_{x_1}\bar{n}(\sin^2\theta, -\sin\theta\cos\theta), \tag{7.49}$$

as well as

$$\partial_{x_1}\bar{n} = \frac{\nabla n \cdot \boldsymbol{t}}{\sin\theta}. \tag{7.50}$$

In fact from (7.50) we easily see

$$\begin{aligned} \partial_{x_1}(\sin\theta\partial_{x_1}\bar{n}) &= (\partial_{x_1}^2 n)\sin\theta + (\partial_{x_1}\partial_{x_2}n)(\partial_{x_1}f)\sin\theta \\ &- (\partial_{x_1}\partial_{x_2}n)\cos\theta - (\partial_{x_2}^2 n)(\partial_{x_1}f)\cos\theta + (\partial_{x_1}n)\cos\theta\partial_{x_1}\theta + (\partial_{x_2}n)\sin\theta\partial_{x_1}\theta, \end{aligned}$$

but by the definition of $\Delta_s n$ we know

$$\Delta_s n = (\partial_{x_1}^2 n)\sin^2\theta - 2(\partial_{x_1}\partial_{x_2}n)\sin\theta\cos\theta + (\partial_{x_2}^2 n)\cos^2\theta,$$

and hence

$$\sin\theta\partial_{x_1}(\sin\theta\partial_{x_1}\bar{n}) = \Delta_s n + (\nabla n \cdot \boldsymbol{\nu})(\partial_{x_1}\theta)\sin\theta. \tag{7.51}$$

Recall now that the definition of the sheath edge requires

$$\nabla n \cdot \boldsymbol{\nu} = 0,$$

and (7.51) simplifies to

$$\sin\theta\partial_{x_1}(\sin\theta\partial_{x_1}\bar{n}) = \Delta_s n. \tag{7.52}$$

Finally substitute (7.48) - (7.50) into the defining "*constitutive relation*" for V:

$$(V+1) + \frac{\mathbf{h}\cdot(\cos\theta,\sin\theta)}{n} = -\frac{1}{n}\nabla_s \cdot (V\nabla_s \ln n),$$

and do the obvious trigonometric simplifications to find the first order O.D.E. for $\bar{V}$:

$$\frac{(\partial_{x_1}\bar{V})(\partial_{x_1}\bar{n})\sin^2\theta(1+\cos^2\theta)}{\bar{n}^2} + \bar{V}\Big(1 + \frac{\sin\theta}{\bar{n}}\partial_{x_1}\Big(\frac{\sin\theta\partial_{x_1}\bar{n}}{\bar{n}}\Big)\Big) + 1 + \frac{\mathbf{h}\cdot(\cos\theta,\sin\theta)}{\bar{n}} = 0.$$

□

Remark 7.1. *These are the same formulas as would have followed from the formulation of Angenent and Gurtin* [3]*.*

For a presumably known $\bar{V}(x_1,t)$, we see that the coefficient matrix of (7.45) yields the characteristic equation:

$$\lambda(\bar{V}\cos\theta - \lambda)^2 = 0$$

yielding eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \bar{V}\cos\theta$$

and unfortunately we have only one linearly independent eigenvector for the pair of eigenvalues λ_2, λ_3, i.e., (7.45) is not strictly hyperbolic. Hence the relation between $\bar{V}$ and $\boldsymbol{t}$ "*input*" given by (7.1.1) is crucial.

The simplest approximation that captures the dependence of $\bar{V}$ on the other fluid variables is to recall that when the sheath interface is a plane, a cylinder and a sphere, surface gradient terms in (7.27) will vanish. Hence for sheath interfaces which are near planes, cylinders, or spheres, dropping surface gradient terms provides a first approximation quasi-linear theory of sheath interface motions, i.e., we take the normal velocity to be given by the approximate relation.

$$\tilde{V} = -1 - \frac{\nabla\zeta\cdot\boldsymbol{\nu}}{n} = -1 - \frac{\cos\theta\partial_{x_1}\zeta + \sin\theta\partial_{x_2}\zeta}{n}.$$

With this approximate speed $\tilde{V}$, the system (7.45) becomes

$$\begin{pmatrix}\partial_t\theta\\ \partial_t n\\ \partial_t f\end{pmatrix} + \begin{pmatrix}\tilde{V}\cos\theta - \sin\theta\partial_\theta\tilde{V} & -\sin\theta\partial_n\tilde{V} & 0\\ 2n\sin\theta & \tilde{V}\cos\theta & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\partial_{x_1}\theta\\ \partial_{x_1}n\\ \partial_{x_1}f\end{pmatrix} = \begin{pmatrix}\sin\theta\partial_{x_1}\tilde{V}\\ 0\\ \frac{\tilde{V}}{\sin\theta}\end{pmatrix}, \tag{7.53}$$

where we have dropped the overbars.

The characteristic equation for the coefficients matrix is given by

$$\lambda\Big[\lambda^2 - (2\tilde{V}\cos\theta - \partial_\theta\tilde{V}\sin\theta)\lambda + (\tilde{V})^2\cos^2\theta - \tilde{V}\partial_\theta\tilde{V}\cos\theta\sin\theta + 2n\sin^2\theta\partial_n\tilde{V}\Big] = 0.$$

But notice that

$$\partial_n\tilde{V} = -\frac{\nabla\zeta\cdot\boldsymbol{\nu}}{n^2},$$

and hence we have

$$n\partial_n\tilde{V} = \frac{\nabla\zeta\cdot\boldsymbol{\nu}}{n} = -\tilde{V} - 1.$$

It follows that λ satisfies

$$\begin{aligned}&\lambda = 0 \qquad \text{or}\\ &\lambda^2 - \Big(2\tilde{V}\cos\theta - \partial_\theta\tilde{V}\sin\theta\Big)\lambda + \tilde{V}^2\cos^2\theta - \tilde{V}\partial_\theta\tilde{V}\cos\theta\sin\theta - 2(\tilde{V}+1)\sin^2\theta = 0,\end{aligned}$$

and hence

$$\begin{aligned}\lambda &= 0 \qquad \text{or}\\ 2\lambda &= (2\tilde{V}\cos\theta - \partial_\theta\tilde{V}\sin\theta)\end{aligned}$$

$$\pm \quad \Big[(2\tilde V\cos\theta-\partial_\theta\tilde V\sin\theta)^2-4(\tilde V^2\cos^2\theta-V\partial_\theta\tilde V\cos\theta\sin\theta-2(\tilde V+1)\sin^2\theta)\Big]^{\frac12}.$$

Thus we have three real distinct eigenvalues when

$$4\tilde V^2\cos^2\theta-4\tilde V\partial_\theta\tilde V\cos\theta\sin\theta+(\partial_\theta\tilde V)^2\sin^2\theta>4\tilde V^2\cos^2\theta-4\tilde V\partial_\theta\tilde V\cos\theta\sin\theta-8(\tilde V+1)\sin^2\theta.$$

Doing the obvious cancellation we see

$$(\partial_\theta\tilde V)^2>-8(\tilde V+1)$$

so that when $\tilde V+1>0$ we have real distinct eigenvalues. Of course a more precise result is that we have real distinct eigenvalues if and only if

$$\frac{(-\sin\theta\partial_{x_1}\zeta+\cos\theta\partial_{x_2}\zeta)^2}{n^2}>8\Big(\frac{\cos\theta\partial_{x_1}\zeta+\sin\theta\partial_{x_2}\zeta}{n}\Big).$$

Thus we may state the following local existence, uniqueness theorem.

Theorem 7.1. *Assume the normal component of the current $\nabla\zeta\cdot\boldsymbol{\nu}$ obtained from solving the interior sheath system (2.6) is known and sufficiently smooth on $\mathbb{R}^2\times[0,T]$ for some $T>0$, and initial data (θ_0,n_{s0},f_0) satisfy the following conditions:*

(i) $(\theta_0,n_{s0},f_0)\in(C^1(\mathbb{R}))^3$ *and* $||\theta_0||_{C^1(\mathbb{R})}+||n_{s0}||_{C^1(\mathbb{R})}+||f_0||_{C^1(\mathbb{R})}<G_0$;
(ii) $n_{s0}>0$;
(iii) $\dfrac{(-\sin\theta_0\partial_{x_1}\zeta_0+\cos\theta_0\partial_{x_2}\zeta_0)^2}{n_{s0}^2}>8\Big(\dfrac{\cos\theta_0\partial_{x_1}\zeta_0+\sin\theta_0\partial_{x_2}\zeta_0}{n_{s0}}\Big)$ *on the interface $\mathcal{S}$,*

*where G_0 is a positive constant. Then there is a time interval $[0,T_*)$ with $T_*>0$, so that the interface equations (**??**)-(7.53) with the approximate constitutive relation $V=\tilde V$ have a unique classical solution $(\theta,n)\in(C^1(\mathbb{R}\times[0,T_*)))^2$.*

Proof. The result follows from the classical local existence theorem for strictly hyperbolic systems (see Courant and Hilbert [19], Douglis [25]). □

Remark 7.2. *1. The time T_* depends on $||\theta_{s0}||_{1,\infty;\mathbb{R}},||n_{so}||_{1,\infty;\mathbb{R}}$ and G_0.*
2. If initial data and all coefficients are $C^k,k\geq2$, then by the standard iteration scheme, there exist unique C^k-solutions to the sheath interface system (see Friedrichs [31]*).*

7.1.2. **Evolving simple closed convex curve in the plane**. In this part, we consider the case where the interface is given by a simple closed convex curve in the plane (see Figure 3).

Consider a portion Γ of a curve which is represented by $x_2=f(x_1,t)$ in x_1-x_2 plane. If we wish to study the evolution of a simple closed convex curve in the plane, it will be convenient to use polar coordinates:

$$x_1=r\cos\beta,\qquad x_2=r\sin\beta, \tag{7.54}$$

so that the evolving curve is represented by $r=r(\beta,t)$, and the portion Γ becomes

$$r\sin\beta=f(r\cos\beta,t), \tag{7.55}$$

First recall the result of Lemma 7.1 to be used in computations below, i.e.

$$\partial_{x_1}f=-\cot\theta,\qquad \partial_tf=\frac{V}{\sin\theta}.$$

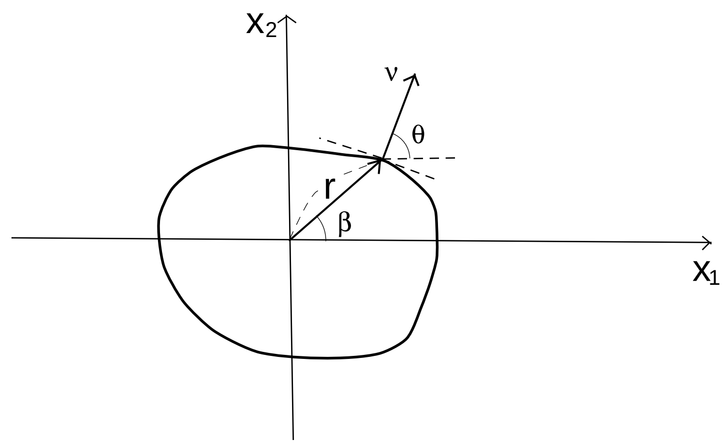

FIGURE 3. Schematic diagram of a sheath interface in the plane

Now differentiate (7.55) with respect to t to get

$$\begin{aligned}\partial_t r \sin\beta &= \partial_{x_1} f(\partial_t r\cos\beta) + \partial_t f \\ &= -\cot\theta(\partial_t r\cos\beta) + \frac{V}{\sin\theta},\end{aligned}$$

and we see

$$\partial_t r = \frac{V}{\cos(\theta-\beta)}. \tag{7.56}$$

Note that we need to know the evolution of θ together with n. In order to use the previous computation in Section 7.1.1, we need to express ∂_{x_1} in terms of ∂_β.

$$\text{We claim: } \partial_{x_1} = -\frac{2\sin\beta}{r}\partial_\beta. \tag{7.57}$$

Proof of the claim. Differentiate (7.55) with respect to r to obtain

$$\partial_r\beta = -\frac{\sin\beta + \cot\theta\cos\beta}{r\cos\beta - r\sin\beta\cot\theta}, \tag{7.58}$$

where we used $\partial_{x_1} f = -\cot\theta$. On the other hand, the chain rule yields

$$\begin{aligned}\partial_{x_1} &= \partial_{x_1} r\partial_r + \partial_{x_1}\beta\partial_\beta \\ &= \partial_{x_1} r\partial_r\beta\partial_\beta + \partial_{x_1}\beta\partial_\beta.\end{aligned} \tag{7.59}$$

Recall that

$$r^2 = x_1^2 + (f(x_1,t))^2 \quad \text{and} \quad \beta = \arctan\Big(\frac{f(x_1,t)}{x_1}\Big),$$

and apply (7.59) to r and β, use the above relation to find

$$\partial_{x_1} r = -\sin\beta\cot\theta + \cos\beta \quad \text{and} \quad \partial_{x_1}\beta = -\frac{(\cos\beta\cot\theta + \sin\beta)}{r}, \tag{7.60}$$

and then use (7.58) and (7.60) to obtain

$$\partial_{x_1} = -\frac{(-\sin\beta\cot\theta + \cos\beta)(\sin\beta + \cot\theta\cos\beta)}{r(\cos\beta - \sin\beta\cot\theta)}\partial_\beta - \frac{(\cos\beta\cot\theta + \sin\beta)}{r}\partial_\beta,$$

i.e.,

$$\partial_{x_1} = -\frac{2\sin\beta}{r}\partial_\beta.$$

This completes the proof of the claim.

On the other hand, note that the convexity of interface can be treated as follows:

$$\partial_{x_1}^2 f = \partial_{x_1}(-\cot\theta) = \frac{\partial_{x_1}\theta}{\sin^2\theta} = -\frac{2\sin\beta\partial_\beta\theta}{r\sin^2\theta} > 0,$$

where we used the above claim and hence the convexity of an interface is represented by the relation

$$\sin\beta\partial_\beta\theta < 0.$$

We combine (7.56) - (7.57) and Lemma 7.2 to obtain the following lemma.

Lemma 7.3. *In the case when the sheath interface is a simple closed curve, (7.27) is equivalent to the system:*

$$\begin{pmatrix} \partial_t\theta \\ \partial_t n_s \\ \partial_t r \end{pmatrix} - \frac{2\sin\beta}{r}\begin{pmatrix} V\cos\theta & 0 & 0 \\ 2n_s\sin\theta & V\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \partial_\beta\theta \\ \partial_\beta n_s \\ \partial_\beta r \end{pmatrix} = \begin{pmatrix} -\frac{2\sin\theta\sin\beta\partial_\beta V}{r} \\ 0 \\ \frac{V}{\cos(\theta-\beta)} \end{pmatrix}$$

with the same constitutive equation for V as in Lemma 7.1.

If we employ the small surface gradient approximation $\tilde{V}$ for V, we can state the following theorem.

Theorem 7.2. *Assume the normal component of the irrotational part of the current $\nabla\zeta\cdot\boldsymbol{\nu}$ obtained from solving the interior sheath system (2.10) is known, sufficiently smooth, 2π-periodic in β and smooth 2π-periodic initial data (θ_0, n_{s0}, r_0) in the sheath interface satisfy the following conditions:*

(i) $(\theta_0, n_{s0}, r_0) \in (C^1(\mathbb{R}))^3$ *and* $||\theta_0||_{C^1(\mathbb{R})} + ||n_{s0}||_{C^1(\mathbb{R})} + ||r_0||_{C^1(\mathbb{R})} < G_0$;
(ii) $n_{s0} > 0, \quad r_0 > 0$;
(iii) $\dfrac{(-\sin\theta_0, \cos\theta_0)\cdot\nabla\zeta_{s0}}{n_{s0}^2} > \dfrac{8(\cos\theta_0, \sin\theta_0)\cdot\nabla\zeta_{s0}}{n_{s0}}$;
(iv) $|\theta_0 - \beta| < \dfrac{\pi}{2} \qquad \sin\beta\partial_\beta\theta_0 < 0.$

Then there is a $T_{**} > 0$ so that the interface equations (7.64) with approximate constitutive relation $V = \tilde{V}$ has a unique classical solution $(\theta, n, r) \in (C^1(\mathbb{R}\times[0, T_{**}]))^3$ which is 2π-periodic in β, and this solution satisfies the estimates

$$\begin{aligned} &||\theta||_{C^1(\mathbb{R}\times[0,T_{**}))} + ||n_s||_{C^1(\mathbb{R}\times[0,T_{**}))} + ||r||_{C^1(\mathbb{R}\times[0,T_{**}))} \\ &\leq 2\Big(||\theta_0||_{C^1(\mathbb{R})} + ||n_{s0}||_{C^1(\mathbb{R})} + ||r_0||_{C^1(\mathbb{R})}\Big), \end{aligned} \tag{7.61}$$

where G_0 is a positive constant, and T_{**} depends on $||\theta_0||_{C^1(\mathbb{R})}, ||n_{s0}||_{C^1(\mathbb{R})}, ||r_0||_{C^1(\mathbb{R})}$ and G_0. Furthermore the sheath interface is convex since $\sin\beta\partial_\beta\theta < 0$.

Proof. The eigenvalues of the coefficient matrix are $\lambda_1, \lambda_2, \lambda_3$, where λ_1, λ_2 are the eigenvalues of (7.45) and $\lambda_3 = 0$. Thus again we have real distinct eigenvalues and again the classical existence and uniqueness theorem in [19, 25] can be applied. Convexity is preserved since we have C^1-continuity of θ in β and t. □

Remark 7.3. *1. If the initial data and all coefficients are $C^k, k \geq 2$, then by the standard iteration scheme, there exists a unique C^k-solution to the sheath interface system satisfying (7.61) with C^1-norms replaced by C^k-norms (see Friedrichs* [31]*).*

2. The above existence theorem can be further generalized into the Hölder space $C^{k,\gamma}(\mathbb{R})$ with the estimate (7.61) replaced by $C^{k,\gamma}(\mathbb{R})$-norm. (See [39]*).*

7.2. **Local existence for the sheath system.** In this subsection we study the local existence of smooth solutions to the sheath system when the boundary of the target is a small perturbation of an infinite cylinder and the initial sheath interface is a smooth simple closed convex curve as in Section 7.1.2. We model the proof on the presentation of Nouri [43] and use an iteration procedure given in the papers of Chen and Feldman [13] and Canic, Keyfitz and Lieberman [12] to construct approximate solutions to the sheath system and then employ the Schauder Fixed Theorem to show the existence of local in time sheath solutions. Since the exact location of the sheath interface is not known *a priori*, at each iteration step, we have to evolve the interface system with the data given by the sheath system as well. This makes the proof technically challenging, yet since the proof outlines a possible approach to numerical implementation, we view it as a crucial part of our presentation.

Below we summarize the main assumptions for initial and boundary data to the sheath and interface systems and then we state the main theorem of this subsection. In what follows, $\delta_{*i}, i = 1, 2$ and δ^* are positive constants representing the lower and upper bounds of the size of data, and ∇, Δ denotes a spatial gradient and a spatial Laplacian respectively.

Notations. We first introduce some efficient notation for norms and partial derivatives:

$$\begin{aligned}
| \cdot | \quad &: \quad \text{the Eulidean norm in } \mathbb{R}^n, \\
[\cdot]_{0,\gamma} \quad &: \quad \text{the Hölder seminorm in a spatial region in } \mathbb{R}^2 \\
|| \cdot ||_{0,\gamma} \quad &: \quad \text{the Hölder norm in spatial region in } \mathbb{R}^2, \\
[[\cdot]]_{0,\gamma} \quad &: \quad \text{the Hölder seminorm in a space-time region in } \mathbb{R}^2 \times [0, T], \\
||| \cdot |||_{0,\gamma} \quad &: \quad \text{the Hölder norm in a space-time region } \mathbb{R}^2 \times [0, T].
\end{aligned}$$

For the definitions of Hölder seminorm and norm, we refer to Evans' book [28]. On the other hand, for calculus in two space variables, we will denote the *multi-indice* α as an ordered pair (α_1, α_2) of nonnegative integers, and use the notation:

$$|\alpha| := \alpha_1 + \alpha_2 \quad \text{and} \quad \partial^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}.$$

Now consider the sheath system **(S)**:

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \Omega_s(t) \times (0, \infty), \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \phi, \\ \Delta \phi = n, \end{cases} \tag{7.62}$$

subject to initial and boundary data:

$$\begin{cases}(n, \mathbf{u}, \phi) = (n_0, \mathbf{u}_0, \phi_0) \quad \text{on } \Omega_s(0) \times \{t = 0\}, \\ \nabla \phi \cdot \boldsymbol{\nu}_0 = g \quad \text{on } \partial\Omega_0 \times [0, \infty). \end{cases} \tag{7.63}$$

We impose several conditions on data and the target boundary: Let $\gamma \in (0, 1)$ be fixed.

- (A1) (Regularity, compatibility)

$$\begin{aligned} &n_0 \in C^{1,\gamma}(\bar{\Omega}_s(0)), \quad \mathbf{u}_0 \in (C^{2,\gamma}(\bar{\Omega}_s(0)))^2, \\ &g \in C^{1,\gamma}(\partial\Omega_0 \times (0, \infty)), \quad \Delta\phi_0 = n_0 \ \text{ on } \Omega_s(0), \\ &\mathbf{u}_0 = -\boldsymbol{\nu} \ \text{ on } \mathcal{S}(0), \qquad \nabla\phi_0 \cdot \boldsymbol{\nu}_0 = g \ \text{ on } \partial\Omega_0 \times \{t = 0\}. \end{aligned}$$

- (A2) (Boundedness)

$$\begin{aligned} &\delta_{*1} \le n_0, \quad \max_{|\alpha| \le 1} ||\partial^\alpha n_0||_{0,\gamma,\bar{\Omega}_s(0)} + |||g|||_{2,\gamma,\partial\Omega_0\times[0,\infty))} \le \delta^*, \\ &\max_{i=1,2} \sum_{0 \le k \le 2} \max_{|\alpha| = k} ||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} \le \delta^*. \end{aligned}$$

- (A3) (Monotonicity and dissipativity) For $\mathbf{x} \in \Omega_s(0)$,

  (1) The real parts of the eigenvalues of $\nabla\mathbf{u}_0(\mathbf{x})$ are non-negative.
  (2) The initial velocity $\mathbf{u}_0$ is strongly dissipative in the sense that

$$\mathbf{u}_0(\mathbf{x}) \cdot \mathbf{x} \le -\eta_0 ||\mathbf{x}||^2 \quad \text{for some positive constant } \eta_0.$$

- (A4) (Consistency)

  Initial data $(n_0, \mathbf{u}_0, \phi_0)$ are given so that initial interface is contracting:

$$-1 < \tilde{V}_0 := -1 - \frac{\nabla\zeta_0 \cdot \boldsymbol{\nu}}{n_0} < 0,$$

  where ζ_0 is the unique solution of the exterior Neumann problem

$$\Delta\zeta_0 = 0 \quad \text{in } \Omega,$$

  with boundary data

$$\begin{cases} \nabla\zeta_0 \cdot \boldsymbol{\nu}_0 = h_0, \quad \text{on } \partial\Omega_0 \times \{t = 0\}, \\ \lim_{|x| \to \infty} \nabla\zeta_0 = \mathbf{0}, \end{cases}$$

  and $h_0 := \partial_t g - n_0(\mathbf{u}_0 \cdot \boldsymbol{\nu}_0)$.

- (A5) (Convexity and regularity of the target boundary)

  $\partial\Omega_0$ is convex and $C^{2,\gamma}$ so that the corresponding normal $\boldsymbol{\nu}_0$ is in $C^{1,\gamma}(\partial\Omega_0 \times [0, \infty))$:

$$\max_{i=1,2} |||\nu_{0i}|||_{1,\gamma,\partial\Omega_0\times[0,T]} \le \delta^*.$$

We next consider the approximate interface system is given as before by:

$$\begin{pmatrix} \partial_t \theta \\ \partial_t n_s \\ \partial_t r \end{pmatrix} - \frac{2\sin\beta}{r} \begin{pmatrix} \tilde{V}\cos\theta & 0 & 0 \\ 2n_s \sin\theta & \tilde{V}\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_\beta \theta \\ \partial_\beta n_s \\ \partial_\beta r \end{pmatrix} = \begin{pmatrix} -\frac{2\sin\theta\sin\beta\partial_\beta\tilde{V}}{r} \\ 0 \\ \frac{\tilde{V}}{\cos(\theta-\beta)} \end{pmatrix} \tag{7.64}$$

subject to initial data:

$$(\theta, n, r)(\beta, 0) = (\theta_0, n_{s0}, r_0)(\beta), \quad \beta \in \mathbb{R}.$$

We impose following conditions on initial data (B):

- (B1) (Regularity, boundedness and compatibility)
$$\begin{aligned} &(\theta_0, n_{s0}, r_0) \in (C^{3,\gamma}(\mathbb{R}))^3, \quad n_0 = n_{s0} \quad \text{on } \mathcal{S}(0), \\ &\delta_{*1} \le \min_{\beta\in\mathbb{R}} n_{s0}(\beta), \quad ||\theta_0||_{3,\gamma,\mathbb{R}} + ||n_{s0}||_{3,\gamma;\mathbb{R}} + ||r_0||_{3,\gamma;\mathbb{R}} \le \delta^*, \end{aligned}$$
where δ_{*1} and δ^* are positive constants.

- (B2) (Location of the initial sheath interface)
The target and initial sheath interface are sufficiently separated in the sense that
$$2r_b < \delta_{*2} \le \min_{\beta\in\mathbb{R}} r_0(\beta),$$
where r_b denotes the radius of the smallest circle with center 0 containing a target Ω_0 and δ_{*2} is a positive constant.

- (B3) (Convexity of initial interface)
$$|\theta_0 - \beta| \le \frac{\pi}{2} \quad \text{and} \quad \sin\beta\partial_\beta\theta_0 > 0.$$
- (B4) (Initial strict hyperbolicity)
$$\frac{\nabla\zeta_{s0}\cdot(-\sin\theta_0, \cos\theta_0)}{n_{s0}^2} > \frac{8\nabla\zeta_{s0}\cdot(\cos\theta_0, \sin\theta_0)}{n_{s0}}.$$
- (B5) (Consistency with initial data)
$$-(\cos\theta_0, \sin\theta_0)\cdot(r_0\cos\beta, r_0\sin\beta) \le -\eta_0 r_0^2, \qquad \beta \in \mathbb{R}.$$

Let r_1 be a sufficiently small positive number satisfying

$$0 < r_1 < \min\Big\{\delta^*, \frac{\delta_{*2}}{2} - r_b\Big\} \quad \text{and we set}$$

$$\begin{aligned} K_0 &:= 151\Big(\Big[\frac{4\pi\delta^*}{r_1}\Big] + 2\Big), \\ r_a &:= \text{the radius of the largest circle with a center 0 contained} \\ &\quad\ \text{inside the target } \Omega_0. \end{aligned}$$

We set T to be a sufficiently small positive constant satisfying the *a priori* bound:

$$0 < T \ll \min\Big\{1, \frac{1}{6K_0\delta^*}\Big(\frac{\delta_{*2}}{2} - r_b\Big), \frac{\eta_0 r_a^2}{2(6K_0(\delta^*)^2 + R_3)}\Big\}.$$

where R_3 is a positive constant depending only on $K_0, \delta^*, \delta_{*i}, i = 1, 2$ to be specified explicitly in Lemma A.9. In what follows, we use a simplified notation for space-time regions: For the positive constant T chosen as above,

$$\Omega_1 := B(0, 3\delta^*) - \Omega_0, \quad \Omega_* = \{\mathbf{x} : \frac{\delta_{*2}}{2} < |\mathbf{x}| < \delta^*\} \quad \text{and} \quad \Lambda(T) := \Omega_1 \times [0, T]. \tag{7.65}$$

Here $B(0, 3\delta^*)$ denotes the ball with a radius $3\delta^*$ and a center $\mathbf{x} = 0$.

Notice that $\Lambda(T)$ is a bounded region in space-time and the inclusion relation:

$$\Omega_0 \subset \Omega_1 \subset \Omega \quad \text{and} \quad \Omega_* \subset \Omega_1.$$

We define a subset of a Banach space $(C^{1,\gamma}(\bar{\Lambda}(T)))^2$ as follows.

Definition 7.1.

$$\begin{aligned}\mathcal{B}(T) \;:=\; & \{\mathbf{v} = (v_1, v_2) \in (C^{1,\gamma}(\bar{\Lambda}(T)))^2 : \partial^\alpha v_i \in C^{0,\gamma}(\bar{\Lambda}(T)), i = 1, 2, |\alpha| = 2, \\ & \textit{and } \mathbf{v} \textit{ satisfies the conditions } (\mathcal{D}) \textit{ below}\},\end{aligned}$$

$(\mathcal{D}1)$ $\mathbf{v}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ *on* $\Omega_s(0)$;

$(\mathcal{D}2)$ $\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{x} \le -\dfrac{\eta_0}{2}||\mathbf{x}||^2, \qquad (\mathbf{x}, t) \in (B(0, r_b + 6K_0\delta^* T) - \Omega_0) \times [0, T]$;

$(\mathcal{D}3)$ $\displaystyle\max_{i=1,2} \sum_{0\le k\le 2} \max_{|\alpha|=k} |||\partial^\alpha v_i|||_{0,\gamma,\bar{\Lambda}(T)} \le 3K_0\delta^*$;

$(\mathcal{D}4)$ $\displaystyle\max_{i=1,2} |||\partial_t v_i|||_{0,\gamma,\bar{\Lambda}(T)} \le K_0\Big(18(\delta^*)^2 + R_3\Big)$,

Remark 7.4. *1. The definition of the above set $\mathcal{B}(T)$ was motivated by the work of Nouri* [43].
2. $\mathcal{B}(T)$ is a compact convex subset of a Banach space $\mathcal{T}$

$$\mathcal{T} := \{\mathbf{v} \in C^{1,\tau}(\bar{\Lambda}(T)) : \partial^\alpha v_i \in C^{0,\tau}(\bar{\Lambda}(T)), i = 1, 2, |\alpha| = 2\} \tag{7.66}$$

for $\tau \in (0, \gamma)$ via the Arzela-Ascoli Theorem.
3. The dissipative condition $(\mathcal{D}2)$ guarantees that all characteristic curves passing through a point $(\mathbf{x}, t) \in (B(0, r_b + 3K_0\delta^* T) - \Omega_0) \times [0, T]$ reaches the initial sheath region $\Omega_s(0)$ backward in time and the target boundary $\partial\Omega_0$ forward in time.

In the following subsection, we present the construction of an iteration map and then state the main theorem for a local existence of sheath solutions. The detailed proof of the existence of local sheath solution is based on a series of lengthy and technical lemmas and these lemmas will be proved in Appendix A.

7.2.1. **Construction of an iteration map**. Let T be a given small positive number to be determined later. Next we describe the construction of an iteration map step by step.

- *Step 0.* Let $\mathbf{v} \in \mathcal{B}(\Lambda(T))$ be given.

- *Step 1.* Determine $\Lambda_s^1(T; \mathbf{v})$ and n in $\Lambda_s^1(T; \mathbf{v})$:

We solve a linear transport equation for n as follows:

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0, \qquad n(\mathbf{x}, 0) = n_0(\mathbf{x}). \tag{7.67}$$

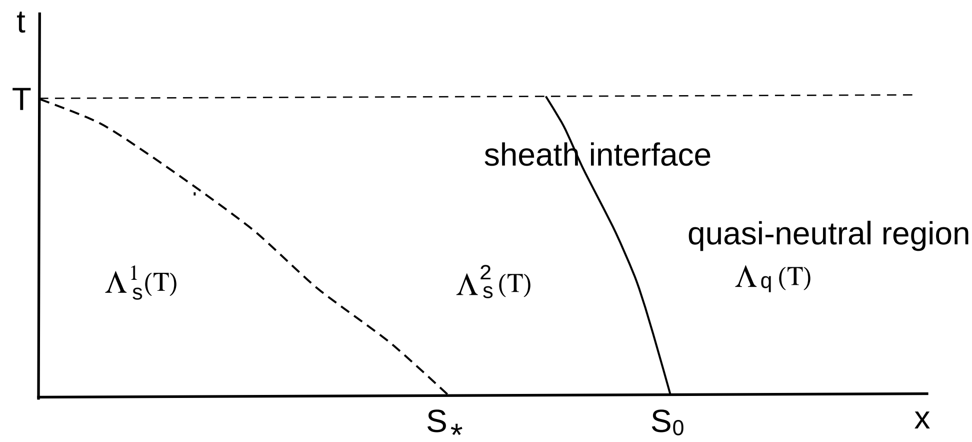

FIGURE 4. Schematic diagram of a space-time region

Define a space-time region $\Lambda_s^1(T;\mathbf{v})$ and $\Omega_s^1(0;\mathbf{v})$ (see Figure 4):

$\Lambda_s^1(T;\mathbf{v}) :=$ the region bounded by $\partial\Omega_0 \times [0,T]$ and backward characteristic surfaces issued from $\partial\Omega_0 \times \{t = T\}$,

$\Omega_s^1(s;\mathbf{v}) :=$ projection of the space-time region $\Lambda_s^1(T;\mathbf{v}) \cap (\mathbb{R}^2 \times \{t = s\})$ onto $\mathbb{R}^2$,

and a characteristic curve passing through $(\mathbf{x}, t)$: For $(\mathbf{x}, t) \in \Lambda_s^1(T;\mathbf{v})$,

$$\frac{d\boldsymbol{\chi}(s)}{ds} = \mathbf{v}(\boldsymbol{\chi}(s), s), \quad \boldsymbol{\chi}(t) = \mathbf{x}, \qquad 0 \le s \le T. \tag{7.68}$$

Since $\mathbf{v}$ is uniformly bounded in $C^{1,\gamma}$-norm, there exists a unique solution $\boldsymbol{\chi}(s, t, \mathbf{x})$ to (7.68), and we set

$$\boldsymbol{\alpha}(\mathbf{x}, t) = \boldsymbol{\chi}(0, t, \mathbf{x}), \quad (\mathbf{x}, t) \in \Lambda_s^1(T;\mathbf{v}).$$

On the other hand, along the characteristic n satisfies

$$\frac{d}{ds} \ln n(\boldsymbol{\chi}(s, t, \mathbf{x}), s) = -(\nabla \cdot \mathbf{v})(\boldsymbol{\chi}(s, t, \mathbf{x}), s) \quad 0 \le s \le T.$$

We integrate the above equation along $\boldsymbol{\chi}(\cdot, t, \mathbf{x})$ to get

$$n(\mathbf{x}, t) = n_0(\boldsymbol{\alpha}(\mathbf{x}, t)) \exp\Big(-\int_0^t (\nabla \cdot \mathbf{v})(\boldsymbol{\chi}(s, t, \mathbf{x}), s) ds\Big), \quad (\mathbf{x}, t) \in \Lambda_s^1(T;\mathbf{v}). \tag{7.69}$$

• *Step 2*: Determine the normal current $\mathbf{h} \cdot \boldsymbol{\nu} = \nabla\zeta \cdot \boldsymbol{\nu}$ on the interface.

We use the given boundary data g and n given by *Step 1* to compute

$$h_0 = \partial_t g - (n\mathbf{v})\Big|_{\partial\Omega_0} \cdot \boldsymbol{\nu}_{\mathbf{0}},$$

and solve the exterior Neumann problem for Laplace's equation: For given $t \in [0, T]$,

$$\begin{cases} \Delta\zeta(\mathbf{x}, t) = 0, & \mathbf{x} \in \Omega, \\ \nabla\zeta \cdot \boldsymbol{\nu}_0 = h_0 & \text{on } \partial\Omega_0 \text{ and } \quad \lim_{|\mathbf{x}|\to\infty} \nabla\zeta = \mathbf{0}. \end{cases} \tag{7.70}$$

- *Step 3*: Determine the location of the interface and the interfacial density.

With $\nabla\zeta$ determined by *Step 2*, we solve the interface system:

$$\begin{pmatrix} \partial_t\theta \\ \partial_t n_s \\ \partial_t r \end{pmatrix} - \frac{2\sin\beta}{r} \begin{pmatrix} \tilde{V}\cos\theta & 0 & 0 \\ 2n_s\sin\theta & \tilde{V}\cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_\beta\theta \\ \partial_\beta n_s \\ \partial_\beta r \end{pmatrix} = \begin{pmatrix} -\frac{2\sin\theta\sin\beta\partial_\beta\tilde{V}}{r} \\ 0 \\ \frac{\tilde{V}}{\cos(\theta-\beta)} \end{pmatrix}$$

subject to $C^{2,\gamma}$-initial data

$$(\theta, n_s, r)(\beta, 0) = (\theta_0, n_{s0}, r_0)(\beta) \in (C^{2,\gamma}(\mathbb{R}))^3,$$

satisfying the assumptions (B1) -(B3) and approximate normal velocity of the interface

$$\tilde{V} = -1 - \frac{\nabla\zeta \cdot (\cos\theta, \sin\theta)}{n_s},$$

where $\nabla\zeta$ is evaluated at $(r\cos\theta, r\sin\theta)$. Since $\nabla\zeta$ is uniformly bounded in $C^{1,\gamma}(\bar{\Lambda}(T))$ and $T \ll 1$, it follows from Theorem 7.2 that there exist $C^{1,\gamma}$ solutions (θ, n_s, r) such that

$$\frac{\delta_{*2}}{2} \le r(\beta, t) \le 2\delta^*, \quad (\beta, t) \in \mathbb{R} \times [0, T].$$

- *Step 4*: Determine $\Lambda_s^2(T; \mathbf{v})$, $\Omega_s^2(0; \mathbf{v})$ and n in $\Lambda_s^2(T; \mathbf{v})$.

In *Step 3*, we have determined the trajectory of the sheath interface $\mathcal{S}(t)$, and an ion density n_s on the interface. We first define $\Lambda_s^2(T; \mathbf{v}), \Omega_s^2(0; \mathbf{v})$ (see Figure 4):

$$\begin{aligned} \Lambda_s^2(T; \mathbf{v}) :=\ & \text{the region bounded by backward characteristic surfaces issued from} \\ & \partial\Omega_0 \times \{t = T\} \text{ and interfaces } \cup_{t\in[0,T]}\mathcal{S}(t), \\ \Omega_s^2(s; \mathbf{v}) :=\ & \text{projection of the space-time region } \Lambda_s^2(T) \cap (\mathbb{R}^2 \times \{t = s\}) \text{ onto } \mathbb{R}^2. \end{aligned}$$

We then repeat the same procedure as in *Step 1* to get n in $\Lambda_s^2(T; \mathbf{v})$, i.e., solve the linear transport equation

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0, \quad (\mathbf{x}, t) \in \Lambda_s^2(T; \mathbf{v}),$$

with initial and boundary data:

$$n(\mathbf{x}, 0) = n_0(\mathbf{x}) \quad \mathbf{x} \in \Omega_s^2(0) \quad \text{ and } \quad n(\mathbf{x}, t) = n_s(\mathbf{x}, t) \quad (\mathbf{x}, t) \in \mathcal{S}(t).$$

Let $(\boldsymbol{\alpha}, t_0)$ be the point on either $\Omega_s(0) \times \{t = 0\}$ or $\mathcal{S}(t_0) \times \{t = t_0\}$. We solve the equations for $\boldsymbol{\chi}$ and n as before:

$$\begin{cases} \frac{d}{ds}\boldsymbol{\chi}(s, t_0, \boldsymbol{\alpha}) = \mathbf{v}(\boldsymbol{\chi}(s, t_0, \boldsymbol{\alpha}), s), \quad s > t_0, \\ \frac{d}{ds}\ln n(\boldsymbol{\chi}(s, t_0, \boldsymbol{\alpha}), s) = -(\nabla \cdot \mathbf{v})(\boldsymbol{\chi}(s, t_0, \boldsymbol{\alpha}), s), \end{cases}$$

subject to initial and boundary data:

$$\boldsymbol{\chi}(t_0, t_0, \boldsymbol{\alpha}) = \boldsymbol{\alpha} \quad \text{and} \quad n(\boldsymbol{\alpha}, t_0) = \begin{cases} n_0(\boldsymbol{\alpha}) & t_0 = 0, \\ n_s(\boldsymbol{\alpha}, t_0) & t_0 > 0. \end{cases}$$

We set the approximate sheath region $\Lambda_s(T; \mathbf{v})$:

$$\Lambda_s(T; \mathbf{v}) := \Lambda_s^1(T; \mathbf{v}) \cup \Lambda_s^2(T; \mathbf{v}) \quad \text{and} \quad \Omega_s(t; \mathbf{v}) := \Omega_s^1(t; \mathbf{v}) \cup \Omega_s^2(t; \mathbf{v}).$$

• *Step 5*: Determine an electric field $\nabla\phi$ in $\Lambda_s(T; \mathbf{v})$.

¿From *Step 4*, we know n in the region $\Lambda_s(T; \mathbf{v})$, so we can now solve the Poisson equation with mixed Dirichlet-Neumann boundary data:

$$\begin{cases} \Delta\phi = n, \quad (\mathbf{x}, t) \in \Lambda_s(T; \mathbf{v}), \\ \nabla\phi \cdot \boldsymbol{\nu}_0 = g \quad \text{on } \partial\Omega_0 \quad \text{and} \quad \phi = -\ln n_s \quad \text{on } \mathcal{S}(t), \quad 0 \le t \le T. \end{cases}$$

• *Step 6*: Determine a new updated velocity $\hat{\mathbf{u}}$.

With the electric field $\nabla\phi$ from *step 5*, solve the inhomogeneous Burgers' equation:

$$\partial_t \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} = \nabla\phi,$$

with initial and boundary data:

$$\hat{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega_s(0), \quad \text{and} \quad \hat{\mathbf{u}} = -\boldsymbol{\nu} \quad \text{on } \mathcal{S}(t), \quad 0 \le t \le T.$$

Let $\hat{\boldsymbol{\chi}}(s, t_0, \boldsymbol{\alpha})$ be the characteristic curve issued from $(\boldsymbol{\alpha}, t_0)$. Then we have

$$\hat{\mathbf{u}}(\hat{\boldsymbol{\chi}}(t, t_0, \boldsymbol{\alpha})) = \begin{cases} \mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \nabla\phi(\hat{\boldsymbol{\chi}}(\mathrm{s}, 0, \boldsymbol{\alpha}), s) ds & t_0 = 0, \\ -\boldsymbol{\nu}(\boldsymbol{\alpha}) + \int_{t_0}^t \nabla\phi(\hat{\boldsymbol{\chi}}(s, t_0, \boldsymbol{\alpha}), s) ds & t_0 > 0. \end{cases}$$

Here $\hat{\boldsymbol{\chi}}$ is a characteristic curve associated with $\hat{\mathbf{u}}$. Since $\nabla\phi$ can be uniformly bounded by initial and boundary data and $T \ll 1$, the above inhomogeneous Burgers' equation is uniquely solvable.

• *Step 7*: Extend $\hat{\mathbf{u}}$ to the whole fixed domain $\Lambda(T)$.

We use Theorem 7.2 and the fact that the boundary data h_0 for (7.70) is uniformly bounded by the initial and boundary data for the sheath system to get

$$\Omega_s(t; \mathbf{v}) \subset B(0, 3\delta^*) \quad \text{for } t \in [0, T].$$

We then extend each component of $\hat{\mathbf{u}}$ to the fixed physical domain Ω_1 (see Appendix B). Finally we set

$$\mathbf{u}(\cdot, t) \equiv \mathcal{E}(\hat{\mathbf{u}}(\cdot, t)|\mathcal{S}(t)), \quad t \in [0, T].$$

and then define an iteration map $\mathcal{F}$ as follows:

$$\mathcal{F} : \mathcal{B}(\Lambda(T)) \to \mathcal{B}(\Lambda(T)); \qquad \mathcal{F}(\mathbf{v}) = \mathbf{u}.$$

Below we state the main theorem of this subsection which provides a local in time existence of sheath solutions to (2.6). The proof of the theorem and relevant lemmas are given in Appendix A.

Theorem 7.3. *Suppose the conditions (A1) - (A5) and (B1)-(B5) stated at the beginning of this subsection hold. Then there exists a sufficiently small positive time T and the sheath region $\Lambda_s(T;\mathbf{u})$ such that the sheath system (7.62) - (7.63) admits a smooth solution $(n,\mathbf{u},\phi)$ satisfying*

$$\begin{aligned}&n \in C^{1,\gamma}(\bar{\Lambda}_s(T;\mathbf{u})), \quad \mathbf{u} \in (C^{1,\gamma}(\bar{\Lambda}_s(T;\mathbf{u})))^2\\ &\partial^\alpha u_i \in C^{0,\gamma}(\bar{\Lambda}_s(T;\mathbf{u})), \quad \nabla\phi \in C^{1,\gamma}(\bar{\Lambda}_s(T;\mathbf{u})), \quad |\alpha|=2, \quad i=1,2.\end{aligned}$$

7.3. **Local existence for the exterior quasi-neutral system.** Recall the quasi-neutral system is given by the isothermal gas dynamics system:

$$\begin{cases}\partial_t n + \nabla_{\mathbf{x}}\cdot(n\mathbf{u}) = 0, \quad (\mathbf{x},t)\in\Omega_q\times(0,\infty),\\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + \nabla_{\mathbf{x}}(\ln n) = \mathbf{0}.\end{cases} \tag{7.71}$$

subject to initial and boundary data

$$\begin{aligned}(n,\mathbf{u})(\mathbf{x},0) &= (n_{q0},\mathbf{u}_{q0})(\mathbf{x}), \qquad \mathbf{x}\in\Omega_q(0),\\ (n,\mathbf{u})(\mathbf{x},t) &= (n_s,\mathbf{u}_s)(\mathbf{x},t) \qquad \text{on } \mathcal{S}(t),\end{aligned}$$

in the quasi-neutral region. We continue our discussions of Section 7.1 where the sheath edge is described by a curve:

$$x_2 = f(x_1,t),$$

in the $x_1 - x_2$ plane. Theorem 7.1 provided local existence and uniqueness in time for the sheath edge dynamics when V was replaced by the local "*small gradient*" approximation $\tilde{V}$. Thus Theorem 7.1 provides a smooth boundary $x_2 = f(x_1,t)$ and data:

$$n = n_s, \qquad \mathbf{u} = -\boldsymbol{\nu} \quad \text{on } x_2 = f(x_1,t), \quad t>0, \tag{7.72}$$

with $\boldsymbol{\nu} = (-\partial_{x_1} f, 1)\Big(1+(\partial_{x_1} f)^2\Big)^{-\frac{1}{2}}$ for the exterior quasi-neutral system.
In this section, we will sketch the proof of the following theorem for local existence-uniqueness of initial-boundary value problem for the exterior isothermal gas dynamics quasi-neutral system.

Theorem 7.4. *The exterior isothermal gas dynamics quasi-neutral system (7.71) with boundary data (7.72) and initial data*

$$n(x_1,x_2,0) = n_0(x_1,x_2), \quad \mathbf{u}(x_1,x_2,0) = \mathbf{u}_0(x_1,x_2), \quad x_2 > f(x_1,0),$$

has a unique classical smooth solution on $[0,T]$ for any compact subset of $\Omega_q = \{(x_1,x_2) : x_2 \geq f(x_1,t)\}$ where $T>0$ sufficiently small, provided the following conditions are satisfied:

(1) *the initial data for the sheath edge problem of Theorem 7.1 and the initial data for the exterior quasi-neutral problem are consistent and sufficiently smooth at $x_2 = f(x_1,0)$;*
(2) *the initial data is sufficiently smooth;*
(3) *$\tilde{V}$ is not -1 or 0 at $t=0$.*

Proof. The proof follows from the theorem of S. Schochet (Appendix A2 in [54]). We will apply Schochet's theorem via the obvious change of variables. First set

$$z = x_2 - f(x_1,t), \qquad U_1(x_1,z,t) = u_1(x_1, z + f(x_1,t), t),$$
$$U_2(x_1,z,t) = u_2(x_1, z + f(x_1,t), t), \qquad N(x_1,z,t) = n(x_1, z + f(x_1,t), t).$$

and let U_{1e}, U_{2e} and N_e be any smooth extension of the boundary data in the region $z = x_2 - f(x_1,t) > 0$. Next set

$$\bar{N} = N - N_e, \quad \bar{U}_1 = U - U_{1e}, \quad \bar{U}_2 = U - U_{2e}, \quad W = (\bar{N}, \bar{U}_1, \bar{U}_2).$$

A straightforward application of the chain rule shows the quasi-neutral system (7.71) becomes

$$A_0 \partial_t W + A_1 \partial_{x_1} W + A_2 \partial_z + BW = F,$$

where A_0, A_1, A_2 and B are 3×3 symmetric matrices that depend smoothly on W, x_1, z, t for $z \geq 0, t > 0, x_1 \in \mathbb{R}$ and W in a neighborhood of 0. F is a smooth function of (x_1, z, t) taking values in $\mathbb{R}^3$:

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\bar{N}+N_e)^2 & 0 \\ 0 & 0 & (\bar{N}+N_e)^2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \bar{U}_1 + U_{1e} & \bar{N}+N_e & 0 \\ \bar{N}+N_e & \bar{U}_1 + U_{1e} & 0 \\ 0 & 0 & (\bar{U}_1+U_{1e})(\bar{N}+N_e)^2 \end{pmatrix}$$

and let us set

$$Z \equiv -\partial_t f - (\bar{U}_1 + U_{1e}) \partial_{x_1} f + (\bar{U}_2 + U_{2e}).$$

Then

$$A_2 = \begin{pmatrix} Z & -\partial_{x_1}(\bar{N}+N_e) & \bar{N}+N_e \\ -\partial_{x_1}(\bar{N}+N_e) & Z(\bar{N}+N_e)^2 & 0 \\ \bar{N}+N_e & 0 & Z(\bar{N}+N_e)^2 \end{pmatrix}$$

and B, F are defined in the obvious way. The boundary condition (7.52) becomes

$$W = 0 \quad \text{at } z = 0, t > 0, x_1 \in \mathbb{R}.$$

Notice the boundary $z = 0$ is C^∞ and Schochet's boundary matrix $\mathcal{M}$ on $z = 0$ (see [54]) is just the identity whose null space is 0. Thus Schochet's conditions (ii) - (viii) are satisfied if we only consider the boundary $z = 0$. Also on the boundary $z = 0$, the boundary matrix of (v) is simply $-A_2$ and $\det A_2 = -n_s^4[1 + (\partial_{x_1} f)^2]^{\frac{3}{2}} \bar{V}(1 + \bar{V})$ when $W = 0$. Hence if $\bar{V} \neq 0, -1$, condition (v) of Schochet's theorem is also satisfied in the neighborhood of $W = 0$. Unfortunately Schochet's theorem as stated above requires a bounded domain with C^∞ boundary. However since the proof is based on his Theorem A1 which holds for planar boundaries, Theorem A2 holds in our case as well [53]. □

8. **Dynamics of the sheath interface: Extended bulk interface system**

In this section we suggest another approach to compute the sheath interface motion based on only bulk quantities. The idea is simple and follows from the general level set ideas recently summarized in the monograph of Osher and Fedkiw [44]. Recall that all quantities in the sheath interface evolution (5.24) can be extended as bulk variables defined on $\mathbb{R}^3 \times [0, \infty)$, say $\boldsymbol{\nu} = \frac{\nabla\psi}{|\nabla\psi|}$,etc. Hence if we solve the bulk system, location of the level set $\psi = 0$ will provide the true location of the sheath interface given the known value of $\nabla\zeta$ obtained from solving the interior sheath problem. This idea is explained below in further detail.

8.1. **Extended bulk interface system (EBI).** We follow the program noted above and introduce the extended bulk interface system (5.24) whose solution on the interface reduces to the original solution to the interface system. The bulk system for the extended bulk quantities ψ and n are:

$$\begin{cases} \partial_t \psi + V|\nabla\psi| = 0, \\ \partial_t n + V\boldsymbol{\nu}\cdot\nabla n = n\nabla\cdot\boldsymbol{\nu}, \\ V + 1 + \dfrac{\mathbf{h}\cdot\boldsymbol{\nu}}{n} = -\dfrac{1}{n}\Big[\nabla\cdot(V\nabla\ln n) - \nabla\cdot\Big((V\nabla\ln n\cdot\boldsymbol{\nu})\boldsymbol{\nu}\Big)\Big] \\ \qquad\qquad -\dfrac{1}{n}\Big[\nabla(V\nabla\ln n)(\boldsymbol{\nu},\boldsymbol{\nu}) - \nabla\Big((V\nabla\ln n\cdot\boldsymbol{\nu})\boldsymbol{\nu}\Big)(\boldsymbol{\nu},\boldsymbol{\nu})\Big], \end{cases} \tag{8.73}$$

where $\nabla\zeta$ is known from solving the interior sheath system and $\boldsymbol{\nu} = \dfrac{\nabla\psi}{|\nabla\psi|}$. Notice on level sets $\psi = 0$, (5.24) and (8.73) are identical. The advantage of (8.73) is that it is defined via "*bulk*" quantities and the evolution of the level set $\psi = 0$ may be obtained by solving (8.73) on $(\mathbb{R}^2 - \Omega_0)\times[0,\infty)$.

8.1.1. **Special solutions to EBI**. In this part, we show consistency of the **(EBI)** approach with the earlier results of [37] for planar, cylindrical and spherical symmetric motions. Hence we first look for the special solutions of (8.73) which are planar and radially symmetric.

We first consider planar solutions with the following ansatz:

$$\psi(\mathbf{x},t) = x_1 - s(t) \quad n(\mathbf{x},t) = n(r,t) \quad \text{and} \quad \mathbf{h} = (h,0,0).$$

In this case, since all surface gradients and a curvature term $\nabla\cdot\boldsymbol{\nu}$ vanish, **EBI** system (8.73) becomes

$$\frac{\delta\psi}{\delta t} = 0, \quad \frac{\delta n}{\delta t} = 0 \quad \text{and} \quad V = -1 - \frac{h}{n}.$$

The relation $\dfrac{\delta}{\delta t} = \partial_t + V\boldsymbol{\nu}\cdot\nabla$ yields a system for planar solutions (ψ, n, h):

$$-\dot{s} + V = 0, \qquad \partial_t n + \dot{s}\partial_{x_1} n = 0 \quad \text{and} \quad V = -1 - \frac{h}{n}. \tag{8.74}$$

It is easy to see that

$$n(\mathbf{x},t) = n_0(\text{ constant }), \quad \dot{s}(t) = -1 - \frac{h}{n_0}, \quad \text{and} \quad \psi(\mathbf{x},t) = x_1 - s(t). \tag{8.75}$$

are solutions of (8.74) and this is consistent with [37].

Next we consider radially symmetric solutions so that (ψ, n, h) only depends on the radial distance r and time t. We take the ansatz ψ as:

$$\psi(r,t) = r - s(t).$$

Under the above ansatz for ψ, the system (8.73) becomes

$$V = \dot{s}(t), \qquad \partial_t n + V\partial_r n = \frac{2n}{r}, \qquad \dot{s} + 1 + \frac{h_r}{n} = 0,$$

where h_r is the radial component of the current $\mathbf{h}$. We combine the first two equations to get

$$\partial_t n + \dot{s}\partial_r n = \frac{2n}{r}, \qquad \dot{s} + 1 + \frac{h_r}{n} = 0. \tag{8.76}$$

We define $\hat{s}(\alpha,t)$ by the characteristic curve (particle path) issued from α corresponding to the first equation of (8.76), i.e.,

$$\frac{d\hat{s}(\alpha,t)}{dt} = \dot{s}(t), \qquad \hat{s}(\alpha,0) = \alpha.$$

Then along the characteristic $r = \hat{s}(\alpha,\cdot)$, the system (8.76) becomes

$$\frac{dn(\hat{s}(t),t)}{dt} = \frac{2n(\hat{s}(t),t)}{\hat{s}(t)}, \qquad \dot{\hat{s}}(t) + 1 + \frac{h_r(\hat{s}(t),t)}{n(\hat{s}(t),t)} = 0. \tag{8.77}$$

We now take the time-derivative of the second equation in (8.77) to get

$$\ddot{\hat{s}}(t) + \frac{2(\dot{\hat{s}}(t)+1)}{\hat{s}(t)} - \frac{(\dot{\hat{s}}(t)+1)(\partial_r h_r \dot{\hat{s}}(t) + \partial_t h_r)}{n} = 0.$$

Here h_r and n are evaluated at the particle path $(\hat{s}(t),t)$. We take the ansatz for the $h_r(r,t) = \frac{h(t)}{r^2}$. By direct calculation we have

$$\ddot{\hat{s}}(t) + \frac{2(\dot{\hat{s}}(t)+1)^2}{\hat{s}(t)} - \frac{\dot{h}(t)(\dot{\hat{s}}(t)+1)}{h(t)} = 0.$$

and hence on the interface $\psi(r,t) = 0$ we recover the 2nd order equation for $\hat{s}(t)$ which is identical with the result in [37]. The cylindrical case can be done analogously.

8.2. **Approximate extended bulk interface system.** While we view the **(EBI)** approach as a promising method for numerical computation, in this subsection we give a more modest application and derive an approximate solution of the bulk interface motion when the target is a small "*transversal*" perturbation of an infinite plane. We employ a formal asymptotic expansion around the special solution (ψ_0, n_0, h_0) satisfying (8.75).

8.2.1. **Small transversal perturbation of a planar target**. Let us choose a small parameter μ and we formally expand ψ, n and $\mathbf{h}$ in terms of power series of μ:

$$\begin{aligned} \psi(\mathbf{x},t) &= x_1 - s(t) + \mu\psi_1(x_2,x_3,t) + \cdots, \\ n(\mathbf{x},t) &= n_0 + \mu n_1(x_2,x_3,t) + \cdots, \\ \mathbf{h}(\mathbf{x},t) &= (h_0(t),0,0) + \mu\mathbf{h}_1(x_2,x_3,t) + \cdots, \end{aligned}$$

where n_0 is a positive constant and $\mathbf{h}_1 = (h_{11}, h_{12}, h_{13})$ and

$$\psi_0(\mathbf{x},t) = x_1 - s(t), \quad n(\mathbf{x},t) = n_0 \quad \text{and} \quad h_0(t)$$

are exact solutions to the extended **EBI** with the relation:

$$\dot{s}(t) = -1 - \frac{h_0(t)}{\bar{n}_0}.$$

Below, we formally denote the higher-order terms by "$\cdots$".

Next we derive a system for $(\psi_1, n_1, \mathbf{h}_1)$:

$$\begin{cases} \partial_t \psi_1 - \dfrac{h_{11}(x_2,x_3,t)}{n_0} - \dfrac{h_0(t)n_1}{n_0} + \dfrac{1}{n_0^2}\Big(1+\dfrac{h_0(t)}{n_0}\Big)\Delta n_1 = 0, \\ \partial_t n_1 = n_0(\partial^2_{x_2}\psi_1 + \partial^2_{x_3}\psi_1). \end{cases} \tag{8.78}$$

or equivalently, we have a linear plate equation for n_1:

$$\partial_t^2 n_1 + \frac{h_0}{n_0}(\partial^2_{x_2}n_1 + \partial^2_{x_3}n_1) + \frac{1}{n_0}\Big(1+\frac{h_0}{n_0}\Big)(\partial^2_{x_2}+\partial^2_{x_3})^2 n_1 = \partial^2_{x_2}h_{11} + \partial^2_{x_3}h_{11}.$$

- *Derivation of (8.78).* We claim:

$$\boldsymbol{\nu}\cdot\nabla n = \mathcal{O}(\mu^2), \qquad \frac{\mathbf{h}\cdot\boldsymbol{\nu}}{n} = \frac{h_0}{n_0} + \mu\Big(\frac{h_{11}}{n_0} - \frac{h_0 n_1}{n_0^2}\Big) + \mathcal{O}(\mu^2),$$
$$V = -1 - \frac{h_0}{n_0} - \mu\frac{h_{11}}{n_0} - \mu\frac{h_0}{n_1}n_0^2 + \frac{\mu}{n_0^2}\Big(1+\frac{h_0}{n_0}\Big)\Delta n_1 + \mathcal{O}(\mu^2).$$

Below we will check the above claim. It follows from the ansatz for ψ that

$$\nabla\psi = \mathbf{e}_1 + \mu\nabla\psi_1 + \cdots,$$
$$\partial_t\psi = -\dot{s}(t) + \mu\partial_t\psi_1 + \cdots,$$

where $\mathbf{e}_1$ is the unit coordinate vector in x_1-axis, i.e., $\mathbf{e}_1 = (1,0,0)$. Since $\partial_{x_1}\psi_1 = 0$,

$$|\nabla\psi| = 1 + \mathcal{O}(\mu), \qquad \boldsymbol{\nu} = \mathbf{e}_1 + \mu(0, \partial_{x_2}\psi_1, \partial_{x_3}\psi_1) + \mathcal{O}(\mu^2),$$
$$\frac{\mathbf{h}\cdot\boldsymbol{\nu}}{n} = \frac{h_0(t)}{n_0} + \mu\frac{h_{11}}{n_0} - \mu\frac{h_0 n_1}{n_0^2} + \mathcal{O}(\mu^2).$$

By direct calculation we have

$$\begin{aligned} \boldsymbol{\nu}\cdot\nabla n &= \Big(\mathbf{e}_1 + \mu(0,\partial_{x_2}\psi_1,\partial_{x_3}\psi_1) + \mathcal{O}(\mu^2)\Big)\cdot\Big(\mu\nabla n_1 + \mathcal{O}(\mu^2)\Big) \\ &= \mu\partial_{x_1}n_1 + \mathcal{O}(\mu^2) = \mathcal{O}(\mu^2). \end{aligned}$$

Recall that

$$\nabla_s = \nabla - \boldsymbol{\nu}(\boldsymbol{\nu}\cdot\nabla),$$

i.e.,

$$\nabla_s = \nabla - \Big(\mathbf{e}_1 + \mu\nabla\psi_1\Big)(\partial_{x_1} + \mu\nabla\psi_1\cdot\nabla) + \mathcal{O}(\mu^2),$$

where $\nabla\psi_1 = (0, \partial_{x_2}\psi_1, \partial_{x_3}\psi_1)$. Hence for any scalar quantity $w(x_2,x_3,t)$, we have

$$\nabla_s w = \nabla w - (\partial_{x_1}w)\mathbf{e}_1 - \mu\Big((\partial_{x_1}w)\nabla\psi_1 + (\nabla\psi\cdot\nabla w)\mathbf{e}_1\Big) + \mathcal{O}(\mu^2). \tag{8.79}$$

We apply the above equation (8.79) to $w = \ln n$ to obtain

$$\nabla_s \ln n = \frac{\nabla_s n}{\bar{n}} = \mu\frac{\nabla n_1}{n_0} + \mathcal{O}(\mu^2).$$

Hence we have

$$-\frac{1}{n}\nabla_s(V\nabla_s\ln n) = -\frac{\mu}{n_0^2}\nabla\Big[\Big(-1-\frac{h_0(t)}{n_0}\Big)\nabla n_1\Big] + \mathcal{O}(\mu^2) \quad \text{and}$$
$$V = -1 - \frac{h_0}{n_0} - \mu\frac{h_{11}}{n_0} - \mu\frac{h_0}{n_1}n_0^2 + \frac{\mu}{n_0^2}\Big(1+\frac{h_0}{n_0}\Big)\Delta n_1 + \mathcal{O}(\mu^2).$$

We now equate order μ-terms in our **EBI** system to see

$$\partial_t\psi_1 - \frac{h_{11}}{n_0} - \frac{h_0 n_1}{n_0} + \frac{1}{n_0^2}\Big(1+\frac{h_0}{n_0}\Big)\Delta_{(x_2,x_3)}n_1 = 0, \tag{8.80}$$

$$\partial_t n_1 = n_0(\partial^2_{x_2}\psi_1 + \partial^2_{x_3}\psi_1). \tag{8.81}$$

Finally we apply the differential operators $\Delta_{(x_2,x_3)}$ and ∂_t to (8.80) and (8.81) respectively to obtain

$$\partial_t^2 n_1 + h_0\Delta_{(x_2,x_3)}n_1 + \frac{1}{n_0}\Big(1+\frac{h_0}{n_0}\Big)\Delta^2_{(x_2,x_3)}n_1 = \Delta_{(x_2,x_3)}h_{11}. \tag{8.82}$$

Since $h_0 < 0$, the above equation has the structure of an equation for a vibrating plate. Note it is well-posed if and only if

$$1+\frac{h_0}{n_0} > 0.$$

The motion of ψ is constructed by the insertion of the solution $\bar{n}_1$ of (8.82) into the first equation of (8.78). The level set $\psi = 0$ tracks the motion of the "*vibrating plate*" via the relation:

$$x_1 = s(t) - \mu\psi_1(x_2,x_3,t) + \mathcal{O}(\mu^2).$$

Remark 8.1. *Notice here we have not made the "small gradient" approximation $V = \tilde{V}$ and the role of the surface derivatives in the original implicit constitutive equation for V becomes apparent.*

Appendix A. **Local existence of sheath solutions**

In this appendix, we present a series of a priori estimates for the approximate solutions constructed in *Step 0 - Step 7* in Section 7 and then give the proof Theorem 7.3.

A.1. **Basic a priori estimates.**

In this part, we give *a priori* estimates for the approximate solutions constructed in *Step 0 - Step 7*.

A.1.1. **A priori estimates for** *Step1*.

In Lemmas A1-A3, we will give a proof of the existence, uniqueness and regularity for n as given in *Step 1* of Section 7.2.

We first consider the equation for a characteristic curve. For given $(\mathbf{x}, t)$,

$$\partial_s \boldsymbol{\chi}(s,t,\mathbf{x}) = \mathbf{v}(\boldsymbol{\chi}(s,t,\mathbf{x}), s), \quad \boldsymbol{\chi}(t,t,\mathbf{x}) = \mathbf{x}, \qquad 0 \le s \le T. \tag{A.83}$$

In what follows, we will use calculus type estimates for the Hölder seminorm. For $f_i \in C^{0,\gamma}(\bar{\Lambda}_s(T;\mathbf{v}))$ $i = 1, 2$, we have

$$[[f_1 f_2]]_{0,\gamma} \le [[f_1]]_{0,\gamma} |||f_2|||_0 + |||f_1|||_0 [[f_2]]_{0,\gamma}, \tag{A.84}$$

$$[[e^f]]_{0,\gamma} \le e^{|||f|||_0} [[f]]_{0,\gamma}. \tag{A.85}$$

Here $[[\cdot]]_{0,\gamma}$ and $|||\cdot|||_0$ denote the Hölder and *esssup* norms defined on the same space-time region.

In the following Lemma, we use simplified notation for balls in $\mathbb{R}^2$:

$$B_1 := B(0, r_b + 3K_0\delta^* T_0) \qquad \text{and} \qquad B_2 := B(0, r_b + 6K_0\delta^* T_0).$$

Lemma A.1. *There exists a sufficiently small constant $T_0 > 0$ and a unique solution $\boldsymbol{\chi}$ to the equation (A.83) satisfying the following estimates: For $0 < T \le T_0$, $\mathbf{v} \in \mathcal{B}(T)$,*

1. *The forward characteristic curve $\boldsymbol{\chi}(s, 0, \mathbf{x})$, $s \ge 0$, $\mathbf{x} \in \Omega_s^1(0;\mathbf{v}) \subset (B_1 - \Omega_0)$ hits the target boundary $\partial\Omega_0$ and the ion-density in the region $\Lambda_s^1(T;\mathbf{v})$ is given by*
$$n(\boldsymbol{\chi}(t,0,\mathbf{x}),t) = n_0(\mathbf{x}) \exp\Big(- \int_0^t (\nabla\cdot\mathbf{v})(\boldsymbol{\chi}(s,0,\mathbf{x}),s)ds\Big), \quad \mathbf{x} \in \Omega_s^1(0;\mathbf{v}).$$
2. *$\boldsymbol{\chi}(s,t,\mathbf{x}) \in C^{1,\gamma}([0,T]\times[0,T]\times\mathbb{R}^2)$ and $\sup_{s,t,\mathbf{x}} \max_{i,j=1,2} |\partial_{x_j}\chi^i| \le 2.$*
3. *Suppose that $\mathbf{v}_i \to \mathbf{v}$ in $C^{1,\gamma}(\bar{\Lambda}(T))$ and let $\boldsymbol{\chi}_i$ and $\boldsymbol{\chi}$ be the characteristic curves corresponding to $\mathbf{v}_i$ and $\mathbf{v}$ respectively. Then for $(\mathbf{x}, t) \in \Lambda_s^1(T;\mathbf{v})$,*
$$\boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) \to \boldsymbol{\chi}(\cdot,t,\mathbf{x}) \quad \textit{in } C^{1,\gamma}([0,T]).$$
4. *$\boldsymbol{\alpha}(\mathbf{x},t) := \boldsymbol{\chi}(0,t,\mathbf{x})$ is Lipschitz continuous in $(\mathbf{x},t) \in \Lambda(T)$ with a Lipschitz constant 4, i.e.*
$$|\boldsymbol{\alpha}(\mathbf{x},t) - \boldsymbol{\alpha}(\mathbf{y},s)| \le 4|(\mathbf{x},t) - (\mathbf{y},s)|.$$

Proof. (i) It follows from the dissipative condition ($\mathcal{D}2$) in the definition of $\mathcal{B}(T)$, we have

$$\mathbf{v}(\mathbf{x},t)\cdot\mathbf{x} \le -\frac{\eta_0}{2}|\mathbf{x}|^2, \quad (\mathbf{x},t) \in (B_2 - \Omega_0)\times[0,T].$$

Then we have for $(\mathbf{x},t) \in (B_2 - \Omega_0)\times[0,T]$,

$$\frac{d}{ds}|\boldsymbol{\chi}(s,t,\mathbf{x})|^2 = 2\langle \mathbf{v}(\boldsymbol{\chi}(s,t,\mathbf{x}),s), \boldsymbol{\chi}(s,t,\mathbf{x})\rangle \le -\eta_0 |\boldsymbol{\chi}(s,t,\mathbf{x})|^2.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^2$. Hence the characteristic $\boldsymbol{\chi}(s, t, \mathbf{x})$ satisfies

$$|\boldsymbol{\chi}(s, 0, \mathbf{x})| \le e^{-\frac{\eta_0 s}{2}} |\boldsymbol{\chi}(0, 0, \mathbf{x})| = e^{-\frac{\eta_0 s}{2}} |\mathbf{x}|.$$

So $\boldsymbol{\chi}(s, 0, \mathbf{x})$ has decreasing magnitude and must hit the target at some positive s.

Let $T \le T_0$ and we define the subregions $\Lambda_s^1(T; \mathbf{v}), \Omega_s^1(T; \mathbf{v})$ of $\Lambda(T)$ and $\Omega_s(0)$ as in *Step 1* of Section 7.2.1. Then the characteristic curve $\boldsymbol{\chi}(s, 0, \mathbf{x}), \quad (\mathbf{x}, 0) \in \Omega_s^1(0) \times \{t = 0\}$ hits the target boundary $\partial\Omega_0$ and will provide the ion density n at the target boundary, i.e.,

$$n(\boldsymbol{\chi}(t, 0, \mathbf{x}), t) = n_0(\mathbf{x}) \exp\Big(- \int_0^t (\nabla \cdot \mathbf{v})(\boldsymbol{\chi}(s, 0, \mathbf{x}), s) ds\Big), \quad \mathbf{x} \in \Omega_s^1(0; \mathbf{v}).$$

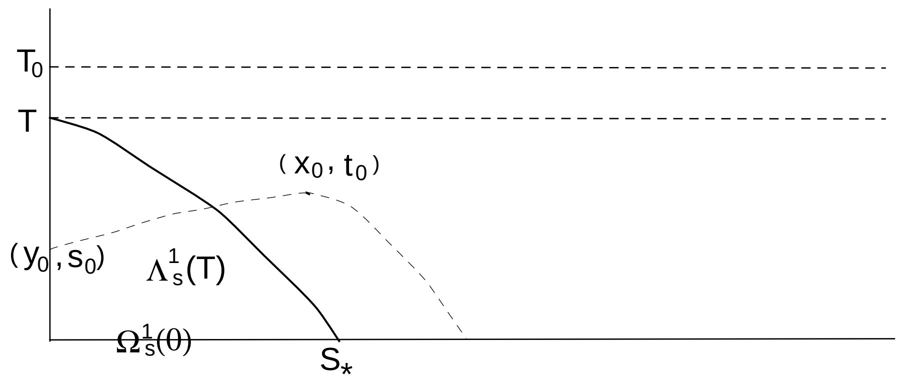

FIGURE 5. Schematic diagram of the geometry of characteristic curves

Remark A.1. *We briefly summarize the geometry of characteristic curves in the space-time region* $\Lambda(T)$ *(see Figure 5).*

The region $\Lambda_s^1(T)$ *will be completely covered by the characteristic curves* $\boldsymbol{\chi}(s, 0, \mathbf{x}), (\mathbf{x}, 0) \in \Omega_s^1(0; \mathbf{v}) \times \{t = 0\}$ *and they are pointing toward the target for positive* s*. On the other hand, all backward characteristic curves* $\boldsymbol{\chi}(s, t, \mathbf{x}), 0 \le s \le t, (\mathbf{x}, t) \in \Lambda(T) - \Lambda_s^1(T)$ *will either hit the initial region* $(B(0, 3\delta^*) - \Omega_0) \times \{t = 0\}$ *at* $s = 0$ *or the target boundary* $\partial\Omega_0$ *at some* $s \in [0, t)$ *(see Figure 5). However the latter situation will not happen, for example, suppose the backward characteristic curve* $\boldsymbol{\chi}(s, t_0, \mathbf{x}_0), 0 \le s \le t_0, (\mathbf{x}_0, t_0) \in \Lambda(T) - \Lambda_s^1(T)$ *hits the target boundary at* $s = s_0$ *at* $\mathbf{y}_0$*:*

$$\mathbf{y}_0 := \boldsymbol{\chi}(s_0, t_0, \mathbf{x}_0).$$

Then forward characteristic curve $\boldsymbol{\chi}(s, s_0, \mathbf{y}_0), s \in [s_0, t_0]$ *will have the same image of a trajectory as* $\boldsymbol{\chi}(s, t_0, \mathbf{x}_0), s \in [s_0, t_0], (\mathbf{x}_0, t_0) \in \Lambda(T) - \Lambda_s^1(T)$*, but this is impossible since by the strong dissipation assumption* $\mathcal{D}2$ *in the Definition 7.1, no forward characteristic curves will be issued from the target boundary.*

(ii) The first part of the proof for (1) follows from the standard theory of ordinary differential equations. In fact we gain regularity in the s-variable, i.e.,

$$\boldsymbol{\chi}(\cdot, t, \mathbf{x}) \in C^{2,\gamma}([0, T]).$$

We differentiate (A.83) with respect to x_j to get

$$\begin{cases} \partial_s \partial_{x_j} \chi^k(s,t,\mathbf{x}) = \nabla v_k(\boldsymbol{\chi}(s,t,\mathbf{x}), s) \cdot \partial_{x_j} \boldsymbol{\chi}(s,t,\mathbf{x}), \quad 0 \le s \le T \quad k,j \in \{1,2\}, \\ \partial_{x_j} \chi^k(t,t,\mathbf{x}) = \delta_{jk}. \end{cases}$$

Here δ_{jk} is a Kronecker delta function and χ^k is the k-th component of $\boldsymbol{\chi}$, k=1,2.

We integrate the above equation along the characteristic curve $\boldsymbol{\chi}$ to see

$$\partial_{x_j} \chi^k(\xi,t,\mathbf{x}) = \delta_{jk} - \int_\xi^t \nabla v_k(\boldsymbol{\chi}(s,t,\mathbf{x}), s) \cdot \partial_{x_j} \boldsymbol{\chi}(s,t,\mathbf{x}) ds.$$

The above relation implies

$$\sup_{s,t,\mathbf{x}} \max_{k,j=1,2} |\partial_{x_j} \chi^k| \le 1 + 6K_0 \delta^*(t-\xi) \sup_{s,t,\mathbf{x}} \max_{k,j=1,2} |\partial_{x_j} \chi^k|.$$

Since $t - \xi \le T \ll 1$, we have

$$\sup_{s,t,\mathbf{x}} \max_{k,j=1,2} |\partial_{x_j} \chi^k| \le 2.$$

(iii) Consider the equations for $\boldsymbol{\chi}_i$ and $\boldsymbol{\chi}$: For $(\mathbf{x}, t) \in \Lambda_s^1(T; \mathbf{v})$,

$$\begin{cases} \partial_\xi \boldsymbol{\chi}_i(\xi,t,\mathbf{x}) = \mathbf{v}_i(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}), \xi), \\ \boldsymbol{\chi}_i(t,t,\mathbf{x}) = \mathbf{x}, \end{cases} \quad \text{and} \quad \begin{cases} \partial_\xi \boldsymbol{\chi}(\xi,t,\mathbf{x}) = \mathbf{v}(\boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi), \\ \boldsymbol{\chi}(t,t,\mathbf{x}) = \mathbf{x}. \end{cases}$$

We use the above equations to calculate $\boldsymbol{\chi}_i(\xi,t,\mathbf{x}) - \boldsymbol{\chi}(\xi,t,\mathbf{x})$, and integrate in ξ from $\xi = s$ to $\xi = t$ to get

$$\begin{aligned} \boldsymbol{\chi}_i(s,t,\mathbf{x}) - \boldsymbol{\chi}(s,t,\mathbf{x}) &= -\int_s^t \Big(\mathbf{v}_i(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}), \xi) - \mathbf{v}(\boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi) \Big) d\xi \\ &= -\int_s^t \Big(\mathbf{v}_i(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}), \xi) - \mathbf{v}_i(\boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi) \Big) d\xi \\ &- \int_s^t \Big(\mathbf{v}_i(\boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi) - \mathbf{v}(\boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi) \Big) d\xi. \end{aligned}$$

(A.86)

Here we used $\boldsymbol{\chi}_i(t,t,\mathbf{x}) = \boldsymbol{\chi}(t,t,\mathbf{x}) = \mathbf{x}$ and note that

$$\begin{aligned} &\int_s^t \Big(\mathbf{v}_i(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}), \xi) - \mathbf{v}_i(\boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi) \Big) d\xi \\ &= \int_s^t \int_0^1 \partial_{s_1} \mathbf{v}_i \Big(\boldsymbol{\chi}(\xi,t,\mathbf{x}) + s_1(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}) - \boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi \Big) ds_1 d\xi \\ &= \int_s^t \int_0^1 \nabla_{\mathbf{x}} \mathbf{v}_i \Big(\boldsymbol{\chi}(\xi,t,\mathbf{x}) + s_1(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}) - \boldsymbol{\chi}(\xi,t,\mathbf{x}), \xi \Big) \cdot \Big(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}) - \boldsymbol{\chi}(\xi,t,\mathbf{x}) \Big) ds_1 d\xi. \end{aligned}$$

(A.87)

We now take the $\mathbb{R}^2$-norm in (A.86) and use (A.87) to see

$$\begin{aligned} &|\boldsymbol{\chi}_i(s,t,\mathbf{x}) - \boldsymbol{\chi}(s,t,\mathbf{x})| \\ &\qquad \le |||\nabla \mathbf{v}_i|||_{0,\bar{\Lambda}(T)} \int_s^t |\boldsymbol{\chi}_i(\xi,t,\mathbf{x}) - \boldsymbol{\chi}(\xi,t,\mathbf{x})| d\xi + |||\mathbf{v}_i - \mathbf{v}|||_{0,\bar{\Lambda}(T)} (t-s). \end{aligned}$$

Note that Gronwall's inequality yields

$$|\boldsymbol{\chi}_i(s,t,\mathbf{x}) - \boldsymbol{\chi}(s,t,\mathbf{x})| \\ \leq |||\mathbf{v}_i - \mathbf{v}|||_{0,\bar{\Lambda}(T)}(t-s)\Big(1 + |||\nabla \mathbf{v}_i|||_{0,\bar{\Lambda}(T)}(t-s)e^{|||\nabla \mathbf{v}_i|||_{0,\bar{\Lambda}(T)}(t-s)}\Big). \tag{A.88}$$

By hypothesis (2) of this lemma, we have $\mathbf{v}_i \to \mathbf{v}$ in $C^{1,\gamma}(\bar{\Lambda}(T))$ as $t \to \infty$, and this implies from (A.88) that

$$||\boldsymbol{\chi}_i(s,t,\mathbf{x}) - \boldsymbol{\chi}(s,t,\mathbf{x})||_{0,[0,T]} \to 0 \quad \text{as } i \to \infty. \tag{A.89}$$

Next we show

$$||\partial_s \boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \partial_s \boldsymbol{\chi}(\cdot,t,\mathbf{x})||_{0,[0,T]} \to 0 \quad \text{as } i \to \infty. \tag{A.90}$$

Note that (A.83) implies

$$||\partial_s \boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \partial_s \boldsymbol{\chi}(\cdot,t,\mathbf{x})||_{0,[0,T]} \\ \leq ||\mathbf{v}_i(\boldsymbol{\chi}_i(\cdot,t,\mathbf{x}),\cdot) - \mathbf{v}_i(\boldsymbol{\chi}(\cdot,t,\mathbf{x}),\cdot)||_{0,[0,T]} + ||\mathbf{v}_i(\boldsymbol{\chi}(\cdot,t,\mathbf{x}),\cdot) - \mathbf{v}(\boldsymbol{\chi}(\cdot,t,\mathbf{x}),\cdot)||_{0,[0,T]} \\ \leq |||\nabla \mathbf{v}_i|||_{0,\bar{\Lambda}(T)}||\boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \boldsymbol{\chi}(\cdot,t,\mathbf{x})||_{0,[0,T]} + |||\mathbf{v}_i - \mathbf{v}|||_{0,\bar{\Lambda}(T)} \to 0 \quad \text{as } i \to \infty.$$

We use (A.90) to show

$$[\boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \boldsymbol{\chi}(\cdot,t,\mathbf{x})]_{0,\gamma,[0,T]} \to 0, \quad \text{as } i \to \infty. \tag{A.91}$$

By direct calculation we have

$$\frac{|(\boldsymbol{\chi}_i - \boldsymbol{\chi})(s_1,t,\mathbf{x}) - (\boldsymbol{\chi}_i - \boldsymbol{\chi})(s_2,t,\mathbf{x})|}{|s_1 - s_2|^\gamma} \\ \leq ||\partial_s \boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \partial_s \boldsymbol{\chi}(\cdot,t,\mathbf{x})||_{0,[0,T]}|s_1 - s_2|^{1-\gamma} \\ \leq ||\partial_s \boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \partial_s \boldsymbol{\chi}(\cdot,t,\mathbf{x})||_{0,[0,T]}T^{1-\gamma} \to 0 \quad \text{as } i \to \infty.$$

Next we show

$$[\partial_s \boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \partial_s \boldsymbol{\chi}(\cdot,t,\mathbf{x})]_{0,\gamma,[0,T]} \to 0, \quad \text{as } i \to \infty. \tag{A.92}$$

It follows from (A.83) and (A.84) that

$$[\partial_s \boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \partial_s \boldsymbol{\chi}(\cdot,t,\mathbf{x})]_{0,\gamma,[0,T]} \\ \leq [\mathbf{v}_i(\boldsymbol{\chi}_i(\cdot,t,\mathbf{x}),\cdot) - \mathbf{v}_i(\boldsymbol{\chi}(\cdot,t,\mathbf{x}),\cdot)]_{0,\gamma,[0,T]} + [\mathbf{v}_i(\boldsymbol{\chi}(\cdot,t,\mathbf{x}),\cdot) - \mathbf{v}(\boldsymbol{\chi}(\cdot,t,\mathbf{x}),\cdot)]_{0,\gamma,[0,T]} \\ \leq |||\nabla \mathbf{v}_i|||_{0,\gamma,\bar{\Lambda}(T)}||\boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) - \boldsymbol{\chi}(\cdot,t,\mathbf{x})||_{0,\gamma,[0,T]} + |||\mathbf{v}_i - \mathbf{v}|||_{0,\gamma,\bar{\Lambda}(T)} \to 0 \quad \text{as } i \to \infty.$$

Here we used (A.89) and (A.91).

Finally we combine the estimates (A.89) - (A.92) to get

$$|||\boldsymbol{\chi}_i(s,t,\mathbf{x}) - \boldsymbol{\chi}(s,t,\mathbf{x})|||_{1,\gamma,[0,T]} \to 0 \quad \text{as } i \to \infty.$$

(iv) By the triangle inequality, we have

$$\frac{|\boldsymbol{\alpha}(\mathbf{x},t) - \boldsymbol{\alpha}(\mathbf{y},s)|}{|(\mathbf{x},t) - (\mathbf{y},s)|} \leq \frac{|\boldsymbol{\alpha}(\mathbf{x},t) - \boldsymbol{\alpha}(\mathbf{y},t)|}{|\mathbf{x} - \mathbf{y}|} + \frac{|\boldsymbol{\alpha}(\mathbf{y},t) - \boldsymbol{\alpha}(\mathbf{y},s)|}{|t-s|}.$$

Here we used (A.89) and hypothesis (2) of this lemma. Next observe that

$$|\boldsymbol{\alpha}(\mathbf{x},t) - \boldsymbol{\alpha}(\mathbf{y},t)| = |\boldsymbol{\chi}(0,t,\mathbf{x}) - \boldsymbol{\chi}(0,t,\mathbf{y})|$$

$$= \Big| \int_0^1 \partial_\xi \boldsymbol{\chi}(0,t,\mathbf{x}+\xi(\mathbf{y}-\mathbf{x}))d\xi \Big| = \Big| \int_0^1 \nabla_{\mathbf{x}} \boldsymbol{\chi}(0,t,\mathbf{x}+\xi(\mathbf{y}-\mathbf{x})) \cdot (\mathbf{y}-\mathbf{x}) d\xi \Big|$$
$$\le \int_0^1 |\nabla_{\mathbf{x}} \boldsymbol{\chi}(0,t,\mathbf{x}+\xi(\mathbf{y}-\mathbf{x}))||\mathbf{y}-\mathbf{x}|d\xi \le 2|\mathbf{y}-\mathbf{x}|.$$

Here we used Lemma A1 (1):

$$|||\nabla_{\mathbf{x}} \boldsymbol{\chi}|||_{0,[0,T]\times[0,T]\times\mathbb{R}^2} \le 2.$$

Similarly, we have

$$|||\boldsymbol{\alpha}(\mathbf{x},t) - \boldsymbol{\alpha}(\mathbf{y},t)|||_{0,[0,T]\times\mathbb{R}^2} \le 2|t-s|.$$

Hence we have

$$|\boldsymbol{\alpha}(\mathbf{x},t) - \boldsymbol{\alpha}(\mathbf{y},s)| \le 4|(\mathbf{x},t)-(\mathbf{y},s)|.$$

□

Lemma A.2. *Suppose f is a scalar valued function defined on $\Lambda_s^1(T;\mathbf{v})$ satisfying*

$$\sup_{0\le t\le T} ||f(\cdot,t)||_{0,\gamma,\bar{\Omega}_s^1(t;\mathbf{v})} < \infty.$$

Then we have

$$\Big[\Big[\int_0^t f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi \Big]\Big]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \le C_1(T)\Big(\sup_{0\le t\le T} ||f(\cdot,t)||_{0,\gamma,\bar{\Omega}_s^1(t;\mathbf{v})} \Big),$$

where $[[\ \cdot\ ]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})}$ is the Hölder seminorm on the space-time region, and

$$C_1(T) := \Big(T^{1-\gamma} + 16^\gamma T\Big) = \mathcal{O}(T^{1-\gamma}).$$

If f is in $C^{0,\gamma}(\bar{\Lambda}_s^1(T;\mathbf{v}))$, then the term $\sup_{0\le t\le T} ||f(\cdot,t)||_{0,\gamma,\bar{\Omega}_s^1(t;\mathbf{v})}$ can be replaced by $|||f|||_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})}$, i.e.,

$$\Big[\Big[\int_0^t f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi \Big]\Big]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \le C_1(T)|||f|||_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})}.$$

Proof. Let $(\mathbf{x},t)$ and $(\mathbf{y},s)$ be two points in $\Lambda_s^1(T;\mathbf{v})$. Without loss of generality, we assume that $s \le t$.

$$\frac{\Big| \int_0^t f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi - \int_0^s f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{y},s)),\xi)d\xi \Big|}{|(\mathbf{x},t)-(\mathbf{y},s)|^\gamma}$$
$$\le \frac{\Big| \int_s^t f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi \Big|}{|(\mathbf{x},t)-(\mathbf{y},s)|^\gamma}$$
$$+\frac{\int_0^s \Big| f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi) - f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{y},s)),\xi) \Big| d\xi}{|(\mathbf{x},t)-(\mathbf{y},s)|^\gamma}. \tag{A.93}$$

The terms on the right hand side of (A.93) can be treated as follows:

- $\dfrac{\Big| \int_s^t f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi \Big|}{|(\mathbf{x},t)-(\mathbf{y},s)|^\gamma} \le (t-s)^{1-\gamma}|||f|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} \le T^{1-\gamma}|||f|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})}$,
- $\Big| f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi) - f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{y},s)),\xi) \Big|$

$$\le \Big(\sup_{0\le\xi\le T}[f(\cdot,\xi)]_{0,\gamma,\bar\Omega_s(\xi)}\Big)|\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t))-\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{y},s))|^\gamma$$
$$\le \Big(\sup_{0\le\xi\le T}[f(\cdot,\xi)]_{0,\gamma,\bar\Omega_s(\xi)}\Big)2^\gamma|\boldsymbol{\alpha}(\mathbf{x},t)-\boldsymbol{\alpha}(\mathbf{y},s)|^\gamma$$
$$\le \Big(\sup_{0\le\xi\le T}[f(\cdot,\xi)]_{0,\gamma,\bar\Omega_s(\xi)}\Big)8^\gamma|(\mathbf{x},t)-(\mathbf{y},s)|^\gamma.$$

Note that

$$\max\Big\{|||f|||_{0,\bar\Lambda^1_s(T;\mathbf{v})},\ \Big(\sup_{0\le t\le T}[f(\cdot,t)]_{0,\gamma,\bar\Omega^1_s(t;\mathbf{v})}\Big)\Big\}\le\sup_{0\le t\le T}||f(\cdot,t)||_{0,\gamma,\bar\Omega^1_s(t;\mathbf{v})}.$$

Hence we have the desired result. Furthermore if f is in $C^{0,\gamma}(\bar\Lambda^1_s(T;\mathbf{v}))$, then the term $\sup_{0\le t\le T}||f(\cdot,t)||_{0,\gamma,\bar\Omega^1_s(t;\mathbf{v})}$ can be replaced by $|||f|||_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v})}$, i.e.,

$$\Big[\Big[\int_0^t f(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi\Big]\Big]_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v})}\le C_1(T)|||f|||_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v})}.$$

□

Lemma A.3. *Let n be the solution of (7.67) given by (7.69). Then there exists a positive constant T_1 such that n satisfies the a priori estimate:*

$$|||n|||_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v})}+\max_{|\alpha|=1}|||\partial^\alpha n|||_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v})}+|||\partial_t n|||_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v})}\le R_1,\quad 0<T\le T_1,$$

where R_1 is a positive constant depending on K_0,δ^* and γ.

Proof. (i) Recall that n satisfies
(A.94)

$$n(\mathbf{x},t)=n_0(\boldsymbol{\alpha}(\mathbf{x},t))\exp\Big(-\int_0^t(\nabla\cdot\mathbf{v})(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi\Big),\qquad\text{for } (\mathbf{x},t)\in\Lambda^1_s(T;\mathbf{v}).$$

Since $\mathbf{v}\in\mathcal{B}_T$, we have

$$|||\nabla\cdot\mathbf{v}|||_{0,\gamma,\bar\Lambda(T)}\le 6K_0\delta^*\ \text{ in } \Lambda(T)$$

and hence (A.94) implies

$$|||n|||_{0,\bar\Lambda^1_s(T;\mathbf{v})}\le e^{6TK_0\delta^*}||n_0||_{0,\bar\Omega^1_s(0;\mathbf{v})}.$$

Furthermore if we assume T_1 is sufficiently small enough to satisfy

$$e^{6T_1K_0\delta^*}\le 2 \tag{A.95}$$

then we have

$$|||n|||_{0,\bar\Lambda^1_s(T;\mathbf{v})}\le 2||n_0||_{0,\bar\Omega^1_s(0;\mathbf{v})}. \tag{A.96}$$

Next we show that $n_0(\boldsymbol{\alpha}(\mathbf{x},t))$ is in $C^{0,\gamma}(\bar\Lambda^1_s(T;\mathbf{v}))$. Let $(\mathbf{x},t)$ and $(\mathbf{y},s)$ be points in $\Lambda^1_s(T;\mathbf{v})$. Without loss of generality, we assume $s\le t$. Then we have

$$\frac{|n_0(\boldsymbol{\alpha}(\mathbf{x},t))-n_0(\boldsymbol{\alpha}(\mathbf{y},s))|}{|(\mathbf{x},t)-(\mathbf{y},s)|^\gamma} = \frac{|n_0(\boldsymbol{\alpha}(\mathbf{x},t))-n_0(\boldsymbol{\alpha}(\mathbf{y},s))|}{|\boldsymbol{\alpha}(\mathbf{x},t)-\boldsymbol{\alpha}(\mathbf{y},s)|^\gamma}\Big(\frac{|\boldsymbol{\alpha}(\mathbf{x},t)-\boldsymbol{\alpha}(\mathbf{y},s)|}{|(\mathbf{x},t)-(\mathbf{y},s)|}\Big)^\gamma \le [n_0]_{0,\gamma,\bar\Omega^1_s(0;\mathbf{v})}4^\gamma,$$

and hence

$$[[n_0(\boldsymbol{\alpha})]]_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v})}\le[n_0]_{0,\gamma,\bar\Omega^1_s(0;\mathbf{v})}4^\gamma. \tag{A.97}$$

On the other hand, it follows from Lemma A.2 and the fact that $\mathbf{v} \in \mathcal{B}(T)$ (see ($\mathcal{D}(2)$) in Definition 7.1) that

$$\Big[\Big[-\int_0^t (\nabla\cdot\mathbf{v})(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi\Big]\Big]_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v}))} \le 6C_1(T)K_0\delta^*.$$

We use (A.85) and (A.95) to get

$$\Big[\Big[\exp\Big(-\int_0^t (\nabla\cdot\mathbf{v})(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi\Big)\Big]\Big]_{0,\gamma,\bar\Lambda^1_s(T;\mathbf{v}))} \le 12C_1(T)K_0\delta^*, \tag{A.98}$$

and then use (A.84), (A.95), (A.97) and (A.98) to find

$$[[n]]_{0,\gamma,\bar\Lambda^1_s(T)} \le 2[n_0]_{0,\gamma,\bar\Omega^1_s(0;\mathbf{v})}4^\gamma + 12||n_0||_{0,\bar\Omega^1_s(0;\mathbf{v})}C_1(T)K_0\delta^*. \tag{A.99}$$

Since $C_1(T_1) = \mathcal{O}(T_1^{1-\gamma})$, we have for T_1 sufficiently small that

$$12C_1(T)K_0\delta^* \le 1, \quad T \le T_1, \tag{A.100}$$

so that (A.99) implies

$$[[n]]_{0,\gamma,\Lambda^1_s(T;\mathbf{v})} \le 2[n_0]_{0,\gamma,\bar\Omega^1_s(0;\mathbf{v})}4^\gamma + ||n_0||_{0,\bar\Omega^1_s(0)}. \tag{A.101}$$

Finally combine (A.96) and (A.101) to get the desired bound

$$\begin{aligned} |||n|||_{0,\gamma,\bar\Lambda^1_s(T)} &\le \max\{2^{2\gamma+1},3\}||n_0||_{0,\gamma,\bar\Omega^1_s(0)}, \quad 0<T\le T_1,\\ &\le \max\{2^{2\gamma+1},3\}\delta^*. \end{aligned} \tag{A.102}$$

(ii) We now need to estimate space derivatives of n. Differentiate the continuity equation

$$\partial_t n + \sum_{i=1}^2 \partial_{x_i}(nv_i) = 0$$

with respect to x_j to find

$$\frac{D(\partial_{x_j}n)}{Dt} = -\Big(\sum_{i=1}^2 \partial^2_{x_ix_j}v_i\Big)n - \Big(\sum_{i=1}^2 \partial_{x_i}v_i\Big)\partial_{x_j}n - \Big(\sum_{i=1}^2 \partial_{x_i}n\partial_{x_j}v_i\Big), \ j=1,2. \tag{A.103}$$

Here $\frac{D}{Dt} = \partial_t + \mathbf{v}\cdot\nabla_x$.

Integrate (A.103) along the characteristic curve $\boldsymbol{\chi}$ to obtain

$$\begin{aligned} \partial_{x_j}n(\mathbf{x},t) &= \partial_{x_j}n_0(\boldsymbol{\alpha}(\mathbf{x},t))\\ &- \sum_{i=1}^2\int_0^t \Big(n\partial^2_{x_ix_j}v_i + \partial_{x_i}v_i\partial_{x_j}n + \partial_{x_i}n\partial_{x_j}v_i\Big)(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi. \end{aligned} \tag{A.104}$$

Since $\mathbf{v}$ satisfies

$$\max_{i=1,2}\Big(\max_{|\alpha|=1}|||\partial^\alpha v_i|||_{0,\gamma,\bar\Lambda^1(T;\mathbf{v})} + \max_{|\alpha|=2}|||\partial^\alpha v_i|||_{0,\gamma,\bar\Lambda(T)}\Big) \le 3K_0\delta^*, \qquad \text{by the definition of } \mathcal{B}(T)$$

and n satisfies

$$|||n|||_{0,\bar\Lambda^1_s(T;\mathbf{v})} \le 2||n_0||_{0,\bar\Omega^1_s(0;\mathbf{v})},$$

for T sufficiently small by (A.96), we have from (A.104) that

$$\max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar\Lambda_s^1(T;\mathbf{v})} \le \max_{|\alpha|=1} ||\partial^\alpha n_0||_{0,\bar\Omega_s^1(0;\mathbf{v})} + 6TK_0\delta^* ||n_0||_{0,\bar\Omega_s^1(0;\mathbf{v})} + 12K_0\delta^* T \max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar\Lambda_s^1(T;\mathbf{v})}. \tag{A.105}$$

We assume T_1 is sufficiently small so that

$$K_0\delta^* T_1 \le \frac{1}{24}, \tag{A.106}$$

then we have from (A.106) that

$$\max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar\Lambda_s^1(T;\mathbf{v})} \le 2\max_{|\alpha|=1} ||\partial^\alpha n_0||_{0,\bar\Omega_s^1(0;\mathbf{v})} + \frac{1}{2}||n_0||_{0,\bar\Omega_s^1(0;\mathbf{v})} \quad \text{for } T \le T_1. \tag{A.107}$$

Next we estimate $[[\partial^\alpha n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})}, |\alpha| = 1$ using Lemma A.2 and (A.84).

By direct calculation, we have following estimates: For $T > 0$ sufficiently small, we have

$$\bullet\ [[\partial_{x_j} n_0(\boldsymbol{\alpha}(\mathbf{x},t))]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} \le 4^\gamma \max_{|\alpha|=1} [\partial^\alpha n_0]_{0,\gamma,\bar\Omega_s^1(0;\mathbf{v})}, \tag{A.108}$$

$$\bullet\ [[\partial^2_{x_i x_j} v_i n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} \le 6K_0(\delta^*)^2 \max\{2^{2\gamma+1}, 3\}, \tag{A.109}$$

$$\bullet\ [[\partial_{x_i} v_i \partial_{x_j} n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} \le 3K_0\delta^* [[\nabla n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} + 3K_0\delta^* |||\nabla n|||_{0,\bar\Lambda_s^1(T;\mathbf{v})}, \tag{A.110}$$

$$\bullet\ [[\partial_{x_i} n \partial_{x_j} v_i]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} \le 3K_0\delta^* \Big([[\nabla n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} + \max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar\Lambda_s^1(T;\mathbf{v})} \Big). \tag{A.111}$$

We combine estimates (A.108) - (A.111) to get

$$\begin{aligned} \max_{|\alpha|=1} [[\partial^\alpha n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} &\le 4^\gamma \max_{|\alpha|=1} [\partial^\alpha n_0]_{0,\gamma,\bar\Omega_s^1(0;\mathbf{v})} + 12K_0\delta^* C_1(T) \\ &\times \Big(\delta^* \max\{2^{2\gamma+1}, 3\} + \max_{|\alpha|=1} [[\partial^\alpha n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} + \max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar\Lambda_s^1(T;\mathbf{v})} \Big). \end{aligned} \tag{A.112}$$

We assume T_1 is sufficiently small so that

$$K_0\delta^* C_1(T_1) \le \frac{1}{24} \tag{A.113}$$

and hence for $T \in (0, T_1]$, (A.112) implies

$$\begin{aligned} \max_{|\alpha|=1} [[\partial^\alpha n]]_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} &\le 2^{2\gamma+1} \max_{|\alpha|=1} [\partial^\alpha n_0]_{0,\gamma,\bar\Omega_s^1(0;\mathbf{v})} + \delta^* \max\{24^\gamma, 3\} \\ &+ 2\max_{|\alpha|=1} ||\partial^\alpha n_0||_{0,\bar\Omega_s^1(0;\mathbf{v})} + \frac{1}{2}||n_0||_{0,\bar\Omega_s^1(0;\mathbf{v})}. \end{aligned} \tag{A.114}$$

We combine (A.107) and (A.114) to obtain

$$\max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} \le 2\delta^* \max\{2^{2^{\gamma+1}}, 4\}. \tag{A.115}$$

(iii) Now we estimate the time derivative of n. Recall that n satisfies

$$\partial_t n + \nabla\cdot(n\mathbf{v}) = 0. \tag{A.116}$$

Next we use (A.102), (A.115) and (A.116) to see

$$\begin{aligned} ||\partial_t n||_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} &\leq 2\max_{|\alpha|=1}||\partial^\alpha n||_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v})}\cdot |||\mathbf{v}|||_{0,\gamma,\bar\Lambda(T)} + ||n||_{0,\gamma,\bar\Lambda_s^1(T;\mathbf{v}))}||\nabla\cdot\mathbf{v}||_{0,\gamma,\bar\Lambda(T))} \\ &\leq 15K_0(\delta^*)^2\max\{2^{2\gamma+1},4\}. \end{aligned} \tag{A.117}$$

Finally we set

$$R_1(K_0,\delta^*,\gamma) := \max\{2^{2\gamma+1},3\}\delta^* + 2\delta^*\max\{2^{2\gamma+1},4\} + 15K_0(\delta^*)^2\max\{2^{2\gamma+1},4\}$$

to see that (A.102), (A.115) and (A.117) imply the desired result. □

A.1.2. **A priori estimates for** *Step 2.* In this part, we will give existence, uniqueness and regularity for the function ζ as given in *Step 2* of Section 7.2.

Recall from *Step 2*, ζ satisfies the exterior Neumann problem for Laplace's equation at given time $t\in[0,T]$:

$$\begin{cases} \Delta\zeta(\cdot,t)=0, & \mathbf{x}\in\Omega_1, \\ \nabla\zeta\cdot\boldsymbol{\nu}_0 = h_0, & \mathbf{x}\in\partial\Omega_0 \quad \text{and} \quad \lim_{|x|\to\infty}\nabla\zeta = \mathbf{0}. \end{cases} \tag{A.118}$$

Lemma A.4. *$h_0\in C^{1,\gamma}(\partial\Omega_0\times[0,T])$ and satisfies*

$$|||h_0|||_{1,\gamma,\partial\Omega_0\times[0,T]} \leq \delta^* + 6\bar C_0K_0R_1(\delta^*)^2,$$

where $\bar C_0$ is a positive constant.

Proof. Recall that $h_0 = \partial_t g - (n\mathbf{v})\cdot\boldsymbol{\nu}_0$. Then

$$\begin{aligned} |||h_0|||_{1,\gamma,\partial\Omega_0\times[0,T]} &\leq |||\partial_t g|||_{1,\gamma,\partial\Omega_0\times[0,\infty)} \\ &+ |||nv_1\nu_{01}|||_{1,\gamma,\partial\Omega_0\times[0,T]} + |||nv_2\nu_{02}|||_{1,\gamma,\partial\Omega_0\times[0,T]}. \end{aligned} \tag{A.119}$$

Since the product of Hölder continuous functions is again Hölder continuous (see [33], pg. 53), we have

$$\begin{aligned} &|||nv_1\nu_{01}|||_{1,\gamma,\partial\Omega_0\times[0,T]} + |||nv_2\nu_{02}|||_{1,\gamma,\partial\Omega_0\times[0,T]} \\ &\leq \bar C_0\Big(|||n|||_{1,\gamma,\partial\Omega_0\times[0,T]}|||v_1|||_{1,\gamma,\partial\Omega_0\times[0,T]}|||\nu_{01}|||_{1,\gamma,\partial\Omega_0\times[0,T]} \\ &+|||n|||_{1,\gamma,\partial\Omega_0\times[0,T]}|||v_2|||_{1,\gamma,\partial\Omega_0\times[0,T]}|||\nu_{02}|||_{1,\gamma,\partial\Omega_0\times[0,T]}\Big) \\ &\leq 6\bar C_0K_0R_1(\delta^*)^2. \end{aligned} \tag{A.120}$$

Here $\bar C_0$ is a positive constant and we used Lemma A.3, $\max_{i=1,2}|||v_i|||_{1,\gamma,\bar\Lambda(T)} \leq 3K_0\delta^*$, and inequalities

$$|||n|||_{1,\gamma,\partial\Omega_0\times[0,T]} \leq |||n|||_{1,\gamma,\bar\Lambda_s^1(T;\mathbf{v})} \quad \text{and} \quad |||v_i|||_{1,\gamma,\partial\Omega_0\times[0,T]} \leq |||v_i|||_{1,\gamma,\bar\Lambda(T)}.$$

Hence in (A.119), we use (A.120) and the assumption (A2) of Section 7.2 to get

$$|||h_0|||_{1,\gamma,\partial\Omega_0\times[0,T]} \leq \delta^* + 6\bar C_0K_0R_1(\delta^*)^2. \tag{A.121}$$

□

Recall the annulus region (7.65) in Section 7.2:

$$\Omega_* = \{\mathbf{x} \in \mathbb{R}^2 : \frac{\delta_{*2}}{2} < |\mathbf{x}| < 2\delta^*\}.$$

The following existence and uniqueness result of two-dimensional exterior Neumann problem (A.118) is due to the results of Bers [11] and Finn-Gilbarg [29].

Lemma A.5. *Suppose the boundary data h_0 is in $C^{1,\gamma}(\partial\Omega_0 \times [0,\infty))$ as provided by Lemma A.4. Then there exists a unique solution ζ up to constant of (A.118) satisfying the following estimates: For the compactly supported subset Ω_* of Ω_1, we have*

(1)

$$|||\nabla\zeta|||_{1,\gamma,\bar{\Omega}_*\times[0,T]} + \sup_{0\le t\le T}\max_{|\alpha|=3} |||\partial^\alpha\zeta(\cdot,t)|||_{0,\gamma,\bar{\Omega}_*} \le \bar{R}_1, \quad 0 \le t \le T,$$

where $\bar{R}_1$ is a positive constant which depends only on Ω_0, δ^*.

(2) *Let $h_0^{(n)} \in C^{1,\tau}(\partial\Omega_0 \times [0,\infty))$ be a sequence of boundary data satisfying the bound (A.121) and*

$$h_0^{(n)} \to h_0 \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0,\infty)) \quad \text{as } n \to \infty.$$

Suppose $\nabla\zeta^{(n)}$ and $\nabla\zeta$ are the corresponding solutions to the above exterior Neumann problem (A.118) for data $h_0^{(n)}, h_0$ respectively. Then we have

$$\nabla\zeta^{(n)} \to \nabla\zeta \quad \text{in } (C^{1,\tau}(\bar{\Omega}_* \times [0,T]))^2 \quad \text{as } n \to \infty.$$

Proof. (i) Since Ω_* is compactly supported in Ω_1, it follows from the interior Schauder estimates ([33], Section 6.1), we have

$$\begin{aligned} &||\nabla\zeta(\cdot,t)||_{1,\gamma,\bar{\Omega}_*} + \max_{|\alpha|=3} ||\partial^\alpha\zeta(\cdot,t)||_{0,\gamma,\bar{\Omega}_*} \\ &\qquad \le C_0(\Omega_0,\Omega_*)|||h_0|||_{1,\gamma,\partial\Omega_0\times[0,T]} \\ &\qquad \le C_0(\Omega_0,\Omega_*)(\delta^* + 6\bar{C}_0K_0R_1(\delta^*)^2), \quad 0 \le t \le T. \end{aligned} \tag{A.122}$$

Let $0 \le s < t$. Then it follows from Laplace's equation and the boundary condition that

$$\begin{cases} \Delta\left(\frac{\zeta(\mathbf{x},t)-\zeta(\mathbf{x},s)}{|t-s|^\gamma}\right) = 0, & \mathbf{x} \in \Omega_1, \quad 0 \le s < t \le T, \\ \nabla\left(\frac{\zeta(\mathbf{x},t)-\zeta(\mathbf{x},s)}{|t-s|^\gamma}\right) \cdot \boldsymbol{\nu}_0 = \frac{h_0(x,t)-h_0(\mathbf{x},s)}{|t-s|^\gamma}, & \mathbf{x} \in \partial\Omega_0. \end{cases}$$

We now apply the global Schauder estimate ([33], Section 6.2) to get

$$\begin{aligned} &\frac{|\nabla\zeta(\mathbf{x},t) - \nabla\zeta(\mathbf{x},s)|}{|t-s|^\gamma} \\ &\qquad \le C_1(\Omega_0,\Omega_*)|||h_0|||_{1,\gamma,\partial\Omega_0\times[0,T]} \\ &\qquad \le C_1(\Omega_0,\Omega_*)(\delta^* + 6\bar{C}_0K_0R_1(\delta^*)^2), \quad \mathbf{x} \in \Omega_1. \end{aligned} \tag{A.123}$$

We take the supremum over $t \ne s$ to get

$$\sup_{t\ne s} \frac{|\nabla\zeta(\mathbf{x},t) - \nabla\zeta(\mathbf{x},s)|}{|t-s|^\gamma} \le C_1(\Omega_0,\Omega_*)(\delta^* + 6\bar{C}_0K_0R_1(\delta^*)^2), \qquad \mathbf{x} \in \Omega_1. \tag{A.124}$$

Let $(\mathbf{x},t)\neq(\mathbf{y},s)$ and without loss of generality, assume that $\mathbf{x}\neq\mathbf{y}, t\neq s$. From (A.122) and (A.125), the Hölder quotient satisfies

$$\begin{aligned}\frac{|\nabla\zeta(\mathbf{x},t)-\nabla\zeta(\mathbf{y},s)|}{|(\mathbf{x},t)-(\mathbf{y},s)|^\gamma} &\le \frac{|\nabla\zeta(\mathbf{x},t)-\nabla\zeta(\mathbf{x},s)|}{|t-s|^\gamma}+\frac{|\nabla\zeta(\mathbf{x},s)-\nabla\zeta(\mathbf{y},s)|}{|\mathbf{x}-\mathbf{y}|^\gamma}\\ &\le [\nabla\zeta(\mathbf{x},\cdot)]_{0,\gamma,[0,T]}+[\nabla\zeta(\cdot,s)]_{0,\gamma,\Omega_s(s)}\\ &\le (C_0(\Omega_0,\Omega_*)+C_1(\Omega_0,\Omega_*))(\delta^*+6\bar{C}_0K_0R_1(\delta^*)^2).\end{aligned}$$

Taking sup over the time-space region $\Omega_*\times[0,T]$, we have

$$[\nabla\zeta]_{0,\gamma,\bar{\Omega}_*\times[0,T]}\le (C_0(\Omega_0,\Omega_*)+C_1(\Omega_0,\Omega_*))(\delta^*+6\bar{C}_0K_0R_1(\delta^*)^2). \tag{A.125}$$

Similarly we can estimate $\max\limits_{|\alpha|=2}|||\partial^\alpha\zeta|||_{0,\gamma,\bar{\Omega}_*\times[0,T]}$ to get

$$\max_{|\alpha|=2}|||\partial^\alpha\zeta|||_{0,\gamma,\bar{\Omega}_*\times[0,T]}\le C_2(\Omega_0,\Omega_*)(\delta^*+6\bar{C}_0K_0R_1(\delta^*)^2). \tag{A.126}$$

Finally we combine (A.122), (A.125) and (A.126) to get

$$|||\nabla\zeta|||_{1,\gamma,\bar{\Omega}_*\times[0,T]}+\sup_{0\le t\le T}\max_{|\alpha|=3}|||\partial^\alpha\zeta(\cdot,t)|||_{0,\gamma,\bar{\Omega}_*}\le\bar{R}_1,$$

where $\bar{R}_1:=(C_0(\Omega_0,\Omega_*)+C_1(\Omega_0,\Omega_*)+C_2(\Omega_0,\Omega_*))(\delta^*+6\bar{C}_0K_0R_1(\delta^*)^2)$.

(ii) The difference $\zeta^{(n)}-\zeta$ satisfies

$$\begin{cases}\Delta(\zeta^{(n)}(\cdot,t)-\zeta(\cdot,t))=0, & \mathbf{x}\in\Omega_1,\\ \nabla(\zeta^{(n)}-\zeta)\cdot\boldsymbol{\nu}_0=h_0^{(n)}-h_0, & \mathbf{x}\in\partial\Omega_0 \quad\text{and}\quad \zeta^{(n)}-\zeta=0 \quad \text{on } \partial B(0,3\delta^*).\end{cases}$$

By the Schauder estimates (Section 6.2 in [33]), we have

$$||\nabla\zeta^{(n)}(\cdot,t)-\nabla\zeta(\cdot,t)||_{1,\tau,\bar{\Omega}_1}\le C||h_0^{(n)}-h_0||_{1,\tau,\partial\Omega_0}.$$

Letting $n\to\infty$, it follows from the above inequality and hypothesis (2) of this lemma that

$$\nabla\zeta^{(n)}(\cdot,t)\to\nabla\zeta(\cdot,t)\quad \text{in } (C^{1,\tau}(\bar{\Omega}_1))^2,\quad 0\le t\le T. \tag{A.127}$$

For the time-estimates we apply the same method as in (i) to get

$$\nabla\zeta^{(n)}(\mathbf{x},\cdot)\to\nabla\zeta(\mathbf{x},\cdot)\quad \text{in } (C^{1,\tau}([0,T]))^2,\quad \mathbf{x}\in\bar{\Omega}_*. \tag{A.128}$$

We combine (A.127) and (A.128) to see

$$\nabla\zeta^{(n)}\to\nabla\zeta\quad \text{in } (C^{1,\tau}(\bar{\Omega}_1\times[0,T]))^2.$$

This yields

$$\nabla\zeta^{(n)}\to\nabla\zeta\quad \text{in } (C^{1,\tau}(\bar{\Omega}_*\times[0,T]))^2.$$

□

A.1.3. **A priori estimates for** *Step 3.* We now need to give existence, uniqueness and regularity for the interface of *Step 3* of Section 7.2.

Lemma A.6. *1. Assume that ζ satisfies estimate (1) of Lemma A.5:*

$$||\nabla\zeta||_{1,\gamma,\bar{\Omega}_*\times[0,T]} + \sup_{0\le t\le T}\max_{|\alpha|=3}||\partial^\alpha\zeta(\cdot,t)||_{0,\gamma,\bar{\Omega}_*} \le \bar{R}_1.$$

Then there exists a unique solution for the interface system (7.64) satisfying

$$(\theta, n_s, r) \in (C^{1,\gamma}(\mathbb{R}\times[0,T]))^3 \quad \textit{and} \quad (\partial_\beta\theta, \partial_\beta n_s, \partial_\beta r)(\cdot,t) \in (C^{2,\gamma}(\mathbb{R}))^3, \quad t\in[0,T].$$

Moreover, we have

- $||\theta||_{1,\gamma,\mathbb{R}\times[0,T]} + ||n_s||_{1,\gamma,\mathbb{R}\times[0,T]} + ||r||_{1,\gamma,\mathbb{R}\times[0,T]} \le 2\Big(||\theta_0||_{1,\gamma,\mathbb{R}} + ||n_{s0}||_{1,\gamma,\mathbb{R}} + ||r_0||_{1,\gamma,\mathbb{R}}\Big),$
- $||\partial_\beta\theta(\cdot,t)||_{2,\gamma,\mathbb{R}} + ||\partial_\beta n_s(\cdot,t)||_{2,\gamma,\mathbb{R}} + ||\partial_\beta r(\cdot,t)||_{2,\gamma,\mathbb{R}} \le 2\Big(||\partial_\beta\theta_0||_{2,\gamma,\mathbb{R}} + ||\partial_\beta n_{s0}||_{2,\gamma,\mathbb{R}} + ||\partial_\beta r_0||_{2,\gamma,\mathbb{R}}\Big).$

2. Let

$$\nabla\zeta^{(n)} \to \nabla\zeta \quad \textit{in } (C^{1,\tau}(\bar{\Omega}_*\times[0,T]))^2 \quad \textit{as given by (2) of Lemma A.4},$$

and let $(\theta^{(n)}, n_s^{(n)}, r^{(n)})$ and (θ, n_s, r) be the solutions of the sheath system corresponding to $\nabla\zeta^{(n)}$ and $\nabla\zeta$ respectively. Then we have

$$(\theta^{(n)}, n_s^{(n)}, r^{(n)}) \to (\theta, n_s, r) \quad \textit{in } (C^{1,\tau}(\mathbb{R}\times[0,T]))^3 \quad \textit{as } n\to\infty.$$

Proof. The result is just continuity with respect to data for the hyperbolic system (7.64). The proof of convergence in $C^1(\mathbb{R}\times[0,T])$ follows from the argument in [25]. The proof of Hölder norms $C^{1,\gamma}(\mathbb{R}\times[0,T])$ is similar to that of [25] (see [39]. □

A.1.4. **A priori estimates for** *Step 4.* We next present the existence, uniqueness and regularity of the ion density n in the region $\Lambda_s^2(T)$ given by *Step 4* of Section 7.2.

Lemma A.7. *Let n be the ion-density obtained from Step 4. Then the formulas in Step 4, namely n satisfies the differential equations:*

$$\begin{cases} \dfrac{d}{ds}\boldsymbol{\chi}(s,t_0,\boldsymbol{\alpha}) = \mathbf{v}(\boldsymbol{\chi}(s,t_0,\boldsymbol{\alpha}),s), \quad s>t_0,\\ \dfrac{d}{ds}\ln n(\boldsymbol{\chi}(s,t_0,\boldsymbol{\alpha}),s) = -(\nabla\cdot\mathbf{v})(\boldsymbol{\chi}(s,t_0,\boldsymbol{\alpha}),s), \end{cases}$$

subject to initial and boundary data:

$$\boldsymbol{\chi}(t_0,t_0,\boldsymbol{\alpha}) = \boldsymbol{\alpha} \quad \textit{and} \quad n(\boldsymbol{\alpha},t_0) = \begin{cases} n_0(\boldsymbol{\alpha}) & t_0 = 0,\\ n_s(\boldsymbol{\alpha},t_0) & t_0>0, \end{cases}$$

are indeed valid. Furthermore for sufficiently small T, the following estimates hold:

$$n\in C^{1,\gamma}(\Lambda_s^2(T;\mathbf{v})) \quad \textit{and} \quad ||n||_{1,\gamma;\Lambda_s^2(T;\mathbf{v})} \le R_2,$$

where R_2 is a positive constant depending only on K_0, δ^*, γ.

Proof. The proof follows from Remark A.1, i.e. since backward characteristics starting at a point $(\mathbf{x},t) \in \Lambda_s^2(T;\mathbf{v})$ can be traced back to a point $(\boldsymbol{\alpha},0)$ in the absence of the sheath interface, the presence of the sheath interface means backward characteristics must hit either a point in $\Omega_s^2(0;\mathbf{v})$ or a point in the sheath interface. Furthermore, the segment of backwards characteristic between $(\mathbf{x},t)$ and $(\boldsymbol{\alpha},0)$ can hit the sheath interface at most once. Indeed, the backwards characteristic can enter but not exit the domain $\Lambda_s(T;\mathbf{v}) = \Lambda_s^1(T;\mathbf{v}) \cap \Lambda_s^2(T;\mathbf{v})$ through the interface surface at time $0 < t < T$. This is because initially $\mathbf{u}_0 = -\boldsymbol{\nu}$ on $\mathcal{S}(0)$, hence $|\mathbf{v}\cdot\boldsymbol{\nu}+1| < \varepsilon$ on $\mathcal{S}(t)$ for $0 < t < T$ by (A1) and Theorem **??**, where $T > 0$ is sufficiently small depending only on initial data, boundary data, and $\varepsilon > 0$. By choosing $\varepsilon > 0$ sufficiently small, the vector field for the characteristic $\dfrac{d\boldsymbol{\chi}}{dt} = \mathbf{v}(\boldsymbol{\chi},t)$ always points into the domain $\Lambda_s(T;\mathbf{v})$ at the point of intersection with the sheath interface.

Hence the formulas follow from (7.67). Furthermore the regularity estimates in the statement of the lemma can be obtained in a similar manner as in Lemma A.3. ☐

We combine Lemma A.3 and Lemma A.7 to get the regularity result for n in the sheath region.

Lemma A.8. *For sufficiently small T, we have*

$$n \in C^{1,\gamma}(\bar{\Lambda}_s(T;\mathbf{v})) \quad \textit{and} \quad ||n||_{1,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \le R_1 + R_2.$$

A.1.5. **A priori estimates for** *Step 5.* We next give the existence, uniqueness and regularity for the function ϕ defined in *Step 5.*

Consider Poisson's equation on the space-time sheath region $\Lambda_s(T;\mathbf{v})$: Let $t \in [0,T]$ be given and ϕ satisfy

$$\begin{cases} \Delta\phi = n \quad \text{in } \Omega_s(t;\mathbf{v}), \\ \nabla\phi\cdot\ \boldsymbol{\nu}_0 = g \quad \text{on } \partial\Omega_0 \quad \text{and} \quad \phi = -\ln n_s \quad \text{on } \mathcal{S}(t). \end{cases} \tag{A.129}$$

Lemma A.9. *Let n be an ion density in the sheath region $\Lambda_s(T;\mathbf{v})$ and satisfy the a priori estimate in Lemma A.8. Then Poisson's equation (A.129) has a unique solution ϕ satisfying the following estimate:*

$$\max_{1\le|\alpha|\le 2} |||\partial^\alpha\phi|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} + \sup_{0\le t\le T}\max_{|\alpha|=3} ||\partial^\alpha\phi(\cdot,t))||_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \le R_3.$$

Here R_3 is a positive constant only depending on $K_0, \delta_{*i}, i = 1,2$ and δ^* respectively.

Proof. (i) Differentiation of (A.129) with respect to t shows that $\partial_t\phi$ satisfies the mixed Dirichlet-Neumann problem for Poisson's equation.

$$\begin{cases} \Delta\partial_t\phi = -\text{div}(n\mathbf{v}) \quad \text{in } \Omega_s(t;\mathbf{v}), \\ \nabla\partial_t\phi\cdot\ \boldsymbol{\nu}_0 = \partial_t g \quad \text{on } \partial\Omega_0 \quad \text{and} \quad \partial_t\phi = -\partial_t \ln n_s \quad \text{on } \mathcal{S}(t), \end{cases}$$

where we used $\nabla\phi\cdot\ \boldsymbol{\nu} = 0$ and $\nabla n\cdot\ \boldsymbol{\nu} = 0$ on the interface $\mathcal{S}(t)$, $0 \le t \le T$.

By the direct application of Hölder estimates of the first derivatives given in ([33], Section 8), we have

$$||\partial_t\phi||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \le \bar{C}_1\Big(||\partial_t g||_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + |||n\mathbf{v}|||_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + ||\partial_t \ln n_s||_{2,\gamma,\mathbb{R}}\Big).$$

Here $\bar{C}_1$ depends on Ω_0 and $\mathcal{S}(t)$, but we can choose uniform $\bar{C}_1$ independent of t and depending only on δ_* and δ^* for sufficiently small T, $0 \le t \le T$.

On the other hand, since $\mathbf{v} \in \mathcal{B}(T)$, we have

- $|||n\mathbf{v}|||_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq \bar{C}_2 ||n||_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} ||\mathbf{v}||_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq \bar{C}_3 K_0 (R_2 + R_3)\delta^*$,
- $||\partial_t g||_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq \delta^*$ by the assumption (A2) in Section 7.2,

where $\bar{C}_2$ and $\bar{C}_3$ are some positive constants.

It follows from the interface equation (7.64) that

$$\begin{aligned} \frac{\partial_t n_s}{n_s} &= -\Big(\frac{4\sin\beta\sin\theta\theta}{r}\Big)\partial_\beta\theta - \Big(\frac{2\sin\beta\tilde{V}\cos\theta}{rn_s}\Big)\partial_\beta n_s, \\ \tilde{V} &= -1 - \frac{\nabla\zeta\cdot(\cos\theta,\sin\theta)}{n_s}. \end{aligned}$$

We use the above relation and the estimates from Lemma A.5 (1) to obtain

$$||\partial_t \ln n_s||_{2,\gamma,\mathbb{R}} \leq C(\delta_{*1}, \delta^*).$$

Here $C(\delta_{*1}, \delta^*)$ is a positive constant depending only on δ_{*1}, δ^*. Hence we have

$$\sup_{0\leq t\leq T} ||\partial_t\phi(\cdot,t)||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq R_{3,0}(K_0,\delta_{*1},\delta_{*2},\delta^*) \quad \text{for } t\in[0,T]. \tag{A.130}$$

(ii) It follows from the Schauder estimates (Section 6.2 in [33]) that

$$\begin{aligned} ||\phi(\cdot,t)||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} &\leq \bar{C}_4\Big(||n(\cdot,t)||_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + ||g(\cdot,t)||_{1,\gamma,\partial\Omega_0} + ||\ln n_s(\cdot,t)||_{2,\gamma,\mathbb{R}}\Big) \\ &\leq R_{3,1}(K_0,\delta_{*1},\delta_{*2},\delta^*), \quad 0\leq t\leq T. \end{aligned} \tag{A.131}$$

Here $\bar{C}_4$ depends only on the Ω_0 and $\mathcal{S}(t)$, but again we can choose $\bar{C}_4$ depending only on δ_* and δ^* for sufficiently small $T, 0 \leq t \leq T$.

Let $(\mathbf{x},t)$ and $(\mathbf{y},s)$ be any points in $\Lambda_s(T;\mathbf{v})$. Without loss of generality, we assume that $0 \leq s < t$. By assumption (A4) of Section 7.2, we have a contracting interface so that

$$\Omega_s(t;\mathbf{v}) \subset \Omega_s(s;\mathbf{v}), \quad 0\leq s < t \leq T \ll 1.$$

Hence $\mathbf{x} \in \Omega_s(t;\mathbf{v})$. Then inequality (A.130) implies, for $\mathbf{x} \in \Omega_s(t;\mathbf{v})$

$$\max_{1\leq|\alpha|\leq 2} ||\partial^\alpha\phi(\mathbf{x},\cdot)||_{0,\gamma,[0,T]} \leq R_{3,0}(K_0,\delta_{*1},\delta_{*2},\delta^*)T^{1-\gamma}. \tag{A.132}$$

We combine (A.131) and (A.132) and choose T sufficiently small to get

$$\max_{1\leq|\alpha|\leq 2} |||\partial^\alpha\phi|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq R_{3,2}(K_0,\delta_{*1},\delta_{*2},\delta^*). \tag{A.133}$$

(In fact the above argument holds for the expanding interfaces as well).

(iii) On the other hand, $\partial_{x_i}\phi, i = 1,2$ satisfies

$$\begin{cases} \Delta\partial_{x_i}\phi = \partial_{x_i} n \ \text{ in } \Omega_s(t;\mathbf{v}), \\ \nabla\partial_{x_i}\phi\cdot\ \boldsymbol{\nu}_0 = \partial_{x_i} g \quad \text{on } \partial\Omega_0 \quad \text{and} \quad \partial_{x_i}\phi = -\partial_{x_i}\ln n_s \quad \text{on } \mathcal{S}(t). \end{cases} \tag{A.134}$$

Again, it follows from the Poisson equation and the Schauder estimates (Section 6.2 in [33]) that

$$\begin{aligned} &||\partial_{x_i}\phi(\cdot,t)||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \\ &\leq \bar{C}_5\Big(||\partial_{x_i} n(\cdot,t)||_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + ||\partial_{x_i} g(\cdot,t)||_{1,\gamma,\partial\Omega_0} + ||\partial_{x_i}\ln n_s(\cdot,t)||_{2,\gamma,\mathbb{R}}\Big). \end{aligned} \tag{A.135}$$

Here $\bar{C}_5$ depends on $\mathcal{S}(t)$, but we can choose $\bar{C}_5$ depending only on $\delta_{*i}, i = 1, 2$ and δ^* for sufficiently small $T, 0 \leq t \leq T$.

The first two terms in the right hand side of (A.135) can be bounded by a quantity depending on δ^* using Lemma A.7 and assumptions (A1)-(A2) of the boundary data in Section 7.2, i.e.,

$$||\partial_{x_i} n(\cdot,t)||_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + ||\partial_{x_i} g(\cdot,t)||_{1,\gamma,\partial\Omega_0} \leq \bar{C}_6. \tag{A.136}$$

Here $\bar{C}_6$ is a positive constant depending only on δ_{*1} and δ^*.

Now we estimate the third term $||\partial_{x_i} \ln n_s||_{2,\gamma,\mathbb{R}}$ as follows. It follows from (7.57) that we have

$$\partial_{x_1} = -\frac{2\sin\beta}{r}\partial_\beta$$

and similarly we can express ∂_{x_2} in terms of ∂_β. Therefore we have

$$||\partial_{x_i} \ln n_s(\cdot,t)|||_{2,\gamma,\mathbb{R}} = \Big|\Big|\frac{\partial_{x_i} n_s(\cdot,t)}{n_s(\cdot,t)}\Big|\Big|_{2,\gamma,\mathbb{R}} \leq \bar{C}_7 \quad i = 1, 2. \tag{A.137}$$

Here $\bar{C}_7$ is a positive constant depending only on δ_{*1} and δ^*.

Combining estimates (A.136) and (A.137), we obtain

$$\max_{|\alpha|=1} ||\partial^\alpha \phi(\cdot,t)||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) \quad \text{ for } t \in [0,T].$$

The above inequality implies

$$\sup_{0\leq t\leq T} \max_{|\alpha|=1} ||\partial^\alpha \phi(\cdot,t)||_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) \quad \text{ for } t \in [0,T].$$

In particular we have

$$\sup_{0\leq t\leq T} \max_{|\alpha|=3} ||\partial^\alpha \phi(\cdot,t)||_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) \quad \text{ for } t \in [0,T]. \tag{A.138}$$

We set $R_3(K_0, \delta_{*1}, \delta_{*2}\delta^*) := R_{3,2}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) + R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*)$ and use (A.135) and (A.138) to get the desired result. □

A.1.6. **A priori estimates for** *Step 6.* In this part, we give the existence, uniqueness and regularity for the ion velocity $\hat{\mathbf{u}}$ defined in *Step 6* of Section 7.2.

Consider the Burgers' equation with a known source $\nabla\phi$:

$$\partial_t \hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} = \nabla\phi, \qquad (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}_+. \tag{A.139}$$

Lemma A.10. *Suppose the source $\nabla\phi$ satisfies the estimates obtained in Lemma A.9. Also assume initial data $\mathbf{u}_0$ satisfy the assumption (A3) of Section 7.2 so that*

(1) $\nabla\phi \in C^{1,\gamma}(\bar{\Lambda}_s(T;\mathbf{v}))$ *and* $\nabla\phi(\cdot,t) \in C^{2,\gamma}(\Omega_s(t;\mathbf{v}))$;
(2) *for each $\boldsymbol{\alpha} \in \mathbb{R}^2$, the real parts of the eigenvalues of $\nabla\mathbf{u}_0(\boldsymbol{\alpha})$ are non-negative;*

the there is a positive constant T_2 such that (A.139) has a unique solution $\hat{\mathbf{u}} \in C^{1,\gamma}(\bar{\Lambda}_s(T;\mathbf{v}))$ satisfying

$$\begin{aligned} &\det \Gamma(\boldsymbol{\alpha}, t) > 0 \quad \textit{and} \\ &\hat{\mathbf{u}}(\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})) = \mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \nabla\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)ds, \quad t \in [0,T_2], \\ &\hat{\mathbf{u}}(\mathbf{x},t)\cdot\mathbf{x} \le -\frac{\eta_0}{2}|\mathbf{x}|^2, \quad (\mathbf{x},t) \in (B(0, r_b + 6K_0\delta^* T_2) - \Omega_0) \times [0,T_2], \end{aligned} \tag{A.140}$$

where T_2 is a positive constant and

$$\frac{d\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})}{dt} = \hat{\mathbf{u}}(\boldsymbol{\chi}(t,0,\boldsymbol{\alpha}),t) \quad \textit{and} \quad \Gamma(\boldsymbol{\alpha},t) = \nabla\hat{\mathbf{u}}(\boldsymbol{\alpha},t).$$

Proof. (i) Along the particle path $\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})$, system (A.139) becomes

$$\frac{D\hat{\mathbf{u}}}{Dt} = \nabla\phi, \quad \text{where } \frac{D}{Dt} = \partial_t + \hat{\mathbf{u}}\cdot\nabla \tag{A.141}$$

Any smooth solution of (A.140) will satisfy

$$\frac{d^2\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})}{dt^2} = \nabla\phi(\boldsymbol{\chi}(t,0,\boldsymbol{\alpha}),t); \qquad \boldsymbol{\chi}(\boldsymbol{\alpha},0) = \boldsymbol{\alpha}, \quad \frac{d\boldsymbol{\chi}(0,0,\boldsymbol{\alpha})}{dt} = \mathbf{u}_0(\boldsymbol{\alpha}). \tag{A.142}$$

Since $\nabla\phi(\cdot,t)$ is Lipschitz continuous and uniformly bounded, there exists a unique characteristic curve $\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})$ satisfying (A.142) locally in time t. Now we integrate (A.142) to get

$$\frac{d\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})}{dt} = \mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \nabla\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)ds. \tag{A.143}$$

and integration of the above equation yields

$$\begin{aligned} \boldsymbol{\chi}(t,0,\boldsymbol{\alpha}) &= \boldsymbol{\chi}(0,0,\boldsymbol{\alpha}) + t\mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t\int_0^{t_1} \nabla\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)dsdt_1 \\ &= \boldsymbol{\alpha} + t\mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t\int_0^{t_1} \nabla\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)dsdt_1. \end{aligned} \tag{A.144}$$

Next we differentiate (A.144) with respect to $\boldsymbol{\alpha}$ to get

$$\Gamma(\boldsymbol{\alpha},t) = I + t\nabla\mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t\int_0^{t_1} (\nabla\otimes\nabla)\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)\Gamma(\boldsymbol{\alpha},s)dsdt_1. \tag{A.145}$$

Set

$$y(t) = |\Gamma(\boldsymbol{\alpha},t) - I - t\nabla\mathbf{u}_0(\boldsymbol{\alpha})|,$$

where $|\cdot|$ denotes any norm on 2×2 matrices so that we have

$$y(t) \le \int_0^t\int_0^{t_1} |(\nabla\otimes\nabla)\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)||\Gamma(\boldsymbol{\alpha},s)|dsdt_1.$$

Since

$$\begin{aligned} |\Gamma(\boldsymbol{\alpha},t)| &= |\Gamma(\boldsymbol{\alpha},t) - I - t\nabla\mathbf{u}_0(\boldsymbol{\alpha})| + |I + t\nabla\mathbf{u}_0(\boldsymbol{\alpha})| \\ &\le y(t) + 1 + d_1 t, \end{aligned}$$

where $\sup\limits_{\boldsymbol{\alpha}\in\Omega(0)} |\nabla \mathbf{u}_0(\boldsymbol{\alpha})| \le d_1$, we have from (A.145) that

$$y(t) \le \int_0^t \int_0^{t_1} d_2(y(s) + 1 + d_1 s) ds dt_1,$$

where $|\nabla \otimes \nabla \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha})| \le d_2$ for $\boldsymbol{\alpha} \in \Omega(0), 0 \le s \le T$. Hence we have

$$y(t) \le d_2\Big(\frac{t^2}{2} + d_1 \frac{t^3}{3}\Big) + d_2 \int_0^t \int_0^{t_1} y(s) ds dt_1$$

and by Appendix B

$$y(t) \le d_3 t^2 \qquad \text{on } 0 \le t \le T,$$

for sufficiently small T, i.e.,

$$|\Gamma(\boldsymbol{\alpha}, t) - I - t\nabla \mathbf{u}_0(\boldsymbol{\alpha})| \le d_3 t^2, \quad 0 \le t \le T.$$

Define

$$tD(t, \boldsymbol{\alpha}) := \Gamma(\boldsymbol{\alpha}, t) - I - t\nabla \mathbf{u}_0(\boldsymbol{\alpha}).$$

Then we have

$$|D(t, \boldsymbol{\alpha})| \le d_3 t \quad \text{ for some constant } d_3 > 0$$

and

$$\det(\Gamma(\boldsymbol{\alpha}, t)) = \det(I + t(\nabla \mathbf{u}_0(\boldsymbol{\alpha}) + D(t, \boldsymbol{\alpha}))).$$

Let $\lambda_i(\boldsymbol{\alpha}, t)$ and $\lambda_i(\boldsymbol{\alpha}), i = 1, 2$ be the eigenvalues of a matrix $\nabla \mathbf{u}_0(\boldsymbol{\alpha}) + D(t, \boldsymbol{\alpha})$ and $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$ respectively. Then we can see that

$$\lambda_i(\boldsymbol{\alpha}, t) = \lambda_i(\boldsymbol{\alpha}) + \mathcal{O}(t).$$

By the Cayley-Hamilton theorem, we have

$$\begin{aligned} \det(\Gamma(\boldsymbol{\alpha}, t)) &= (t\lambda_1(\boldsymbol{\alpha}, t) + 1)(t\lambda_2(\boldsymbol{\alpha}, t) + 1) \\ &= (t\lambda_1(\boldsymbol{\alpha}) + 1)(t\lambda_2(\boldsymbol{\alpha}) + 1) + \mathcal{O}(t^2) \\ &= \det(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha})) + \mathcal{O}(t^2). \end{aligned}$$

As long as $0 \le t \le T \ll 1$, the sign of $\det(\Gamma(\boldsymbol{\alpha}, t))$ will be determined by $\det\Big(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha})\Big)$.

Next we calculate $\det(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha}))$. Let us set the characteristic polynomial of $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$ by $P(\boldsymbol{\alpha}, \lambda)$. Then we have

$$P(\boldsymbol{\alpha}, \lambda) \equiv \det(\nabla \mathbf{u}_0(\boldsymbol{\alpha}) - \lambda I) = (\lambda_1(\boldsymbol{\alpha}) - \lambda)(\lambda_2(\boldsymbol{\alpha}) - \lambda),$$

where $\lambda_i(\boldsymbol{\alpha})$ are the eigenvalues of $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$. Hence

$$\begin{aligned} \det\Big(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha})\Big) &= t^2 \det\Big(\nabla \mathbf{u}_0(\boldsymbol{\alpha}) + t^{-1} I\Big) = t^2 P(\boldsymbol{\alpha}, -t^{-1}) \\ &= t^2(\lambda_1(\boldsymbol{\alpha}) + t^{-1})(\lambda_2(\boldsymbol{\alpha}) + t^{-1}) \\ &= (t\lambda_1(\boldsymbol{\alpha}) + 1)(t\lambda_2(\boldsymbol{\alpha}) + 1). \end{aligned}$$

Since by assumption (2) above, real parts of the eigenvalues of the Jacobian matrix $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$ are nonnegative, we have

$$\det \Gamma(\boldsymbol{\alpha}, t) \ge \Pi_{q=1}^2 [1 + t \text{ Re } \lambda_q(\boldsymbol{\alpha})] + \mathcal{O}(t^2) > 0, \quad 0 \le t \le T \ll 1.$$

Hence the Lagrangian map is a C^1-diffeomorphism locally in time.

(ii) It follows from (A.140) that

$$\hat{\mathbf{u}}(\boldsymbol{\chi}(t,0,\boldsymbol{\alpha}),t) = \mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \nabla\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)ds.$$

We take an inner product with $\boldsymbol{\chi}(t,0,\boldsymbol{\alpha}),t)$ to get

$$\begin{aligned}
&\langle \hat{\mathbf{u}}(\boldsymbol{\chi}(t,0,\boldsymbol{\alpha}),t), \boldsymbol{\chi}(t,0,\boldsymbol{\alpha})\rangle \\
&= \langle \mathbf{u}_0(\boldsymbol{\alpha}), \boldsymbol{\chi}(t,0,\boldsymbol{\alpha})\rangle + \int_0^t \langle \nabla\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}), \boldsymbol{\chi}(t,0,\boldsymbol{\alpha})\rangle ds \\
&= \langle \mathbf{u}_0(\boldsymbol{\alpha}), \boldsymbol{\chi}(t,0,\boldsymbol{\alpha}) - \boldsymbol{\alpha}\rangle + \langle \mathbf{u}_0(\boldsymbol{\alpha}), \boldsymbol{\alpha}\rangle + \int_0^t \langle \nabla\phi(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}), \boldsymbol{\chi}(t,0,\boldsymbol{\alpha})\rangle ds.
\end{aligned} \tag{A.146}$$

Since

$$|\boldsymbol{\chi}(t,0,\boldsymbol{\alpha}) - \boldsymbol{\alpha}| = \Big| \int_0^t \mathbf{v}(\boldsymbol{\chi}(s,0,\boldsymbol{\alpha}),s)ds \Big| \le 6K_0\delta^* T,$$
$$\text{and} \quad |||\nabla\phi|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} \le R_3,$$

Hence in (A.146), we have

$$\langle \hat{\mathbf{u}}(\boldsymbol{\chi}(\boldsymbol{\alpha},t),t), \boldsymbol{\chi}(\boldsymbol{\alpha},t)\rangle \le 6K_0(\delta^*)^2 T - \eta_0||\boldsymbol{\alpha}||^2 + R_3 T.$$

Now we choose T sufficiently small so that

$$\Big(6K_0(\delta^*)^2 + R_3\Big)T \le \frac{\eta_0 r_a^2}{2} \le \frac{\eta_0}{2}||\boldsymbol{\alpha}||^2, \qquad \boldsymbol{\alpha} \in \Omega_s(0).$$

Then we have

$$\langle \hat{\mathbf{u}}(\boldsymbol{\chi}(\boldsymbol{\alpha},t),t), \boldsymbol{\chi}(\boldsymbol{\alpha},t)\rangle \le -\frac{\eta_0}{2}||\boldsymbol{\alpha}||^2.$$

On the other hand, since

$$\frac{d}{ds}|\boldsymbol{\chi}(0,0,\boldsymbol{\alpha})|^2 \le -2\eta_0|\boldsymbol{\chi}(0,0,\boldsymbol{\alpha})|^2 = -2\eta_0|\boldsymbol{\alpha}|^2,$$

we have

$$|\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})| \le |\boldsymbol{\alpha}| \quad \text{for } t \le T \ll 1.$$

Therefore we obtain

$$\langle \hat{\mathbf{u}}(\boldsymbol{\chi}(t,0,\boldsymbol{\alpha}),t), \boldsymbol{\chi}(t,0,\boldsymbol{\alpha})\rangle \le -\frac{\eta_0}{2}|\boldsymbol{\chi}(t,0,\boldsymbol{\alpha})|^2.$$

□

A.1.7. **A priori estimates for** *Step 7* **of Section 7.2**. Finally we prove the existence of a linear extension map and some estimates of the extension.

Lemma A.11. *Let $\mathcal{S}(t), t \in [0,T]$ be the $C^{2,\gamma}$-regular simple closed convex curve in $\mathbb{R}^2$ provided by Lemma A.6 such that $\mathcal{S}(t)$ lies inside the annulus Ω_* and $\Omega_s(t;\mathbf{v})$ is the corresponding sheath region $0 \le t \le T$, T sufficiently small. Then there exists a bounded linear operator $\mathcal{E}(\cdot;\mathcal{S}(t)) : C^{2,\gamma}(\Omega_s(t;\mathbf{v})) \to C^{2,\gamma}(\Omega_1)$ satisfying*

(a) $\mathcal{E}(\hat{\mathbf{u}}|\mathcal{S}(t)) = \hat{\mathbf{u}}$, in $\Omega_s(t;\mathbf{v})$,
(b) $\mathcal{E}(\hat{\mathbf{u}}|\mathcal{S}(t))$ has support in $B(0,3\delta^*)$,
(c) $|||\mathcal{E}(\hat{\mathbf{u}}|\mathcal{S}(t))|||_{2,\gamma,\bar{\Omega}_1} \le K_0|||\hat{\mathbf{u}}|||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})}$,

where K_0 is independent of $t \in [0,T]$ and Ω_* is the annulus region (7.65).

Proof. Since the proof is rather long, we delay its proof until Appendix C. □

A.2. **Continuity of $\mathcal{F}$.** In this part, we establish the continuity of $\mathcal{F}$ which in turn imply the existence of a fixed point of $\mathcal{F}$.

Lemma A.12. *Let f be a continuous function such that*

$$f(t) \leq C_0 + C_1 (f(t))^2, \quad t \geq 0,$$
$$f(0) \leq C_0 \quad \text{and} \quad C_0 C_1 \leq \frac{1}{8},$$

where C_0 and C_1 are positive constants independent of t. Then we have

$$f(t) \leq 2C_0.$$

Proof. Define

$$F(k) = C_1 k^2 - k + C_0.$$

Then by direct calculation, we have

$$\min F(k) = \frac{-1 + 2C_0 C_1}{2C_1} < 0 \quad \text{at } k = \frac{1}{2C_1}.$$

Now we denote r_1 and r_2 by the roots of $F(k) = 0$ such that $r_1 < r_2$. Then by direct calculation, the smallest root r_1 satisfies

$$C_0 \leq r_1 = \frac{2C_0}{1 + \sqrt{1 - 4C_0 C_1}} \leq 4(\sqrt{2} - 1) C_0 \leq 2C_0.$$

On the other hand, since $F(f(t)) \geq 0$, we have two cases:

$$\text{either } f \leq r_1 \quad \text{or} \quad f \geq r_2,$$

however since $f(0) \leq C_0 \leq r_1$ and $f(t)$ is continuous, we have

$$f \leq r_1 \leq 2C_0.$$

□

Proposition A.1. *There exists a positive constant T such that the map $\mathcal{F}$ with $\mathcal{F}(\mathbf{v}) := \mathcal{E}(\hat{\mathbf{u}}; \mathcal{S}(t))$ is a well-defined map from $\mathcal{B}(T)$ to $\mathcal{B}(T)$.*

Proof. For the time being, we assume T sufficiently small so that

$$T \leq \min\{T_1, T_2\}. \tag{A.147}$$

So all estimates in the previous lemmas hold.

(i) By the construction of $\hat{\mathbf{u}}$ in the sheath region $\Lambda_s(T; \mathbf{v})$, we have from solving (A.139) along the characteristic

$$\hat{\mathbf{u}}(\mathbf{x}, t) = \begin{cases} \mathbf{u}_0(\boldsymbol{\alpha}(\mathbf{x}, t)) + \int_0^t \nabla\phi(\hat{\boldsymbol{\chi}}(s, t, \mathbf{x}), s) ds & t_0 = 0, \\ -\boldsymbol{\nu}(\boldsymbol{\alpha}(\mathbf{x}, t)) + \int_{t_0}^t \nabla\phi(\hat{\boldsymbol{\chi}}(s, t, \mathbf{x}), s) ds & t_0 > 0. \end{cases} \tag{A.148}$$

In (A.148), we have

$$||\hat{u}_i||_{0, \bar{\Lambda}_s(T; \mathbf{v})} \quad \leq \quad \begin{cases} ||u_{i0}||_{0, \bar{\Omega}_s(0)} + t|||\nabla\phi|||_{0, \bar{\Lambda}_s(T; \mathbf{v})} & t_0 = 0, \\ 1 + (t - t_0)|||\nabla\phi|||_{0, \bar{\Lambda}_s(T; \mathbf{v})} & t_0 > 0. \end{cases}$$

This of course implies

$$\begin{aligned} ||\hat{u}_i||_{0, \bar{\Lambda}_s(T; \mathbf{v})} &\leq ||u_{i0}||_{0, \bar{\Omega}_s(0)} + T|||\nabla\phi|||_{0, \bar{\Lambda}_s(T; \mathbf{v})} \\ &\leq ||u_{0i}||_{0, \bar{\Omega}_s(0)} + TR_3 \qquad \text{by Lemma A.8.} \end{aligned} \tag{A.149}$$

On the other hand, we use Lemma A.2 to obtain

$$[[\hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq [u_{0i}]_{0,\gamma,\bar{\Omega}_s(0)} + C_1(T)R_3. \tag{A.150}$$

We combine (A.149) and (A.150) to get

$$||\hat{u}_i||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq ||u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + TR_3 + C_1(T)R_3. \tag{A.151}$$

We assume T sufficiently small so that

$$TR_3 + C_1(T)R_3 \leq \frac{\delta^*}{3}. \tag{A.152}$$

Here we used $C_1(T) = \mathcal{O}(T^{1-\gamma})$. Hence we have from (A.152) that

$$\max_{i=1,2} ||\hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq \max_{i=1,2} ||u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}. \tag{A.153}$$

(ii) We differentiate the momentum equation

$$\partial_t \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} = \nabla \phi$$

with respect to x_k to find

$$\frac{D(\partial_{x_k} \hat{u}_i)}{Dt} + \sum_{j=1}^{3} (\partial_{x_k} \hat{u}_j)(\partial_{x_i} \hat{u}_j) = \partial_{x_k}(\partial_{x_i} \phi), \tag{A.154}$$

where $\frac{D}{Dt} = \partial_t + \hat{\mathbf{u}} \cdot \nabla$. We integrate (A.154) along the characteristic to get

(i) if $t_0 = 0$,

$$\partial_{x_k} \hat{u}_i(\mathbf{x}, t) = \partial_{x_k} u_{0i}(\boldsymbol{\alpha}(\mathbf{x}, t)) - \sum_{j=1}^{2} \int_0^t \Big((\partial_{x_k} \hat{u}_j)(\partial_{x_i} \hat{u}_j) \Big)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}), s) ds + \int_0^t \partial_{x_k}(\partial_{x_i} \phi)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}), s) ds;$$

(ii) and if $t_0 > 0$,

$$\begin{aligned} \partial_{x_k} \hat{u}_i(\mathbf{x}, t) = -\partial_{x_k} \nu_i(\boldsymbol{\alpha}(\mathbf{x}, t)) - \sum_{j=1}^{2} \int_{t_0}^t \Big((\partial_{x_k} \hat{u}_j)(\partial_{x_i} \hat{u}_j) \Big)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}), s) ds \\ + \int_{t_0}^t \partial_{x_k}(\partial_{x_i} \phi)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}), s) ds. \end{aligned} \tag{A.155}$$

The above equalities yield

$$\begin{aligned} \max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha \hat{u}_i|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} &\leq \max_{i=1,2} \max_{|\alpha|=1} ||\partial^\alpha u_{0i}||_{0,\bar{\Omega}_s(0)} + 2T \Big(\max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha \hat{u}_i|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} \Big)^2 \\ &+ T \max_{|\alpha|=2} |||\partial^\alpha \phi|||_{0,\bar{\Lambda}_s(T;\mathbf{v})}, \quad 0 \leq t \leq T. \end{aligned}$$

Since $T \ll 1$, it follows from Lemma A.12 that

$$\begin{aligned} \max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha \hat{u}_i|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} &\leq 2 \Big(\max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha u_{0i}||_{0,\bar{\Omega}_s(0)} + T \max_{|\alpha|=2} |||\partial^\alpha \phi|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} \Big) \\ &\leq 2 \max_{i=1,2} \max_{|\alpha|=1} ||\partial^\alpha u_{0i}||_{0,\bar{\Omega}_s(0)} + 2TR_3. \end{aligned} \tag{A.156}$$

On the other hand, it follows from the inequalities (A.155) and (A.84) that

$$\begin{aligned}&\max_{i=1,2}\max_{|\alpha|=1}[[\partial^\alpha \hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\\ &\qquad\le \max_{i=1,2}\max_{|\alpha|=1}[\partial^\alpha u_{0i}]_{0,\gamma,\bar{\Omega}_s(0)} + 4C_1(T)\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha \hat{u}_{0i}||_{0,\bar{\Omega}_s(0)} + 2TR_3\Big)\\ &\qquad\times \max_{i=1,2}\max_{|\alpha|=1}[[\partial^\alpha \hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} + C_1(T)R_3.\end{aligned}$$

We assume that

$$4C_1(T)\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\bar{\Lambda}_s(T;\mathbf{v})} + 2TR_3\Big) \le \frac{1}{2}.$$

Here we used $C_1(T) = \mathcal{O}(T^{1-\gamma})$.

Then we have

$$\max_{i=1,2}\max_{|\alpha|=1}[[\partial^\alpha \hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \le 2\max_{i=1,2}\max_{|\alpha|=1}[[\nabla u_{0i}]]_{0,\gamma,\bar{\Omega}_s(0)} + 2C_1(T)R_3. \tag{A.157}$$

We combine (A.156) and (A.157) to get

$$\max_{i=1,2}\max_{|\alpha|=1}|||\partial^\alpha \hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \le 2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + 2TR_3 + 2C_1(T)R_3.$$

We assume again that T is sufficiently small so that

$$2TR_3 + 2C_1(T)R_3 \le \frac{\delta^*}{3}. \tag{A.158}$$

Then we have

$$\max_{i=1,2}|||\nabla \hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \le 2\max_{i=1,2}||\nabla u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}. \tag{A.159}$$

(iii) We differentiate (A.154) with respect to x_l to obtain

$$\begin{aligned}&\frac{D(\partial^2_{x_k x_l}\hat{u}_i)}{Dt} + \sum_{j=1}^{3}\Big[(\partial_{x_l}\hat{u}_j)(\partial^2_{x_j x_k}\hat{u}_i) + (\partial_{x_i}\hat{u}_j)(\partial^2_{x_k x_l}\hat{u}_j) + (\partial_{x_k}\hat{u}_j)(\partial^2_{x_i x_l}\hat{u}_j)\Big]\\ &\qquad = \partial^2_{x_k x_l}(\partial_{x_i}\phi).\end{aligned} \tag{A.160}$$

We integrate the equation (A.160) along the characteristic curve to find

$$
\begin{aligned}
&(i) \text{ if } t_0 = 0,\\
&\partial^2_{x_k x_l}\hat{u}_i(\mathbf{x},t) = \partial^2_{x_k x_l} u_{0i}(\boldsymbol{\alpha}(\mathbf{x},t))\\
&\qquad -\sum_{j=1}^{2}\int_0^t \Big((\partial_{x_l}\hat{u}_j)(\partial^2_{x_j x_k}\hat{u}_i) + (\partial_{x_i}\hat{u}_j)(\partial^2_{x_k x_l}\hat{u}_j) + (\partial_{x_k}\hat{u}_j)(\partial^2_{x_i x_l}\hat{u}_j)\Big)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}),s)ds\\
&\qquad +\int_0^t \partial_{x_k}\partial_{x_l}(\partial_{x_i}\phi)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}),s)ds;\\
&(ii) \text{ if } t_0 > 0,\\
&\partial_{x_k}\partial_{x_l}\hat{u}_i(\mathbf{x},t) = -\partial^2_{x_k x_l}\nu_i(\boldsymbol{\alpha}(\mathbf{x},t))\\
&\qquad -\sum_{j=1}^{2}\int_{t_0}^t \Big((\partial_{x_l}\hat{u}_j)(\partial^2_{x_j x_k}\hat{u}_i) + (\partial_{x_i}\hat{u}_j)(\partial^2_{x_k x_l}\hat{u}_j) + (\partial_{x_k}\hat{u}_j)(\partial^2_{x_i x_l}\hat{u}_j)\Big)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}),s)ds\\
&\qquad +\int_{t_0}^t \partial^2_{x_k x_l}(\partial_{x_i}\phi)(\hat{\boldsymbol{\chi}}(s,t,\mathbf{x}),s)ds.
\end{aligned}
\tag{A.161}
$$

Then it follows from (A.161) that

$$
\begin{aligned}
&\max_{i=1,2}\max_{|\alpha|=2}|||\partial^\alpha\hat{u}_i|||_{0,\bar{\Lambda}_s(T;\mathbf{v})}\\
&\quad \le \max_{i=1,2}\max_{|\alpha|=2}||\partial^\alpha\hat{u}_{0i}||_{0,\bar{\Omega}_s(0)} + 6T\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}\Big)\\
&\quad \times \max_{i=1,2}\max_{|\alpha|=2}|||\partial^\alpha\hat{u}_i|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} + TR_3.
\end{aligned}
$$

We choose T sufficiently small so that

$$
6T\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}\Big) \le \frac{1}{2} \quad \text{and} \quad TR_3 \le \frac{\delta^*}{12}. \tag{A.162}
$$

Then we have

$$
\max_{i=1,2}\max_{|\alpha|=2}|||\partial^\alpha\hat{u}_i|||_{0,\bar{\Lambda}_s(T;\mathbf{v})} \le 2\max_{i=1,2}\max_{|\alpha|=2}||\partial^\alpha u_{0i}||_{0,\bar{\Omega}_s(0)} + \frac{\delta^*}{6}. \tag{A.163}
$$

We need to check the Hölder seminorm of $\partial^\alpha\hat{u}_i$. Again we use (A.161) to find

$$
\begin{aligned}
&\max_{i=1,2}\max_{|\alpha|=2}[[\partial^\alpha\hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\\
&\quad \le \max_{i=1,2}[[\partial^\alpha u_{0i}]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} + 6C_1(T)\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}\Big)\\
&\quad \times \max_{i=1,2}\max_{|\alpha|=2}[[\partial^\alpha\hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} + 6C_1(T)\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}\Big)\\
&\quad \times\Big(2\max_{i=1,2}\max_{|\alpha|=2}||\partial^\alpha u_{0i}||_{0,\bar{\Omega}_s(0)} + \frac{\delta^*}{6}\Big) + C_1(T)R_3.
\end{aligned}
$$

Here we have used (A.84).

We assume that T sufficiently is sufficiently small that

$$6C_1(T)\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)}+\frac{\delta^*}{3}\Big)\le\frac{1}{2}\quad\text{and}\tag{A.164}$$

$$\begin{aligned}&6C_1(T)\Big(2\max_{i=1,2}\max_{|\alpha|=1}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)}+\frac{\delta^*}{3}\Big)\\&\times\Big(2\max_{i=1,2}\max_{|\alpha|=2}||\partial^\alpha \hat{u}_{0i}||_{0,\Omega_s(0)}+\frac{\delta^*}{6}\Big)+C_1(T)R_3\le\frac{\delta^*}{12}.\end{aligned}\tag{A.165}$$

Hence we have

$$\max_{i=1,2}\max_{|\alpha|=2}[[\partial^\alpha\hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\le 2\max_{i=1,2}\max_{|\alpha|=2}[\partial^\alpha u_{0i}]_{0,\gamma,\bar{\Omega}_s(0)}+\frac{\delta^*}{6}.\tag{A.166}$$

We combine (A.163) and (A.166) to get

$$\max_{i=1,2}\max_{|\alpha|=2}|||\partial^\alpha\hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\le 2\max_{i=1,2}\max_{|\alpha|=2}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)}+\frac{\delta^*}{3}.\tag{A.167}$$

We combine (A.153), (A.159) and (A.167) to get

$$\max_{i=1,2}\sum_{0\le k\le 2}\max_{|\alpha|=k}|||\partial^\alpha\hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\le 2\max_{i=1,2}\sum_{0\le k\le 2}\max_{|\alpha|=k}||\partial^\alpha u_{0i}||_{0,\gamma,\bar{\Omega}_s(0)}+\delta^*\le 3\delta^*.\tag{A.168}$$

(iv) It follows from the Burgers' equation (A.139) that

$$\begin{aligned}\max_{i=1,2}|||\partial_t\hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}&\le 2\Big(\max_{i=1,2}|||\hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\Big)\Big(\max_{i=1,2}\max_{|\alpha|=1}|||\partial^\alpha\hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\Big)\\&+|||\nabla\phi|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})}\\&\le 18(\delta^*)^2+R_3.\end{aligned}\tag{A.169}$$

Finally we combine all estimates (A.168) and (A.169) to get

$$\max_{i=1,2}\Big(\max_{|\alpha|\le 2}|||\partial^\alpha\hat{u}_i|||_{0,\gamma,\bar{\Lambda}(T)}\Big)\le 3\delta^*\quad\text{and}\quad\max_{i=1,2}|||\partial_t\hat{u}_i|||_{0,\gamma,\bar{\Lambda}(T)}\le\Big(18(\delta^*)^2+R_3\Big).$$

By the construction of extension of $\hat{\mathbf{u}}$, we find

$$\text{(a)}\ \max_{i=1,2}\Big(\max_{|\alpha|\le 2}|||\partial^\alpha u_i|||_{0,\gamma,\bar{\Lambda}(T)}\Big)\le 3K_0\delta^*,\tag{A.170}$$

$$\text{(b)}\ \max_{i=1,2}|||\partial_t u_i|||_{0,\gamma,\bar{\Lambda}(T)}\le K_0\Big(18(\delta^*)^2+R_3\Big).\tag{A.171}$$

Here we notice that the norm $||\cdot||_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})}$ in Appendix C can be generalized to the space-time norm $|||\cdot|||_{0,\gamma,\bar{\Lambda}(T)}$.

Finally the estimates (A.170) and (A.171) show that $\mathbf{u}\in\mathcal{B}(T)$. □

We set

$$\Lambda_s(T;r):\ \text{the sheath region determined by the interface } r\ ,$$

and recall that an interface $\mathcal{S}(t)$ is represented by the radial function $r(\cdot,t)$.

Lemma A.13. ([13]) *Let* $\tau < \gamma$,

$r_i \to r$ *in* $C^{1,\tau}(\mathbb{R} \times [0,T])$ *as given by Lemma A.5 and*
$\hat{\mathbf{u}}_i \in C^{1,\tau}(\Lambda_s(T; r_i))$: *be associated solutions of (A.139) for each i as given by Lemma A.10.*

Let $\mathcal{E}(\hat{\mathbf{u}}_i(\cdot,t); r_i(t))$ *be the extension of* $\hat{\mathbf{u}}_i(\cdot,t)$ *with*

$$\mathcal{E}(\mathbf{u}_i(\cdot,t); r_i(t)) \to \mathbf{w} \quad \text{in } C^{1,\tau}(\bar{\Lambda}(T)).$$

Then we have

$$\mathbf{w} = \mathcal{E}(\hat{\mathbf{w}}\Big|_{\Lambda_s(T;r)}).$$

Proof. The proof follows from a straightforward modification of the proof in [13] as hence is omitted. □

Proof of Theorem 7.3

Let $\{\mathbf{v}_i\}$ be a convergent sequence in $\mathcal{B}_T$ in the topology of $\mathcal{T}$ (see (7.66)) such that

$$\mathbf{v}_i \to \mathbf{v} \quad \text{in } C^{1,\tau}(\bar{\Lambda}(T)) \quad \text{and} \quad \partial^\alpha \mathbf{v}_i \to \partial^\alpha \mathbf{v}, \quad \text{in } C^{0,\tau} \quad |\alpha| = 2, \quad 0 < \tau < \gamma.$$

By Proposition A.1, $\mathcal{F}(\mathbf{v}_i)$ is well-defined as an element of $\mathcal{B}(T)$ for each i and the sequence $\{\mathcal{F}(\mathbf{v}_i)\}$ is uniformly bounded in $\mathcal{T}$. Since the Arzela-Ascoli theorem implies the compact imbedding of $C^{1,\gamma}(\bar{\Lambda}(T))$ into $C^{1,\tau}(\bar{\Lambda}(T))$ with $0 < \tau < \gamma$, we have a convergent subsequence which we still denote by $(\mathbf{v}_i, \mathcal{F}(\mathbf{v}_i))$:

$$\mathcal{F}(\mathbf{v}_i) \to \mathbf{w} \qquad \text{in } C^{1,\tau}(\bar{\Lambda}(T)).$$

We claim:

$$\mathcal{F}(\mathbf{v}) = \mathbf{w}. \tag{A.172}$$

Proof of the claim: Let $(\boldsymbol{\chi}_i, n_i, \hat{\mathbf{u}}_i, \mathcal{S}_i, \phi_i)$ and $(\boldsymbol{\chi}, n, \hat{\mathbf{u}}, \mathcal{S}, \phi)$ be the quantities corresponding to $\mathbf{v}_i$ and $\mathbf{v}$ respectively.

Step I. Suppose that

$$\mathbf{v}_i \to \mathbf{v} \text{ in } C^{1,\tau}(\bar{\Lambda}(T)) \text{ as } i \to \infty.$$

Then it follows from Lemma A.1 (2) that $\boldsymbol{\chi}_i(\cdot,t,\mathbf{x}) \to \boldsymbol{\chi}(\cdot,t,\mathbf{x})$ in $C^{1,\gamma}([0,T])$ as $i \to \infty$ and hence since $\mathbf{v} \in C^{1,\tau}(\bar{\Lambda}(T))$, we have

$$\nabla \cdot \mathbf{v}_i(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}),\xi) \to \nabla \cdot \mathbf{v}(\boldsymbol{\chi}(\xi,t,\mathbf{x}),\xi) \quad \text{in } \ C^{1,\tau}(\partial\Omega_0 \times [0,T]) \qquad \text{as } i \to \infty.$$

We use Lemma A.2 to get

$$\int_0^t \nabla \cdot \mathbf{v}_i(\boldsymbol{\chi}_i(\xi,t,\mathbf{x}),\xi)d\xi \to \int_0^t \nabla \cdot \mathbf{v}(\boldsymbol{\chi}(\xi,t,\mathbf{x}),\xi)d\xi \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0,T]) \quad \text{as } i \to \infty. \tag{A.173}$$

On the other hand, since $\boldsymbol{\alpha}_i \to \boldsymbol{\alpha}$ in $C^{1,\tau}(\partial\Omega_0 \times [0,T])$ as $i \to \infty$, we have

$$n_0(\boldsymbol{\alpha}_i(\mathbf{x},t)) \to n_0(\boldsymbol{\alpha}(\mathbf{x},t)) \qquad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0,T]) \quad \text{as } i \to \infty. \tag{A.174}$$

Here we used the fact that n_0 is in $C^{1,\tau}(\mathbb{R}^2)$. Recall the formula for n:

$$n(\mathbf{x},t) = n_0(\boldsymbol{\alpha}(\mathbf{x},t)) \exp\Big(- \int_0^t (\nabla \cdot \mathbf{v})(\boldsymbol{\chi}(\xi,0,\boldsymbol{\alpha}(\mathbf{x},t)),\xi)d\xi \Big).$$

We now combine (A.173) and (A.174) and the above formula to see

$$n_i(\mathbf{x},t) \to n(\mathbf{x},t) \qquad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0,T]) \quad \text{as } i \to \infty,$$

which in turn implies

$$h_{0i} = \partial_t g - (n_i \mathbf{v}_i) \cdot \boldsymbol{\nu} \to h_0 = \partial_t g - (n\mathbf{v}) \cdot \boldsymbol{\nu}, \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0,T]).$$

By Lemma A.5 and Lemma A.6, we have

$$\nabla\zeta_i \to \nabla\zeta \quad \text{in } C^{1,\tau}(\bar{\Omega}_* \times [0,T]) \quad \text{as } i \to \infty, \tag{A.175}$$

$$(\theta_i, r_i, n_i) \to (\theta, r, n) \quad \text{in } C^{1,\tau}(\mathbb{R} \times [0,T]). \tag{A.176}$$

Step II. Let $\Lambda_s(T;\mathbf{v})$ be the sheath region determined by $\mathbf{v}$. Since $\mathcal{F}(\mathbf{v})$ is uniquely determined by $\mathbf{v}$ on the sheath region, once we can show $\mathbf{w}$ satisfies equations (7.62)-(7.63) and the interface conditions:

$$\mathbf{u} = -\boldsymbol{\nu} \qquad \text{and} \qquad \nabla\phi \cdot \boldsymbol{\nu} = 0 \qquad \text{on } \mathcal{S}(t), \tag{A.177}$$

for the orthogonal flow in the sheath region $\Lambda_s(T;\mathbf{v})$, we will have

$$\mathbf{w} = \mathcal{F}(\mathbf{v}) \quad \text{in } \Lambda_s(T;\mathbf{v}).$$

So let us proceed to show that $\mathbf{w}$ satisfies the sheath system (7.62) and boundary conditions (A.177) in $\Lambda_s(T;\mathbf{v})$. Let $\mathcal{O}$ be any open set compactly supported in $\Lambda_s(T;\mathbf{v})$. Then by (A.175) and (A.176), since $r_i \to r$ in $C^{1,\tau}(\mathbb{R} \times [0,T])$, we have

$$\mathcal{O} \subset \Lambda_s(T;\mathbf{v}_i) \quad i \geq N.$$

For $i \geq N$, we know that $(n_i, \mathbf{v}_i, \phi_i, \hat{\mathbf{u}}_i)$ satisfy

$$\begin{cases} \partial_t n_i + \nabla \cdot (n_i \mathbf{v}_i) = 0, \quad (\mathbf{x}, t) \in \mathcal{O}, \\ \Delta\phi_i = n_i, \\ \partial_t \hat{\mathbf{u}}_i + (\hat{\mathbf{u}}_i \cdot \nabla)\hat{\mathbf{u}}_i = \nabla\phi_i, \end{cases}$$

and

$$(n_i, \mathbf{v}_i, \phi_i, \hat{\mathbf{u}}_i) \to (n, \mathbf{v}, \phi, \mathbf{w}) \text{ in } C^{1,\tau}(\bar{\mathcal{O}}),$$

and hence we find in the limit as $i \to \infty$,

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{v}) = 0, \quad (\mathbf{x}, t) \in \mathcal{O}, \\ \Delta\phi = n, \\ \partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} = \nabla\phi. \end{cases}$$

Next we check the boundary conditions on the sheath interface. Since by (A.176) $\theta_i \to \theta$ in $C^{1,\gamma}(\mathbb{R})$ and $\boldsymbol{\nu}_i = (\cos\theta_i, \sin\theta_i)$, we obtain

$$\boldsymbol{\nu}_i \to \boldsymbol{\nu}, \text{ in } C^{1,\gamma}(\mathbb{R} \times [0,T]).$$

On the other hand, we have

$$\nabla\phi_i \cdot \boldsymbol{\nu}_i = 0 \quad \text{and} \quad \hat{\mathbf{u}}_i = -\boldsymbol{\nu}_i \quad \text{on } \mathcal{S}_i.$$

Letting $i \to \infty$, we see

$$\nabla\phi \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad \mathbf{w} = -\boldsymbol{\nu} \quad \text{on } \mathcal{S}.$$

Hence we have shown that $\mathbf{w}$ satisfies the sheath system (7.62) in the sheath region and boundary conditions (A.177). By the uniqueness of the construction, we have

$$\mathcal{F}(\mathbf{v}) = \mathbf{w} \quad \text{on } \Lambda_s(T;\mathbf{v}).$$

Step III. Recall by (A.176) that we have

$$r_i \to r \quad \text{in } C^{1,\tau}(\mathbb{R} \times [0,T]) \quad \text{and} \quad \mathcal{F}(\mathbf{v}_i) \to \mathbf{w}.$$

Then by Lemma A.12, we have

$$\mathbf{w} = \mathcal{E}(\mathbf{w}\Big|_{\Lambda_s(T;\mathbf{v})}) = \mathcal{F}(\mathbf{v}).$$

Hence we showed that $\mathcal{F}$ is continuous in the $C^{1,\tau}$-topology. Since $\mathcal{F}$ is a continuous map on the compact and convex set $\mathcal{B}(T)$ of $C^{1,\gamma}$ space, by the Schauder fixed point theorem, $\mathcal{F}$ has a fixed point $\mathbf{u}$ such that

$$\mathcal{F}(\mathbf{u}) = \mathbf{u}.$$

This $\mathbf{u}$ is a desired smooth solution of the sheath system. This completes the proof.

Appendix B. **Gronwall-Bellman type inequality**

In this appendix, we prove the Gronwall-Bellman type inequality.

Let f be a real valued positive continuous function and suppose a nonnegative real valued function y satisfies the following integral inequality:

$$y(t) \leq f(t) + c^2 \int_0^t \int_0^{t_1} y(\tau) d\tau dt_1.$$

Then y satisfies

$$\begin{aligned} y(t) &\leq f(t) + c^2 \int_0^t \int_0^{t_1} f(s) \exp[c(t - 2t_1 + s)] ds dt_1 \\ &= f(t) + \Big(\max_{\tau \in [0,t]} f(\tau) \Big) \mathcal{O}(t^2), \qquad \text{as } t \to 0. \end{aligned}$$

Proof. Let us set

$$w(t) \equiv \int_0^t \int_0^{t_1} y(\tau) d\tau dt_1.$$

Then we have

$$y(t) \leq f(t) + c^2 w(t). \tag{B.178}$$

It is easy to see that

$$w''(t) = y(t), \qquad w(0) = 0, \qquad w'(0) = 0.$$

In (C.179), we have a differential inequality for w:

$$w''(t) \leq c^2 w(t) + f(t).$$

Now we introduce another dependent variable u defined by

$$w(t) = \exp(ct) u(t).$$

By direct calculation, we obtain a differential inequality for u:

$$u'' + 2cu' \leq f(t) \exp(-ct).$$

We multiply an integrating factor $\exp(2ct)$ to get

$$\Big(\exp(2ct) u' \Big)' \leq f(t) \exp(ct).$$

Next we integrate the above inequality to get

$$u(t) \leq \int_0^t \int_0^{t_1} f(s) \exp[-c(2t_1 - s)] ds dt_1,$$

where we used

$$u(0) = 0, \qquad u'(0) = 0.$$

This implies

$$\begin{aligned} w(t) &= \exp(ct) u(t) \\ &\leq \int_0^t \int_0^{t_1} f(s) \exp[c(t - 2t_1 + s)] ds dt_1. \end{aligned}$$

In (B1), we have

$$\begin{aligned} y(t) &\leq f(t) + c^2 \int_0^t \int_0^{t_1} f(s) \exp[c(t - 2t_1 + s)] ds dt_1 \\ &\leq f(t) + c^2 \Big(\max_{\tau \in [0,t]} f(\tau) \Big) \int_0^t \int_0^{t_1} \exp[c(t - 2t_1 + s)] ds dt_1 \\ &= f(t) + \Big(\max_{\tau \in [0,t]} f(\tau) \Big) \mathcal{O}(t^2) \qquad \text{as } t \to 0, \end{aligned}$$

where we used

$$\int_0^t \int_0^{t_1} \exp[c(t - 2t_1 + s)] ds dt_1 = \frac{1}{c^2} \Big[-1 + \frac{1}{2}(e^{-ct} + e^{ct}) \Big] = \frac{\mathcal{O}(t^2)}{c^2} \qquad \text{as } t \to 0.$$

□

Appendix C. **Extension Theorem**

In this part, we present an extension theorem for $C^{2,\gamma}$-functions defined on the sheath region $\Omega_s(t;\mathbf{v}), t \in [0,T]$ to the bigger domain $\Omega_1 := B(0,3\delta^*) - \Omega_0$.

We first consider an upper bounds for the length of a convex polygon and a simple closed convex curve inside the annulus $A(r_1,r_2)$ defined by

$$A(r_1,r_2) := \{\mathbf{x} \in \mathbb{R}^2 : r_1 < |\mathbf{x}| < r_2\}.$$

Lemma C.1. *Let $\mathcal{P}$ and $\mathcal{C}$ be a convex n-polygon and a convex curve inside the annulus $A(r_1,r_2)$ respectively. Then we have*

$$l(\mathcal{P}) \le 2\pi r_2 \quad \text{and} \quad l(\mathcal{C}) \le 2\pi r_2,$$

where $l(\mathcal{P})$ and $l(\mathcal{C})$ denote the lengths of the polygon $\mathcal{P}$ and the curve $\mathcal{C}$ respectively.

Proof. (i) Let $\mathcal{P} = \mathcal{P}(\mathbf{x}_1,\cdots,\mathbf{x}_n)$ be a convex n-polygon whose vertices are $\mathbf{x}_1,\cdots,\mathbf{x}_n$. We choose any point $\mathbf{c_0}$ inside $\mathcal{P}$, and we set

$$\mathbf{y}_i : \quad \text{the intersection point with a ray } \overrightarrow{\mathbf{c_0x_i}} \text{ and a circle } B(0,r_2).$$

Then it is easy to see that

$$l(\mathcal{P}(\mathbf{x}_1,\cdots,\mathbf{x}_n)) \le l(\mathcal{P}(\mathbf{y}_1,\cdots,\mathbf{y}_n)). \tag{C.179}$$

On the other hand we know that

$$l(\mathcal{P}(\mathbf{y}_1,\cdots,\mathbf{y}_n)) \le l(B(0,r_2)) = 2\pi r_2. \tag{C.180}$$

We combine (C.179) and (C.180) to obtain

$$l(\mathcal{P}) \le 2\pi r_2.$$

(ii) Let $\mathcal{C}$ be a simple closed convex curve lying inside $A(r_1,r_2)$. Note that for any simple closed convex curve there exists some polygon whose sides are parts of supporting lines of the given convex curve. Choose a sufficiently small positive constant $r_0 > 0$. Since the curve $\mathcal{C}$ is compact, there exists a finite open cover of $\mathcal{C}$ consisting of balls with a center $\bar{\mathbf{x}}_i$ and a radius r_0, say,

$$\mathcal{C} \subset \cup_{i=1}^M B(\bar{\mathbf{x}}_i, r_0), \quad \text{where } \bar{\mathbf{x}}_i \in \mathcal{C}.$$

Consider an M-polygon consisting of parts of supporting lines at $\bar{\mathbf{x}}_i, i = 1,\cdots,M$ and denote it by $\bar{\mathcal{P}}$. Then it follows from the result of (i) that

$$l(\mathcal{C}) \le l(\bar{\mathcal{P}}) \le 2\pi r_2.$$

□

Next we present the existence of a continuous linear extension operator from $C^{1,\gamma}(\Omega_s(t;\mathbf{v}))$ to $C^{1,\gamma}(\Omega_1)$. Even though the construction of this extension operator can be found in the literature, see for example [1, 28, 33], we slightly modify the proofs given in books [1, 28, 33] for our purpose.

The proof of Lemma A.10: We first consider the local extension near one generic point on the interface and then glue these local extensions together using the standard partition of unity to get a global extension. Let t be given.

Step I (local extension): Let $\mathbf{x}_0$ be any generic point on the interface $\mathcal{S}(t)$. Then there are two cases: either $\mathcal{S}(t)$ is flat near $\mathbf{x}_0$, lying in the plane or it is not flat near $\mathbf{x}_0$.

Case 1: $\mathcal{S}(t)$ is flat near $\mathbf{x}_0$ lying on some line.

For simplicity, we assume $\mathbf{x}_0 = (a_1, a_2)$ and the plane is $\{x_2 = a_2\}$. We choose an open ball $B(\mathbf{x}_0, r)$ such that

$$\begin{cases} B^+ := B(\mathbf{x}_0, r) \cap \{x_2 \geq a_2\} \subset B(0, 3\delta^*) - \Omega_s(t; \mathbf{v}), \\ B^- := B(\mathbf{x}_0, r) \cap \{x_2 \leq a_2\} \subset \bar{\Omega}_s(t; \mathbf{v}). \end{cases}$$

Let f be any $C^{2,\gamma}$-function defined on $\Omega_s(t; \mathbf{v})$. We extend f to the ball $B^+ \cup B^-$ as follows.

$$\bar{f}(x_1, x_2) := \begin{cases} 6f(x_1, 2a_2 - x_2) - 8f(x_1, 3a_2 - 2x_2) + 3f(x_1, 4a_2 - 3x_2), & \text{if } (x_1, x_2) \in B^+, \\ f(x_1, x_2), & \text{if } (x_1, x_2) \in B^-. \end{cases}$$

Notice this choice of $\bar{f}$ is not the same as given by Evans [28], since he only desired C^1-regularity. We have used a special case of the result given in [1].

We claim: $\bar{f}$ is $C^{2,\gamma}$ in the ball B.

We need to show all partial derivatives are continuous at $\mathbf{x}_0 = (a_1, a_2)$. Let us write $f^- := \bar{f}\Big|_{B^-}$, $f^+ := \bar{f}|_{B^+}$. By direct calculation we obtain

- $\partial_{x_1}^k f^-(x_1, x_2)$
$= 6\partial_{x_1}^k f(x_1, 2a_2 - x_2) - 8\partial_{x_1}^k f(x_1, 3a_2 - 2x_2) + 3\partial_{x_1}^k f(x_1, 4a_2 - 3x_2), \ k = 0, 1, 2,$
- $\partial_{x_2} f^-(x_1, x_2)$
$= -6\partial_{x_2} f(x_1, 2a_2 - x_2) + 16\partial_{x_2} f(x_1, 3a_2 - 2x_2) - 9\partial_{x_2} f(x_1, 4a_2 - 3x_2),$
- $\partial_{x_2}^2 f^-(x_1, x_2)$
$= 6\partial_{x_2}^2 f(x_1, 2a_2 - x_2) - 32\partial_{x_2}^2 f(x_1, 3a_2 - 2x_2) + 27\partial_{x_2}^2 f(x_1, 4a_2 - 3x_2),$
- $\partial_{x_1}\partial_{x_2} f^-(x_1, x_2)$
$= -6\partial_{x_1}\partial_{x_2} f(x_1, 2a_2 - x_2) + 16\partial_{x_1}\partial_{x_2} f(x_1, 3a_2 - 2x_2) - 9\partial_{x_1}\partial_{x_2} f(x_1, 4a_2 - 3x_2)$

Now evaluate the above identities on the line $\{x_2 = a_2\}$ to see that extended function $\bar{f}$ is C^2 in the ball B and we have

$$[\partial_{x_1}^2 f^-]_{0,\gamma,\bar{B}^+} \leq 31[\partial_{x_1}^2 f]_{0,\gamma,\bar{B}^-}, \qquad [\partial_{x_1}\partial_{x_2} f^-]_{0,\gamma,\bar{B}^+} \leq 119[\partial_{x_1}\partial_{x_2} f]_{0,\gamma,\bar{B}^-}$$
$$\text{and} \quad [\partial_{x_2}^2 f^-]_{0,\gamma,\bar{B}^+} \leq 151[\partial_{x_2}^2 f]_{0,\gamma,\bar{B}^-}.$$

Hence we have

$$||\bar{f}||_{2,\gamma,\bar{B}} \leq 151||f||_{1,\gamma,\bar{B}^-}.$$

Case 2: $\mathcal{S}(t)$ is not flat near $\mathbf{x}_0$.

Since the interface $\mathcal{S}(t)$ is $C^{2,\gamma}$-regular, we can find a $C^{2,\gamma}$-mapping Φ with inverse Φ^{-1} such that Φ straightens out $\mathcal{S}(t)$ near $\mathbf{x}_0$. We write $\mathbf{y} = \Phi(\mathbf{x}), f'(\mathbf{y}) := f(\Phi^{-1}(\mathbf{y}))$. We choose a small ball B as before. Then as in *Case 1*, we extend f' from B^- to B and get

$$||\bar{f}'||_{2,\gamma,\bar{B}} \leq 151||f'||_{2,\gamma,\bar{B}^-}.$$

Let $W := \Phi^{-1}(B)$ and $W^{\pm} := \Phi^{-1}(B^{\pm})$. Then we have

$$||\bar{f}||_{2,\gamma,\bar{W}} \leq 151||f||_{2,\gamma,\bar{W}^-}.$$

Now we glue local extensions together using the partition of unity to get a global extension.

Step II (Global extension): We will extend f defined on $\Omega_s(t;\mathbf{v})$ to the bigger domain Ω_1 such that the extended $\bar{f}$ has support in Ω_1. Let r_1 be a sufficiently small number satisfying

$$0 < r_1 < \min\Big\{\delta^*, 0.5\delta_{*2} - r_b\Big\}.$$

Then for such r_1, we choose points $\mathbf{x}_i (i = 1, \cdots, M(t))$ on the curve $\mathcal{S}(t)$ such that neighboring $\mathbf{x}_i$'s are located by the part of curve with length r except one pair of points, i.e.,

$$l(\text{part of an interface curve connecting } \mathbf{x}_i \text{ and } \mathbf{x}_{i+1}) = r_0, \quad i = 1, \cdots, M(t) - 1,$$
$$l(\text{part of an interface curve connecting } \mathbf{x}_{M(t)} \text{ and } \mathbf{x}_1) \leq r.$$

Then the number $M(t)$ of such points are bounded by

$$M(t) \leq \Big[\frac{l(\mathcal{S}(t))}{r_1}\Big] + 1,$$

where the bracket is the greatest integer function. Then by Lemma C.1, we know that

$$M(t) \leq \Big[\frac{4\pi\delta^*}{r_1}\Big] + 1, \quad t \in [0, T_*].$$

As in *Step I*, we extend f to $B(\mathbf{x}_i, r)$ for each i, and denote $\bar{f}_i$ by the extended function. Now take an open set W_0 whose closure is a compact subset of $\Omega_s(t)$ and $\Omega_s(t) \subset W_0(t) \cup \Big(\cup_{i=0}^{M(t)} W_i(t)\Big)$. Let $\{\kappa_i\}$ be a partition of unity corresponding to the open covering $\{W_i(t)\}_{i=0}^{M(t)}$ of $\Omega_s(t;\mathbf{v})$ and define

$$\bar{f} := \sum_{i=0}^{M(t)} \kappa_i \bar{f}_i, \quad \bar{f}_0 = f.$$

It follows from *Step I* that

$$||\bar{f}||_{2,\gamma,\bar{\Omega}_1} \leq 151(M(t)+1)||f||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})}.$$

We take K_0 to be $151\Big(\Big[\frac{4\pi\delta^*}{r_1}\Big] + 2\Big)$ to obtain the desired result.

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References

[1] R.A. Adams, *Sobolev spaces,* Pure and Applied Mathematics, vol 65, Academic press, New York, 1975.

[2] J.G. Andrews, and R.H. Varey, *Sheath growth in a low pressure plasma,* Physics of Fluids, 14 (1971), pp. 339-343.

[3] S.B. Angenent and M.E. Gurtin, *Anisotropic motion of a phase interface. Well-posedness of the initial value problem and qualitative properties of the interface,* J. Reine Angew. Math. 446 (1994), pp. 1-47.

[4] S. B. Angenent and M. E. Gurtin, *Multiphase thermomechanics with interfacial struture. II. Evolution of an isothermal interface,* Arch. Rational Mech. Anal., 108 (1989), pp. 323-391.

[5] S. Agmon, A.Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I,* Comm. Pure and Appl. Math. 12(1959), pp. 623-727.

[6] G. Auchmuty and J.C. Alexander, L^2 *well-posedness of planar Div-Curl systems,* Arch. Rational Mech. Anal. 160 (2001), pp. 91-134.

[7] N. Ben Abdallah, *Convergence of the Child-Langmuir asymptotics of the Boltzmann equation of semiconductors,* SIAM J. Math. Anal. 27 (1996), pp. 92-109.

[8] N. Ben Abdallah, *The Child-Langmuir regime for electron transport in a plasma including a background of positive ions,* Math. Models Methods Appl. Sci. 4 (1994), pp. 409-438.

[9] N. Ben Abdallah, P. Degond, and P. Markowich,*The quantum Child-Langmuir problem,* Nonlinear Anal. 31 (1998), pp. 629-648.

[10] N. Ben Abdallah, and P. Degond, *The Child-Langmuir law for the Boltzmann equation of semiconductor,* SIAM J. Math. Anal. 26 (1995), pp. 364-398.

[11] L. Bers, *Existence and uniqueness of subsonic flow past a given profile,* Comm. Pure Appl. Math. 7 (1954), pp. 441-504.

[12] S. Canic, B.L. Keyfitz, and G.M. Liebermann, *A proof of existence of perturbed steady transonic shocks via a free boundary problem,* Comm. Pure Appl. Math 53 (2003), pp. 484-511.

[13] G.-Q. Chen, and M. Feldman, *Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type,* J. Amer. Math. Soc. 16 (2003), no. 3, pp. 461-494.

[14] A.J. Chorin, and J.E. Marsden, *A mathematical introduction to fluid mechanics,* Springer-Verlag, New York (1993).

[15] J.W. Cipolla, and M.B., Silevitch, *On the temporal development of a plasma sheath,* Plasma Physics 25 (1981), pp. 373-389.

[16] J.R. Conrad, J.R, J. L. Radtke, R.A., Dodd, F.J., Worzaka, and N.C. Tran, *Plasma source ion-implantation technique for surface modification of materials,* J. Appl. Phys. 62 (1987), pp. 4591-4596.

[17] S. Cordier, *Global solutions to the isothermal Euler-Poisson plasma model,* Appl. Math. Lett. 8 (1995), pp. 19-24.

[18] S. Cordier, and E. Grenier, *Quasi-neutral limit of an Euler-Poisson system arising from plasma physics,* Comm. Partial Differential Equations, 25 (2000), pp. 1099-1113.

[19] R. Courant, and D. Hilbert, *Methods of mathematical physics,* vol II, (Partial differential equations), A Wiley-Interscience Publication, New York (1989).

[20] P. Degond, S. Jaffard, F. Poupaud, and P.A. Raviart, *The Child-Langmuir asymptotics of the Vlasov-Poisson equation for cylindrical or spherically symmetric diodes. II. (Analysis of the reduced problem and determination of the Child-Langmuir current),* Math. Methods Appl. Sci., 19 (1996), pp. 313-340.

[21] P. Degond, S. Jaffard, F. Poupaud, and P.A. Raviart, *The Child-Langmuir asymptotics of the Vlasov-Poisson equation for cylindrical or spherically symmetric diodes. I. (Statement of the problem and basic estimates),* Math. Methods Appl. Sci., 19 (1996), pp. 287-312.

[22] P. Degond, C., Parzani and M.-H. Vignal, *Un modèle d'expansion de plasma dans le vide,* Comptes Rendus Academie Sciences (Paris), Series 1, 335(2002), 399-404.

[23] P. Degond, and P.A. Raviart, *On a penalization of the Child-Langmuir emission condition for the one-dimensional Vlasov-Poisson equation.* Asymptotic Analysis, 6 (1992), pp. 1-27.

[24] P. Degond, and R.A. Raviart, *An asymptotic analysis of the one-dimensional Vlasov-Poisson system: the Child-Langmuir law,* Asymptotic Analysis, 4 (1991), pp. 187-214.

[25] A. Douglis, *Some existence theorems for hyperbolic systems of partial differential equations in two independent variables,* Comm. Pure and Appl. Math. 5 (1952), pp. 119-154.

[26] W. E, Yu. G. Rykov, and Ya. G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics,* Commun. Math. Phys. 177 (1996), pp. 349-380.

[27] S. Engelberg, H. Liu, and E. Tadmor, *Critical threshholds in Euler-Poisson equations,* Indiana Univ. Math. J., 50 (2001), pp. 109-157.

[28] C.E. Evans, *Partial differential equations,* Graduate Studies in Mathematics, vol. 19 AMS.

[29] R. Finn and D. Gilbarg, *Asymptotic behavior and uniqueness of plane subsonic flows,* Comm. Pure Appl. Math. 10 (1957), 23-63.

[30] R. N. Franklin, and J.R. Ockendon, *Asymptotic matching of plasma and sheath in an active low pressure discharge,* Journal of Plasma Physics 4 (1970), pp. 3521-3528.

[31] K.O. Friedrichs, *Nonlinear hyperbolic differential equations for functions of two independent variables,* Amer. J. Math., 70 (1948) , pp. 555-589.

[32] J. Gierling, and K.-U. Riemann, *Comparison of a consistent theory of radio frequency sheaths with step models,* Journal of Applied Physics 83 (1988), pp. 3521-3528.

[33] D. Gilbarg, and N.S. Trudinger, *Elliptic partial differential equations of second order,* Grundlehren der mathematischen Wissenschaften, 224, 2nd edition, Berlin, Springer-Verlag, (1983).

[34] V.A. Godyak, and N. Sternberg, *Dynamic model of the electrode sheaths in symmetrically driven of discharges,* Physical Review A., 42 (1990), pp. 2299-2312.

[35] C. Greengard, and P.A. Raviart, *A boundary value problem for the stationary Vlasov-Poisson equations: the plane diode,* Comm. Pure Appl. Math., 43 (1990), pp. 473-507.

[36] M.E. Gurtin, *Configurational forces as basic concepts of continuum physics,* Applied Mathematical Sciences, 137 (2000), New York, Springer-Verlag.

[37] S.-Y Ha, and M. Slemrod, *Global existence of plasma-ion sheaths and their dynamics,* Commun. Math. Phys. 238, 149-186 (2003).

[38] Y.-W. Kim, G.-H. Kim, S. Han, Y. Lee, J. Cho, and S.-Y. Rhee, *Measurement of sheath expansion in plasma source ion implantation,* Surface and Coatings Technology, 136 (2001), pp. 97-101.

[39] T.-T. Li, and W.-C. Yu, *Boundary value problems for quasi-linear hyperbolic systems,* Duke University Mathematics Series 5, Durham, N.C. (1985), p.74.

[40] M.A. Lieberman, and A.J. Lichtenberg, *Principles of plasma discharge and materials processing,* New York, A Wiley-Interscience Publication, (1994).

[41] H. Liu, and M. Slemrod, *KdV dynamics in the plasma-sheath transition,* to appear in Appl. Math. Lett.

[42] H. Liu, and E. Tadmor, *Spectral dynamics of the velocity gradient field in restricted flows,* Commun. Math. Phys., 228 (2002), pp. 435-466.

[43] A. Nouri, *An Euler-Poisson system in plasma,* J. Austral. Math. Soc. Ser: B, 41 (2000), pp. 442-450.

[44] S. Osher, and R. Fedkiw, *Level set methods and dynamic implicit surfaces,* Applied Mathematical Sciences 153, Springer-Verlag, New York (2003).

[45] F. Poupaud, M. Rascle, and J.P. Vila, *Global solutions to the isothermal Euler-Poisson system with arbitrarily large data,* Journal of Differential Equations, 123 (1995), pp. 93-121.

[46] K.-U. Riemann, *Theory of collisional presheath in an oblique magnetic field,* Phys. Plasmas, 1(1994), pp. 552-558.

[47] K.-U. Riemann, *The Bohm criterion and sheath formation,* J. Phys. D: Appl. Phys., 24 (1991), pp. 493-518.

[48] K.-U. Riemann, and Th. Daube, *Analytical model of the relaxation of a collisionless ion matrix sheath,* J. Appl. Phys., 86 (1999), pp. 1201-1207.

[49] M. Slemrod, and N. Sternberg, *Quasi-neutral limit for Euler-Poisson system,* J. Nonlinear Sci. 11 (2001), pp. 193-209.

[50] M. Slemrod, *Shadowing and the plasma-sheath transition layer,* J. Nonlinear Sci., 11 (2001), pp. 397-414.

[51] M. Slemrod, *Monotone increasing solutions of the Painleve 1 equation $y'' = y^2 + x$ and their role in the stability of the plasma-sheath transition,* European J. Applied Mathematics, 13(2002), 663-680.

[52] M. Slemrod, *The radio-frequency driven plasma sheath: asymptotics and analysis,* to appear in SIAM J. Applied Mathematics.

[53] S. Schochet, *Private communication.*

[54] S. Schochet, *The compressible Euler equations in a bounded domain: Existence of solutions and the incompressible limit,* Commum. Math. Phys. 104 (1986), pp 49-75.

[55] N. Sternberg, and V.A. Godyak, *Solving the mathematical model of the electrode sheath in symmetrically driven rf discharges,* Journal of Computational Physics, 111 (1994), pp. 347-353.

[56] M. Widner, I. Alexeff, and W.D. Jones, *Ion acoustic wave excitation and ion sheath evolution,* Physics of Fluids, 13 (1970), pp. 2532-2540.

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