

LAGRANGIAN SOLUTIONS OF SEMIGEOSTROPHIC EQUATIONS IN PHYSICAL SPACE

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1. INTRODUCTION

The semigeostrophic equations are a simple model of large-scale atmosphere/ocean flows, where 'large-scale' is defined to mean that the flow is rotation-dominated, [4]. They are also accurate in the case where one horizontal scale becomes small, allowing them to describe weather fronts and jet streams. Previous work by J.-D. Benamou and Y. Brenier, [2], and Cullen and Gangbo, [5], and Cullen and Maroofi, [6] proves that the semigeostrophic equations can be solved in the case respectively of 3-dimensional incompressible flow between rigid boundaries, vertically-averaged 3-d incompressible flow with a free surface, and fully compressible flow. However, all these results only prove the existence of weak solutions in 'dual' variables, where the dual variables result from the change of variables introduced by [11]. This makes it difficult to relate the solutions to the full Euler or Navier-Stokes equations, or to those of other simple atmosphere/ocean models. We therefore seek to extend the results of [2] and [5] to prove existence of a solution in physical variables. We do this using the Lagrangian form of the equations in physical space. The proof is based on the recent results of L. Ambrosio [1] on transport equations and ODE for BV vector fields. It is not clear whether it is possible to prove existence of a weak solution to the Eulerian form of the equations.

The transport theory of [1] gives uniqueness. However, the analysis of [2], [5], and [6] does not give uniqueness, because of the dependence of the transport velocity on the transported quantity. This question remains open.

The remaining part of the paper is organized as following: in Section 2 we consider 3-dimensional incompressible semi-geostrophic system in a domain with rigid boundary, and in Section 3 we consider the semi-geostrophic shallow water model.

2. LAGRANGIAN SOLUTIONS OF 3-DIMENSIONAL INCOMPRESSIBLE SEMI-GEOSTROPHIC SYSTEM IN A DOMAIN WITH RIGID BOUNDARY IN PHYSICAL SPACE

2.1. Background. Let $\Omega \subset \mathbf{R}^3$ be an open bounded set. We study the semigeostrophic system:

$$(2.1) \quad \begin{aligned} D_t(v_1^g, v_2^g) + (\partial_1 p, \partial_2 p) &= (u_2, -u_1), & (v_1^g, v_2^g) &= (-\partial_2 p, \partial_1 p), \\ D_t \rho &= 0, & \operatorname{div} u &= 0, & \partial_3 p + \rho &= 0, \\ D_t &= \partial_t + u \cdot \nabla, & \nabla &= (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}), \end{aligned}$$

where $u = (u_1, u_2, u_3)$ is the velocity, p is the pressure, and ρ is the density. All these quantities are functions of $(t, x) \in (0, T) \times \Omega$. The initial and boundary data are

$$(2.2) \quad \begin{aligned} u \cdot \nu &= 0, & \text{on } [0, T) \times \partial\Omega, \\ p(0, x) &= p_0(x) & \text{in } \Omega, \end{aligned}$$

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ν is the outward normal to $\partial\Omega$, and $p_0(x)$ is a given function.

Introducing the function

$$P(t, x) = p(t, x) + \frac{1}{2}(x_1^2 + x_2^2),$$

we rewrite (2.1) as the following system of equations for P , $u = (u_1, u_2, u_3)$ depending on $(t, x) \in (0, T) \times \Omega$:

$$(2.3) \quad \begin{aligned} D_t X &= J(X - x), \\ \operatorname{div} u &= 0, \\ X &= \nabla P, \\ u \cdot \nu &= 0, \quad \text{on } [0, T) \times \partial\Omega, \\ P(0, x) &= P_0(x) \quad \text{in } \Omega, \end{aligned}$$

where $P_0(x) = p_0(x) + \frac{1}{2}(x_1^2 + x_2^2)$. Here $J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Note that a solution of (2.3) determines a solution of the original system (2.1), (2.2). Indeed, given a solution (P, u) of (2.3), we have $X = \nabla P$, and we set the function u in (2.1) to be equal to the function u in (2.3), and

$$\begin{aligned} p(t, x) &= P(t, x) - \frac{1}{2}(x_1^2 + x_2^2), \\ \rho(t, x) &= -\partial_3 p(t, x). \end{aligned}$$

Then (2.1), (2.2) holds, in particular the equation $D_t \rho = 0$ follows from the equation $D_t X_3 = 0$ of (2.3). The form (2.3) is more convenient than the original system because it allows the system to be stated in the Lagrangian sense.

The solutions of [2] and [5] were obtained by interchanging dependent and independent variables in (2.3) to give a set of equations in the 'dual' coordinates (t, X) , where $X = (X_1, X_2, X_3)$. We now regard x as a function of (t, X) which is shown in [2] to be given by defining

$$(2.4) \quad P^*(t, X) = \sup_{x \in \Omega} (x \cdot X - P(t, x)) \quad \text{for } (t, X) \in [0, T) \times \mathbf{R}^3,$$

and setting $x = \nabla P^*(t, X)$. The semigeostrophic system then takes the form

$$(2.5) \quad \partial_t \alpha + \nabla \cdot (U \alpha) = 0 \quad \text{in } [0, T) \times \mathbf{R}^3,$$

$$(2.6) \quad \nabla P(t, \cdot) \# \chi_\Omega = \alpha(t, \cdot) \quad \text{for any } t \in [0, T);$$

$$(2.7) \quad U(t, X) = J(X - \nabla P^*(t, X)),$$

$$(2.8) \quad \alpha(0, X) = \alpha_0(X) \quad \text{for a.e. } X \in \mathbf{R}^3.$$

The initial data $\alpha_0(X)$ is chosen by applying (2.6) with $P(0, \cdot) = P_0(x)$. Here (2.6) means that the map $\nabla P(t, \cdot)$ pushes χ_Ω forward to $\alpha(t, \cdot)$, see Definition A.1 in Appendix A below.

The proofs in [2] and [5] depended on the use of the convexity principle introduced by Cullen, [7], which requires that the function $P(t, \cdot)$ is convex. Shutts and Cullen, [12] relate this condition to a physical stability condition required for the semigeostrophic approximation to be appropriate. Equation (2.4) shows that $P^*(t, \cdot)$ also has to be convex, as the Legendre-Fenchel transform of P .

In this paper we are interested in the solution of problem in the original (“physical”) coordinates, i.e. in the solution of problem (2.3). The unknown functions in this problem are the (modified) pressure P and the velocity u . The Eulerian form of the first equation of (2.3) is

$$(2.9) \quad \partial_t X + u \cdot \nabla X = J(X - x).$$

Using $\operatorname{div} u = 0$, this expression can be written in the divergence form, which yields the following definition of weak (Eulerian) solution of (2.3)

Definition 2.1. *Let $u : [0, T) \times \Omega \rightarrow \mathbf{R}^3$ and $P : [0, T) \times \Omega \rightarrow \mathbf{R}^1$ satisfy $u \in L^1([0, T) \times \Omega; \mathbf{R}^3)$, $\nabla P \in L^\infty([0, T) \times \Omega) \cap C([0, T); L^1(\Omega))$, and $P(t, \cdot)$ is convex in Ω for every $t \in [0, \infty)$. The pair (u, P) is a weak Eulerian solution of (2.3) if*

$$(2.10) \quad \int_{(0, T) \times \Omega} \{ \nabla P(t, x) \cdot [\partial_t \phi(t, x) + (u(t, x) \cdot \nabla) \phi(t, x)] \\ + J[\nabla P(t, x) - x] \cdot \phi(t, x) \} dt dx + \int_{\Omega} \nabla P_0(x) \cdot \phi(0, x) dx = 0.$$

for any $\phi \in C_c^1([0, T) \times \Omega; \mathbf{R}^3)$, and

$$(2.11) \quad \int_{[0, T) \times \Omega} u(t, x) \cdot \nabla \psi(t, x) dt dx = 0$$

for any $\psi \in C_c^1([0, T) \times \overline{\Omega})$.

Remark 2.2. *Equality (2.11) is a weak form of the equation $\operatorname{div} u = 0$ in $(0, T) \times \Omega$ with the boundary condition $u_\nu = 0$ on $(0, T) \times \partial\Omega$ in (2.3).*

In order to obtain a solution (u, P) , we can find P and $x(t, X) = \nabla P^*$ by solving the problem (2.5) – (2.8) in the dual coordinates, using the results of [2], [5]. Thus the problem is to find u such that (2.10), (2.11) hold. Since u in physical coordinates is by definition equal to $D_t x(t, X)$, equations (2.7) and (2.4) yield the following expression for u in terms of P and P^* :

$$(2.12) \quad u(t, x) = \partial_t \nabla P^*(t, X) + U \cdot \nabla(\nabla P^*(t, X)) \\ = \partial_t \nabla P^*(t, \nabla P(t, x)) + D^2 P^*(t, \nabla P(t, x)) [J(\nabla P(t, x) - x)],$$

Formally, if (P, α) satisfy (2.5)–(2.8), P^* is defined by (2.4), and u is defined by (2.12), then (P, u) satisfy (2.3). However, because of the low regularity of $P(t, x)$ obtained as a weak solution of the problem (2.5) – (2.8), it is not clear how to make these calculation rigorous. There is a further difficulty. The product terms $u_i \partial_j P$ that appear in (2.10) are not well-defined given that $\nabla P \in L^\infty$ and that u defined by (2.12) is a measure, which is all the regularity of P currently available.

In this paper we circumvent these difficulties by defining Lagrangian solutions of the problem (2.3). We prove existence of such Lagrangian solutions for initial data P_0 satisfying some mild strict convexity conditions. The proof is based on the recent results of L. Ambrosio [1] on transport equations and ODE for BV vector fields.

In the conclusion of this introduction, we show that the time-stepping procedure and estimates of [5] can be applied to the model considered in [2] to improve some estimates of [2]. This is required to allow the solutions in the dual variables to be transferred to real space.

Let $q > 1$. Denote by $C([0, T]; L_w^q(\mathbf{R}^3))$ the set of all measurable functions $v(t, x)$ on $[0, T] \times \mathbf{R}^3$, such that $v_t(\cdot) = v(t, \cdot) \in L^q(\mathbf{R}^3)$ for any $t \in [0, T]$, and for any $\{t_k\}_{k=1}^\infty, t^* \in [0, T]$ satisfying $\lim_{k \rightarrow \infty} t_k = t^*$ there holds $v_{t_k} \rightharpoonup v_{t^*}$ weakly in $L^q(\mathbf{R}^3)$ (weakly-* if $q = \infty$).

Theorem 2.1 ([2], [5]). *Let $\Omega \subset \mathbf{R}^3$ be an open bounded set and $\bar{\Omega} \subset B$ where B is an open ball $B(0, S)$. Let $P_0(x)$ be a convex bounded function in B , satisfying*

$$(2.13) \quad \alpha_0 := DP_0 \# \chi_\Omega \in L^q(\mathbf{R}^3)$$

for some $q > 1$. Then for any $T > 0$ there exist functions $\alpha(t, X), P(t, X)$ on $[0, T] \times \mathbf{R}^3$ such that

(i) α, P satisfy

$$\begin{aligned} \alpha &\in L^\infty([0, T]; L^q(\mathbf{R}^3)) \cap C([0, T]; L_w^q(\mathbf{R}^3)), \\ P &\in L^\infty([0, T]; W^{1,\infty}(\Omega)) \cap C([0, T]; W^{1,r}(\Omega)), \end{aligned}$$

where r is any number in $[1, \infty)$;

(ii) let $R_0 = S(1 + T)$, then

$$(2.14) \quad \text{supp}(\alpha(t, \cdot)) \subset B(0, R_0) \quad \text{for all } t \in [0, T],$$

(iii) $P(t, \cdot)$ is convex in Ω ;

(iv) P^* defined by (2.4) satisfies:

$$\begin{aligned} P^*(t, \cdot) &\text{ is convex in } \mathbf{R}^3 \text{ for any } t \in [0, T] \\ P^* &\in L_{loc}^\infty([0, T] \times \mathbf{R}^3), \\ \nabla P^* &\in L^\infty([0, T] \times \mathbf{R}^3; \mathbf{R}^3) \cap C([0, T]; L^r(B(0, R); \mathbf{R}^3)), \end{aligned}$$

for any $R > 0$, and any $r \in [1, \infty)$. Moreover,

$$(2.15) \quad \|\nabla P^*(t, \cdot)\|_{L^\infty(\mathbf{R}^3)} \leq S \quad \text{for every } t \in [0, T],$$

(v) (α, P, P^*) satisfy (2.5) – (2.8), where the evolution equation (2.5) and the initial condition (2.8) are understood in the weak sense: for any $\phi \in C_c^1([0, T] \times \mathbf{R}^3)$

$$(2.16) \quad \int_{(0,T) \times \mathbf{R}^3} [\partial_t \phi(t, X) + U(t, X) \cdot \nabla \phi(t, X)] \alpha(t, X) dt dX + \int_{\mathbf{R}^3} \alpha_0(X) \phi(0, X) dX = 0.$$

Proof. We sketch the proof.

Since P_0 is convex and bounded in B , then

$$(2.17) \quad \|\nabla P_0\|_{L^\infty(\Omega)} \leq C(B, \Omega, \|P_0\|_{L^\infty(B)}).$$

Thus α_0 defined by (2.13) has compact support.

As in [2, 5], we construct solutions using a time-stepping procedure. Fix $x^* \in \Omega$. Let $h > 0$ be small, chosen so that T/h is an integer. Let $\eta_h(\cdot) = \frac{1}{h^3} \eta(\frac{\cdot}{h})$ be a standard mollifier. Define

$$\alpha_h^0 = (\alpha_0) * \eta_h,$$

and then inductively define the following quantities for $k = 0, 1, \dots, T/h$:

Suppose $\alpha_h^k \in L^q(\mathbf{R}^3)$, with compact support, is defined. Then let P_h^k is the unique convex function satisfying:

$$\nabla P_h^k \# \chi_\Omega = \alpha_h^k, \quad P_h^k(x^*) = P_0(x^*).$$

Existence and uniqueness of such P_h^k follows from Brenier [3].

$$(2.18) \quad \begin{aligned} (P_h^k)^*(X) &= \sup_{x \in \Omega} (x \cdot X - P_h^k(x)) \quad \text{for } X \in \mathbf{R}^3, \\ Q_h^k &= \eta_h * (P_h^k)^*, \\ U_h^k(X) &= J \left[X - \nabla Q_h^k(X) \right] \end{aligned}$$

Since $\Omega \subset B(0, S)$, it follows that

$$(2.19) \quad \|\nabla(P_h^k)^*, \nabla Q_h^k\|_{L^\infty(\mathbf{R}^3)} \leq S.$$

From (2.18), (2.19)

$$\|(P_h^k)^* - Q_h^k\|_{L^\infty(\mathbf{R}^3)} \leq hS.$$

To define α_h^{k+1} we solve

$$(2.20) \quad \frac{\partial \alpha_h}{\partial t} + \operatorname{div}(\alpha_h U_h^k) = 0 \quad \text{in } \mathbf{R}^3 \times [kh, (k+1)h]$$

$$(2.21) \quad \alpha_h(kh, X) = \alpha_h^k(X).$$

and set

$$(2.22) \quad \alpha_h^{k+1}(X) = \alpha_h((k+1)h, X).$$

Similar to [5, Lemma 4.1] we show that $\alpha_h \in L^\infty([kh, (k+1)h], L^q(\mathbf{R}^3))$ and that

$$(2.23) \quad \|\alpha_h(t, \cdot)\|_{L^q(\mathbf{R}^3)} = \|\alpha_0\|_{L^q(\mathbf{R}^3)} \quad \text{for every } t \in [kh, (k+1)h],$$

$$(2.24) \quad \operatorname{supp}(\alpha_h(t, \cdot)) \subset B(0, S(1 + (k+1)h)) \quad \text{for every } t \in [kh, (k+1)h],$$

$$(2.25) \quad W_1(\alpha_h(t_1, \cdot), \alpha_h(t_2, \cdot)) \leq C(T)|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [kh, (k+1)h],$$

where q is from (2.13), and $W_1(\alpha_h(t_1, \cdot), \alpha_h(t_2, \cdot))$ is the 1-Wasserstein distance between $\alpha_h(t_1, \cdot)$ and $\alpha_h(t_2, \cdot)$ (see e.g. [5, Appendix A] for the definition of W_1).

We repeat this procedure for $k = 1, \dots, T/h$. In particular, we thus define a function

$$(2.26) \quad \alpha_h \in L^\infty([0, T], L^q(\mathbf{R}^3)).$$

For every $t \in (0, T)$ define $P_h(t, \cdot)$ to be the unique convex function satisfying:

$$\nabla P_h(t, \cdot) \# \chi_\Omega = \alpha_h(t, \cdot), \quad P_h(t, x^*) = P_0(x^*),$$

and then define $Q_h(t, \cdot)$, $U_h(t, \cdot)$ by using in (2.18) the $P_h(t, \cdot)$ instead of P_h^k .

We also define functions $\bar{P}_h$ on $[0, T] \times \Omega$ and $\bar{\alpha}_h, \bar{Q}_h, \bar{U}_h$ on $[0, T] \times \mathbf{R}^3$ by setting them equal to $P_h^k, \alpha_h^k, Q_h^k, U_h^k$ respectively on the time interval $t \in [kh, (k+1)h]$.

We note the following fact:

Lemma 2.3 ([3]). *Let $p \geq 1$. Let $\Omega \subset \mathbf{R}^n$ be an open bounded set, $x^* \in \Omega$, $a \in \mathbf{R}$. Let $\rho_j, \rho \in L^p(\mathbf{R}^n)$ for $j = 1, \dots$ be such that $\rho_j \rightharpoonup \rho$ weakly in $L^p(\mathbf{R}^n)$ and $\operatorname{supp}(\rho_j, \rho) \subset B(0, R)$. Let ϕ_j, ϕ be the convex functions on $\mathbf{R}^n$ satisfying $\nabla \phi_j \# \chi_\Omega = \rho_j$, $\nabla \phi \# \chi_\Omega = \rho$, $\phi(x^*), \phi_j(x^*) = a$ for $j = 1, \dots$. Then $\phi_j \rightarrow \phi$ in $W^{1,r}(\Omega)$ for any $r \in [1, \infty)$.*

From (2.23)–(2.25), repeating the argument of [5, pp. 263–268], with the use of Lemma 2.3, we obtain a sequence $h_j \rightarrow 0+$ and a function $\alpha \in L^q([0, T] \times \mathbf{R}^3)$, where q is from

(2.13), such that

$$(2.27) \quad \text{supp}(\alpha_{h_j}(t, \cdot)) \subset B(0, S(1+T)) \quad \text{for each } t \in [0, T], \quad j = 1, 2, \dots,$$

$$(2.28) \quad \alpha_{h_j} \rightharpoonup \alpha \quad \text{weakly in } L^q([0, T] \times \mathbf{R}^3),$$

$$(2.29) \quad \alpha_{h_j}(t, \cdot) \rightharpoonup \alpha(t, \cdot) \quad \text{weakly in } L^q(\mathbf{R}^3) \quad \text{for each } t \in [0, T],$$

and such that, denoting by $P(t, \cdot)$, for every $t \in (0, T)$, the unique convex function satisfying:

$$\nabla P(t, \cdot) \# \chi_\Omega = \alpha(t, \cdot), \quad P(t, x^*) = P_0(x^*),$$

and denoting by $P^*(t, \cdot)$ the convex dual of $P(t, \cdot)$ defined by (2.4), we have

$$(2.30) \quad \overline{P}_{h_j}^*(t, \cdot) \rightarrow P^*(t, \cdot) \quad \text{in } C(\overline{B(0, R)}) \quad \text{for each } R > 0, \quad t \in [0, T],$$

$$(2.31) \quad \overline{Q}_{h_j}(t, \cdot) \rightarrow P^*(t, \cdot) \quad \text{in } C(\overline{B(0, R)}) \quad \text{for each } R > 0, \quad t \in [0, T],$$

$$(2.32) \quad \alpha_{h_j} \overline{U}_{h_j} \rightharpoonup \alpha J(id - \nabla P^*) \quad \text{weakly in } L^q([0, T] \times \mathbf{R}^3; \mathbf{R}^3).$$

Then the proof of Theorem is completed as in [5, pp. 268-269]. $\square$

2.2. Statement of results. In this paper we study the system in the "physical" space (t, x) , and define its weak Lagrangian solutions. In order to do that, we first rewrite system (2.3) in terms of F, P , where $F : [0, T] \times \Omega \rightarrow \Omega$ is the (formal) Lagrangian flow corresponding to the full wind velocity $u = (u_1, u_2, u_3)$, and then we define the corresponding weak solution F, P . This gives the following

Definition 2.4. Let $\Omega \subset \mathbf{R}^3$ be an open bounded set, and $T > 0$. Let $P_0(x) \in W^{1,\infty}(\Omega)$ be convex. Let $r \in [1, \infty)$. Let $P : [0, T] \times \Omega \rightarrow \mathbf{R}^1$ satisfy

$$(2.33) \quad P \in L^\infty([0, T]; W^{1,\infty}(\Omega)) \cap C([0, T]; W^{1,r}(\Omega)),$$

$$(2.34) \quad P(t, \cdot) \text{ is convex in } \Omega \text{ for each } t \in [0, T].$$

Let $F : [0, T] \times \Omega \rightarrow \Omega$ be a Borel map, satisfying

$$(2.35) \quad F \in C([0, T]; L^r(\Omega; \mathbf{R}^3)).$$

Then the pair (P, F) is called a weak Lagrangian solution of (2.3) in $[0, T] \times \Omega$ if

- (i) $F(0, x) = x$, $P(0, x) = P_0(x)$ for a.e. $x \in \Omega$,
- (ii) for any $t > 0$ the mapping $F_t = F(t, \cdot) : \Omega \rightarrow \Omega$ is Lebesgue measure preserving, in the sense that $F_t \# \chi_\Omega = \chi_\Omega$;
- (iii) There exists a Borel map $F^* : [0, T] \times \Omega \rightarrow \Omega$ such that for every $t \in (0, T)$ the map $F_t^* = F^*(t, \cdot) : \Omega \rightarrow \Omega$ is Lebesgue measure preserving: $F_t^* \# \chi_\Omega = \chi_\Omega$, and satisfies $F_t^* \circ F_t(x) = x$ and $F_t \circ F_t^*(x) = x$ for a.e. $x \in \Omega$;
- (iv) The function

$$(2.36) \quad Z(t, x) = \nabla P(t, F_t(x))$$

is a weak solution of

$$(2.37) \quad \begin{aligned} \partial_t Z(t, x) &= J[Z(t, x) - F(t, x)] && \text{in } [0, T] \times \Omega, \\ Z(0, x) &= \nabla P_0(x) && \text{in } \Omega, \end{aligned}$$

in the following sense: for any $\varphi \in C_c^1([0, T] \times \Omega; \mathbf{R}^3)$

$$(2.38) \quad \int_{(0, T) \times \Omega} [Z(t, x) \cdot \partial_t \varphi(t, x) + J(Z(t, x) - F(t, x)) \cdot \varphi(t, x)] dt dx + \int_{\Omega} \nabla P_0(x) \cdot \varphi(0, x) dx = 0.$$

Remark 2.5. We comment on Definition 2.4:

- Continuity in time of P, F , considered as maps on $[0, T]$ with values in $W^{1,r}(\Omega)$ and $L^r(\Omega)$ respectively, required in (2.33), (2.35), combined with initial conditions in part (i) of Definition 2.4, imply that

$$(2.39) \quad \lim_{t \rightarrow 0+} \|F_t - Id\|_{L^r(\Omega)} = 0, \quad \lim_{t \rightarrow 0+} \|P_t - P_0\|_{W^{1,r}(\Omega)} = 0,$$

where $Id : \Omega \rightarrow \Omega$ is the identity mapping. Furthermore, the continuity property of F in (2.35) can be interpreted as "generic continuity" of particle paths in physical space.

- The property Definition 2.4(ii) of the flow F is the Lagrangian form of the equation $\operatorname{div} u = 0$ in $(0, T) \times \Omega$ with the boundary condition $u_\nu = 0$ on $(0, T) \times \partial\Omega$ in (2.3).
- (2.38) is a weak Lagrangian form of the first equation of (2.3) with an initial condition for P .

To justify Definition 2.4 we show that a weak Lagrangian solution (F, P) with the additional regularity property $\partial_t F \in L^\infty([0, T] \times \Omega)$ determines a weak Eulerian solutions of (2.3), and that a smooth Lagrangian solution determines a classical solution of (2.3):

Lemma 2.6 (consistency of weak Lagrangian solutions). *Let $\Omega \subset \mathbf{R}^3$ be an open bounded set, and $T > 0$. Let (F, P) be a weak Lagrangian solution of (2.3) in $[0, T] \times \Omega$.*

- (i) *If $\partial_t F \in L^\infty([0, T] \times \Omega; \mathbf{R}^3)$, then the function*

$$(2.40) \quad u(t, x) := (\partial_t F)(t, F_t^*(x))$$

satisfies $u \in L^\infty([0, T] \times \Omega; \mathbf{R}^3)$, and (u, P) is a weak Eulerian solution of (2.3) in $[0, T] \times \Omega$ in the sense of Definition 2.1;

- (ii) *If $(F, F^*, P) \in C^2([0, T] \times \bar{\Omega})$, then the function (2.40) satisfies $u \in C^1([0, T] \times \bar{\Omega}; \mathbf{R}^3)$, and (u, P) is a classical solution of (2.3) in $[0, T] \times \Omega$.*

Proof. We first prove (i). Since F^* is a Borel map, and $\partial_t F \in L^\infty([0, T] \times \Omega)$, then the right-hand side of (2.40) is a bounded measurable function, thus $u \in L^\infty([0, T] \times \Omega)$. It remains to prove that (P, u) is a weak Eulerian solution. We prove first that (2.11) holds. Let $\psi \in C_c^1([0, T] \times \bar{\Omega})$. Fix $t \in (0, T)$. Since $F_t \# \chi_\Omega = \chi_\Omega$, then

$$\int_{\Omega} (\partial_t \psi)(t, F_t(x)) dx = \int_{\Omega} \partial_t \psi(t, x) dx.$$

Integrating with respect to t and using $\partial_t F \in L^\infty([0, T] \times \Omega)$, we get

$$\int_{[0, T] \times \Omega} \{\partial_t [\psi(t, F_t(x))] - \partial_t F_t(x) \cdot (\nabla \psi)(t, F_t(x))\} dt dx = \int_{[0, T] \times \Omega} \partial_t \psi(t, x) dt dx.$$

Using $\psi(T, \cdot) \equiv 0$ and (2.39), we get

$$\int_{[0, T] \times \Omega} \partial_t F_t(x) \cdot (\nabla \psi)(t, F_t(x)) dt dx = 0.$$

Making the change of variables $y = F_t(x)$ and using the properties (ii, iii) of Definition 2.4, we get

$$\int_{[0,T] \times \Omega} (\partial_t F_t)(F_t^* y) \cdot \nabla \psi(t, y) dt dy = 0.$$

Since (2.40) defines u , then (2.11) follows.

Now we prove (2.10). From properties P and F in Definition 2.4 it follows that $Z(t, x)$ defined by (2.36) satisfies $Z \in L^\infty([0, T] \times \overline{\Omega})$. Then, since Ω is a bounded set and $F_t \# \chi_\Omega = \chi_\Omega$ for all $t \in [0, T]$, equation (A.2) of Corollary A.3 allows us to make the change of variables $y = F_t(x)$ in the first integral of (2.38). Then $x = F_t^*(y)$ for a.e. $(t, x) \in [0, T] \times \Omega$ by (iii) in Definition 2.4, and from (2.38) we get for any $\varphi \in C_c^1([0, T] \times \Omega)$

$$(2.41) \quad \int_{(0,T) \times \Omega} [\nabla P(t, y) \cdot (\partial_t \varphi)(t, F_t^*(y)) + J(\nabla P(t, y) - y) \varphi(t, F_t^*(y))] dt dy \\ + \int_{\Omega} \nabla P_0(x) \varphi(0, x) dx = 0.$$

Next we show that (2.41) holds for all $\varphi \in L^\infty([0, T] \times \Omega)$ satisfying $\partial_t \varphi \in L^\infty([0, T] \times \Omega)$ and $\text{supp}(\varphi) \subset [0, T - \varepsilon] \times \overline{\Omega}$ for some $\varepsilon > 0$. Indeed, for such φ , we construct an approximating sequence $\varphi_j \in C_c^1([0, T] \times \Omega)$ as following. Extend φ to $[0, \infty) \times \Omega$ by defining $\varphi(t, \cdot) \equiv 0$ for $t \geq T$, and further extend φ to $(-\infty, \infty) \times \Omega$ by defining $\varphi(t, x) = \varphi(-t, x)$ for $t < 0$, $x \in \Omega$. Let $h > 0$ and $\Omega_h = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > h\}$. Now $\varphi \chi_{\Omega_h}$ is defined on $\mathbf{R}^1 \times \mathbf{R}^3$. Let $\eta_h(t, x) = \frac{1}{h^4} \eta(\frac{|(t, x)|}{h})$ where $\eta(\cdot)$ is a standard mollifier. Let $j > \frac{1}{\varepsilon}$ be integer. Then functions $\varphi_j = (\varphi \chi_{\Omega_{4h}}) * \eta_h$ with $h = \frac{1}{j} < \varepsilon$ satisfy: $\varphi_j \in C_c^1([0, T] \times \Omega)$ with $\|\varphi_j, \partial_t \varphi_j\|_{L^\infty([0,T] \times \Omega)} \leq C$, where C does not depend on j , and $(\varphi_j, \partial_t \varphi_j) \rightarrow (\varphi, \partial_t \varphi)$ a.e. on $[0, T] \times \Omega$ as $j \rightarrow \infty$. Since $F_t^* \# \chi_\Omega = \chi_\Omega$ for all t , it follows that $(\varphi_j, \partial_t \varphi_j)(t, F_t^*(y)) \rightarrow (\varphi, \partial_t \varphi)(t, F_t^*(y))$ for a.e. $(t, y) \in [0, T] \times \Omega$. With this, since Ω is bounded, $\nabla P \in L^\infty([0, T] \times \Omega)$, and (2.41) holds for each φ_j , the bounded convergence theorem implies (2.41) for φ .

Let $\varphi(t, x) = \eta(t, F_t(x))$, where $\eta \in C_c^1([0, T] \times \Omega)$. Then $\varphi \in L^\infty([0, T] \times \Omega)$, and $\text{supp}(\varphi) \subset [0, T - \varepsilon] \times \overline{\Omega}$ since $\text{supp}(\eta) \subset [0, T - \varepsilon] \times \overline{\Omega}$ for some $\varepsilon > 0$. Moreover, since $\partial_t F \in L^\infty([0, T] \times \Omega)$, it follows that $\partial_t \varphi \in L^\infty([0, T] \times \Omega)$ with

$$\partial_t \varphi(t, x) = (\partial_t \eta)(t, F_t(x)) + \sum_{i=1}^3 \partial_t F_t^i(x) (\partial_{x_i} \eta)(t, F_t(x)) \quad \text{a.e. in } [0, T] \times \Omega,$$

where we used notation $F(t, x) = (F^1, F^2, F^3)(t, x)$.

Thus (2.41) holds for φ . By the properties (ii, iii) in Definition 2.4

$$\varphi(t, F_t^*(y)) = \eta(t, y), \quad \partial_t \varphi(t, F_t^*(y)) = \partial_t \eta(t, y) + [(\partial_t F)(t, F_t^*(y)) \cdot \nabla \eta(t, y)]$$

for a.e. $(t, y) \in [0, T] \times \Omega$. Thus, inserting $\varphi(t, x) = \eta(t, F_t(x))$ into (2.41), we obtain for every $\eta \in C_c^1([0, T] \times \Omega)$

$$\int_{(0,T) \times \Omega} [\nabla P(t, y) \cdot (\partial_t \eta(t, y) + [(\partial_t F)(t, F_t^*(y)) \cdot \nabla \eta(t, y)]) \\ + J(\nabla P(t, y) - y) \eta(t, y)] dt dy + \int_{\Omega} \nabla P_0(x) \eta(0, F_0(x)) dx = 0.$$

Using the property (i) in Definition 2.4 and (2.40), and changing notations y to x and η to φ , we obtain (2.10). Assertion (i) of Lemma 2.6 is proved.

Now assertion (ii) follows directly from (i). $\square$

Our main result is

Theorem 2.2. *Let $\Omega \subset \mathbf{R}^3$ be an open bounded set and $\bar{\Omega} \subset B$ where B is an open ball $B(0, S)$. Let $P_0(x)$ be a convex bounded function in B . Assume that P_0 satisfies*

$$(2.42) \quad DP_0 \# \chi_\Omega \in L^q(\mathbf{R}^3)$$

for some $q > 1$. Then for any $T > 0$ there exists a weak Lagrangian solution (P, F) of (2.3) in $[0, T) \times \Omega$, where (2.33), (2.35) are satisfied for any $r \in [1, \infty)$. Moreover, the function $Z(t, x)$ defined by (2.36) satisfies $Z(\cdot, x) \in W^{1, \infty}([0, T]; \mathbf{R}^3)$ for a.e. $x \in \Omega$, and (2.37) is satisfied, in addition to the weak form (2.38), in the following sense:

$$(2.43) \quad \begin{aligned} \partial_t Z(t, x) &= J(Z(t, x) - F(t, x)) && \text{for } \mathcal{L}^4\text{-a.e. in } (t, x) \in (0, T) \times \Omega, \\ Z(0, x) &= \nabla P_0(x) && \text{for } \mathcal{L}^3\text{-a.e. in } x \in \Omega. \end{aligned}$$

Remark 2.7. The condition (2.42) is equivalent to

$$\det D^2 P_0^* \in L^q(\nabla P_0(\Omega)),$$

where P_0^* is the Legendre-Fenchel transform of P_0 , i.e. $P_0^*(X) = \sup_{x \in \Omega} [x \cdot X - P_0(x)]$ for $X \in \mathbf{R}^3$. The condition (2.42) is a certain strict convexity condition for P_0 . In particular, if P_0 is uniformly strictly convex in B , in the sense that there exists $\varepsilon > 0$ such that $P_0(x) - \varepsilon|x|^2$ is a convex function in B , then $\det D^2 P_0^* \in L^\infty(\nabla P_0(\Omega))$, and $\nabla P_0(\Omega)$ is a bounded set by (2.17). Thus (2.42) is satisfied.

We prove Theorem 2.2 in sections 2.3 and 2.4.

2.3. Lagrangian flow in the dual space. Let Ω, T, P_0 be as in Theorem 2.2.

Let $\alpha_0 = \nabla P_0 \# \chi_\Omega$. By Theorem 2.1, there exists a solution (α, P, P^*) of the problem (2.5)–(2.7) with initial data α_0 , satisfying all assertions of Theorem 2.1.

Note that the vector field U defined by (2.7) is divergence free. Thus the evolution equation (2.5) and its weak form (2.16) can be seen as the transport equation

$$(2.44) \quad \partial_t \alpha + U \cdot \nabla \alpha = 0.$$

Since $\nabla P^* \in L^\infty([0, T) \times \mathbf{R}^3)$, and $P_t^*(\cdot)$ is convex in $\mathbf{R}^3$ for all $t \in [0, T)$, it follows from (2.7) that

$$U \in L_{loc}^\infty([0, T) \times \mathbf{R}^3), \quad \text{and } U \in L^\infty([0, T); BV(B(0, R))) \quad \text{for all } R > 0.$$

By (2.14) α has compact support in $[0, T] \times \mathbf{R}^3$. Thus we can modify U away from $B(0, R_1)$, where large R_1 will be chosen below so that, in particular, $\text{supp}(\alpha) \subset [0, T] \times \overline{B(0, R_1)}$, and the modified function $\tilde{U}$ satisfies

$$(2.45) \quad \begin{aligned} \tilde{U} &\in L^\infty([0, T) \times \mathbf{R}^3), && \tilde{U} \in L^\infty([0, T); BV(B(0, R))) \quad \text{for all } R > 0; \\ \text{div } \tilde{U}(t, \cdot) &= 0 \quad \text{in } \mathbf{R}^3, && \text{for all } t \in [0, T). \end{aligned}$$

To construct $\tilde{U}$ satisfying (2.45), we choose $\zeta \in C^\infty(\mathbf{R}^1)$ such that

$$(2.46) \quad \zeta \equiv 1 \quad \text{on } \{|s| < R_1\}, \quad \zeta \equiv 0 \quad \text{on } \{|s| > R_1 + 1\}, \quad 0 \leq \zeta \leq 1 \quad \text{on } \mathbf{R}^1.$$

and define, for $X \in \mathbf{R}^3$,

$$(2.47) \quad H(X) = (\zeta(|X_1|)X_1, \zeta(|X_2|)X_2, \zeta(|X_3|)X_3).$$

Define $\tilde{U}$ by

$$(2.48) \quad \tilde{U}(t, X) = J[H(X) - \nabla P^*(t, X)].$$

Then $\tilde{U}$ satisfies (2.45), and from (2.46)–(2.47), (2.15)

$$(2.49) \quad U = \tilde{U} \quad \text{in } B(0, R_1),$$

$$(2.50) \quad \|U(t, \cdot)\|_{L^\infty(\mathbf{R}^3)} \leq S + R_1 + 1 \quad \text{for all } t \in [0, T].$$

Now the theory developed by L. Ambrosio [1] applies to the equation (2.44) with U replaced by $\tilde{U}$:

Lemma 2.8. *Let $\tilde{U}$ be defined by (2.46)–(2.48). There exists a unique locally bounded Borel measurable map $\Phi : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$, satisfying:*

- (i) $\Phi(\cdot, X) \in W^{1,\infty}([0, T]; \mathbf{R}^3)$ for a.e. $X \in \mathbf{R}^3$;
- (ii) $\Phi(0, X) = X$ for $\mathcal{L}^3$ -a.e. $X \in \mathbf{R}^3$;
- (iii) For a.e. $(t, X) \in \mathbf{R}^3 \times (0, T)$

$$(2.51) \quad \partial_t \Phi(t, X) = \tilde{U}(t, \Phi(t, X));$$

- (iv) $\Phi(t, \cdot) : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is a $\mathcal{L}^3$ -measure preserving map for every $t \in [0, T]$,

Proof. Since $\tilde{U}$ satisfies (2.45), by [1, Section 6] there exists a unique regular flow relative to $\tilde{U}$. This flow determines a locally bounded Borel measurable map $\Phi : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ satisfying (i)–(iii) and the following property:

Let $\tilde{U}_\varepsilon : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be a family of approximations to $\tilde{U}$ satisfying:

$$(2.52) \quad \begin{aligned} &\tilde{U}_\varepsilon \in C([0, T] \times \mathbf{R}^3; \mathbf{R}^3), \quad \sup \|\tilde{U}_\varepsilon\|_{L^\infty([0, T] \times \mathbf{R}^3; \mathbf{R}^3)} < \infty, \\ &\tilde{U}_\varepsilon \rightarrow \tilde{U} \quad \text{in } L^1_{loc}([0, T] \times \mathbf{R}^3; \mathbf{R}^3), \\ &\|\nabla \tilde{U}_\varepsilon\|_{L^\infty([0, T] \times B_R; \mathbf{R}^3)} \leq C(\varepsilon, R) < \infty \quad \text{for any } \varepsilon, R > 0, \\ &\operatorname{div} \tilde{U}_\varepsilon = 0. \end{aligned}$$

Let $\Phi_\varepsilon(t, X)$ be the unique solution in $[0, T]$ of the ODE $\frac{d}{dt} \Phi_\varepsilon(t, X) = \tilde{U}_\varepsilon(t, \Phi_\varepsilon(t, X))$ with the initial condition $\Phi_\varepsilon(0, X) = X$. Then

$$(2.53) \quad \lim_{\varepsilon \rightarrow 0} \sup_{B_R} \int_{[0, T]} |\Phi(t, X) - \Phi_\varepsilon(t, X)| dX = 0 \quad \text{for any } R > 0.$$

Note that a family $\tilde{U}_\varepsilon$ satisfying the conditions stated above exists. Indeed, let $\eta_\varepsilon : \mathbf{R}^3 \times \mathbf{R}^1 \rightarrow \mathbf{R}^1$ be a family of mollifiers. Extend $P^*(t, X)$ to the time interval $(-\infty, \infty)$ by setting $P^*(t, X) = P^*(0, X)$ for $t < 0$ and $P^*(t, X) = P^*(T, X)$ for $t > 0$, and let

$$\tilde{U}_\varepsilon(t, X) = J[H(X) - (\nabla P^* * \eta_\varepsilon)(t, X)],$$

where the convolution is with respect to (t, X) -variables. Then all properties (2.52) are satisfied.

Also, since $\tilde{U}_\varepsilon$ is smooth and $\operatorname{div} \tilde{U}_\varepsilon = 0$, it follows that each $\Phi_\varepsilon(t, \cdot)$ is a measure-preserving diffeomorphism. Thus (2.53) implies (iv).

Conversely, a map Φ satisfying (i)–(iv) determines a regular Lagrangian flow relative to $\tilde{U}$. This implies uniqueness of Φ by Theorem 6.4 of [1]. $\square$

Lemma 2.9. *Let $\tilde{U}$ be defined by (2.46)–(2.48), and let Φ be the map defined in Lemma 2.8. Then*

- (i) If R_1 in the definition of $\tilde{U}$ is chosen sufficiently large depending only on Ω , T and $\|\nabla P_0\|_{L^\infty(\Omega)}$, then

$$(2.54) \quad \Phi(t, X) \subset B(0, R_1) \quad \text{for a.e. } (t, X) \in [0, T] \times \nabla P_0(\Omega).$$

In particular,

$$(2.55) \quad \partial_t \Phi(t, X) = U(t, \Phi(t, X)) \quad \text{for a.e. } (t, X) \in [0, T] \times \nabla P_0(\Omega).$$

- (ii) There exists a Borel map $\Phi^* : [0, T] \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ such that for every $t \in (0, T)$ the map $\Phi_t^* : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is $\mathcal{L}^3$ -measure preserving, and such that $\Phi_t^* \circ \Phi_t(x) = x$ and $\Phi_t \circ \Phi_t^*(x) = x$ for a.e. $x \in \mathbf{R}^3$.

Proof. We prove (i). For the family $\tilde{U}_\varepsilon$ satisfying (2.52) constructed in the proof of Lemma 2.8, we have, for $0 < \varepsilon < 1/2$, using (2.15)

$$(2.56) \quad |\tilde{U}_\varepsilon(t, X)| \leq |X| + 1 + S.$$

Thus

$$\frac{1}{2} \frac{d}{dt} |\Phi_\varepsilon(t, X)|^2 = \tilde{U}_\varepsilon(t, \Phi_\varepsilon(t, X)) \cdot \Phi_\varepsilon(t, X) \leq \frac{3}{2} |\Phi_\varepsilon(t, X)|^2 + \frac{1}{2} (1 + S)^2.$$

Thus, by the Gronwall inequality,

$$(2.57) \quad |\Phi_\varepsilon(t, X)| \leq e^{3T/2} (|X|^2 + (1 + S)^2 T^2)^{1/2} \quad \text{for all } (t, X) \in [0, T] \times \mathbf{R}^3.$$

Then same is true for $|\Phi(t, X)|$ for a.e. $(t, X) \in [0, T] \times \mathbf{R}^3$. Now, choosing

$$(2.58) \quad R_1 = 2e^{3T/2} \left(\|\nabla P_0\|_{L^\infty(\Omega)}^2 + (1 + S)^2 T^2 \right)^{1/2} + 1,$$

we obtain (2.54), which implies (2.55).

It remains to prove (ii). This is in fact a general property of regular Lagrangian flows constructed by Ambrosio [1, Section 6]. It can be seen as following. For $t_1, t_2 \in [0, T]$ with $t_1 \geq t_2$ and $x \in \mathbf{R}^3$ denote by $\hat{\Phi}(t_1, t_2, x)$ the regular Lagrangian flow relative to $\tilde{U}$ and starting at time t_1 , i.e. $\hat{\Phi}(t_1, \cdot, \cdot)$ satisfies properties (i), (iii), (iv) of Lemma on time-interval $[t_1, T]$ and $\hat{\Phi}(t_1, t_1, X) = X$ for $\mathcal{L}^3$ -a.e. $X \in \mathbf{R}^3$. In the case $t_2 \leq t_1$, $\hat{\Phi}(t_1, t_2, x)$ denotes the regular flow Lagrangian flow relative to $\tilde{U}$ backwards in time, i.e. $\hat{\Phi}(t_1, \cdot, \cdot)$ satisfies properties (i), (iii), (iv) of Lemma on time-interval $[0, t_1]$ and $\hat{\Phi}(t_1, t_1, X) = X$ for $\mathcal{L}^3$ -a.e. $X \in \mathbf{R}^3$. Repeating the argument of Remark 6.7 of [1], we see that the following semigroup property holds: for every $t_1, t_2, t_3 \in [0, T]$

$$\hat{\Phi}(t_1, t_2, \hat{\Phi}(t_2, t_3, x)) = \hat{\Phi}(t_1, t_3, x) \quad \text{for a.e. } x \in \mathbf{R}^n.$$

Also, from the proof of Theorem 6.6 of [1], one can see that if $\tilde{U}_\varepsilon$ is family of approximations to $\tilde{U}$ satisfying (2.52), and if $\hat{\Phi}^\varepsilon(t_1, t_2, x)$ is the regular Lagrangian flow of $\tilde{U}_\varepsilon$ starting at time t_1 , then $\hat{\Phi}^\varepsilon$ converges to $\hat{\Phi}$ in $L^1_{loc}([0, T] \times [0, T] \times \mathbf{R}^3)$. Thus, possibly after a modification on a negligible set, $\hat{\Phi}$ is a Borel map since it is an a.e. limit of continuous maps. Since $\Phi = \hat{\Phi}(0, \cdot, \cdot)$, the map $\Phi^*(t, x) = \hat{\Phi}(t, 0, x)$ satisfies all properties asserted in (ii). $\square$

Now we prove the following property of transport equations:

Lemma 2.10. *Let $\tilde{U} \in L^\infty((0, T) \times \mathbf{R}^3)$ with $\tilde{U}(t, \cdot) \in BV_{loc}(\mathbf{R}^3)$ and $\operatorname{div} \tilde{U}(t, \cdot) = 0$ in $\mathbf{R}^3$ the sense of distributions for every $t \in (0, T)$. Let a locally bounded Borel measurable map $\Phi : \mathbf{R}^3 \times [0, T) \rightarrow \mathbf{R}^3$ satisfy properties (i)–(iv) of Lemma 2.8. Let $q \in (1, \infty]$, $v_0 \in L^q(\mathbf{R}^3)$ and $\operatorname{supp}(v_0) \subset B(0, R)$. Let $v(t, x) := (\Phi_t \# v_0)(x)$. Then*

$$v \in L^\infty((0, T); L^q(\mathbf{R}^3)) \cap C([0, T]; L_w^q(\mathbf{R}^3)), \quad \operatorname{supp} v \subset B(0, R_T) \times [0, T],$$

where $R_T = R + T\|\tilde{U}\|_{L^\infty}$, and v is a weak solution of

$$(2.59) \quad \partial_t v + \operatorname{div}(v\tilde{U}) = 0$$

on $(0, T) \times \mathbf{R}^3$, with initial data $v|_{t=0} = v_0$, in the sense that for any $\varphi \in C_c^1([0, T) \times \mathbf{R}^3)$

$$\int_{[0, T) \times \mathbf{R}^3} v \left(\partial_t \varphi + \tilde{U} \cdot \nabla \varphi \right) dt dx + \int_{\mathbf{R}^3} v_0(x) \varphi(0, x) dx = 0.$$

Proof. Let p be such that $\frac{1}{p} + \frac{1}{q} = 1$ (where $p = 1$ if $q = \infty$). Fix $t \in [0, T)$. From the definition of v and using that Φ_t is $\mathcal{L}^3$ -measure preserving and locally bounded, we get for all $\varphi \in C_c(\mathbf{R}^3)$

$$\begin{aligned} \int_{\mathbf{R}^3} \varphi(x) v_t(x) dx &= \int_{\mathbf{R}^3} \varphi(\Phi_t(x)) v_0(x) dx \leq \|v_0\|_{L^q} \left[\int_{\mathbf{R}^3} \left| \varphi(\Phi_t(x)) \right|^p dx \right]^{\frac{1}{p}} \\ &= \|v_0\|_{L^q} \left[\int_{\mathbf{R}^3} |\varphi(x)|^p dx \right]^{\frac{1}{p}} = \|v_0\|_{L^q} \|\varphi\|_{L^p}. \end{aligned}$$

Thus $v_t \in L^q(\mathbf{R}^3)$ with $\|v_t\|_{L^q} \leq \|v_0\|_{L^q}$ for any t .

Also, weak continuity of v with respect to t also follows: if $\{t_k\}_{k=1}^\infty, t_* \in [0, T)$ and $\lim_{k \rightarrow \infty} t_k = t_*$, then we get for every $\varphi \in C_c(\mathbf{R}^3)$ using (i), (iv) of Lemma 2.8 and Dominated Convergence Theorem

$$\int_{\mathbf{R}^3} v_{t_k}(x) \varphi(x) dx = \int_{\mathbf{R}^3} v_0(x) \varphi(\Phi_{t_k}(x)) dx \rightarrow \int_{\mathbf{R}^3} v_0(x) \varphi(\Phi_{t_*}(x)) dx = \int_{\mathbf{R}^3} v_{t_*}(x) \varphi(x) dx.$$

The bounds on the support of v_t follow from $|\Phi_t(X)| \leq |X| + T\|\tilde{U}\|_{L^\infty}$, which can be obtained by approximating $\tilde{U}$ by smooth vector fields as in the proof of Lemmas 2.8, 2.9 above.

Now we prove that v is a weak solution of (2.59) with initial data $v|_{t=0} = v_0$. Let $\varphi \in C_c^\infty(\mathbf{R}^3 \times [0, T))$. Then using the definition $v_t = \Phi_t \# v_0$ of v , the properties of Φ stated in (i)–(iv) of Lemma 2.8, we get

$$\begin{aligned} \int_{\mathbf{R}^3 \times [0, T)} v \partial_t \varphi dt dx &= \int_{\mathbf{R}^3 \times [0, T)} v_0(x) (\partial_t \varphi)(t, \Phi_t(x)) dt dx \\ &= \int_{\mathbf{R}^3} v_0(x) \int_0^T \frac{d}{dt} [\varphi(t, \Phi_t(x))] dt dx \\ &\quad - \int_{\mathbf{R}^3 \times [0, T)} v_0(x) \nabla \varphi(t, \Phi_t(x)) \cdot \partial_t \Phi_t(x) dt dx \\ &= - \int_{\mathbf{R}^3} v_0(x) \varphi(0, x) dx - \int_{\mathbf{R}^3 \times [0, T)} v_0(x) \nabla \varphi(t, \Phi_t(x)) \cdot \tilde{U}(t, \Phi_t(x)) dx \\ &= - \int_{\mathbf{R}^3} v_0(x) \varphi(0, x) dx - \int_{\mathbf{R}^3 \times [0, T)} v_t(x) \nabla \varphi(t, x) \cdot \tilde{U}(t, x) dx. \end{aligned}$$

Lemma 2.10 is proved. $\square$

Now we prove that the solution (α, P) of (2.5) – (2.8) satisfies the property that α is a Lagrangian solution of the transport equation (2.5), in the sense of (2.60).

Proposition 2.11. *Let Ω, T, q, P_0 be as in Theorem 2.1. Let (α, P) be the weak solution of (2.5) – (2.8) constructed in Theorem 2.1. Let $\tilde{U}$ be defined by (2.46)–(2.48), and let Φ be the regular Lagrangian flow of $\tilde{U}$ defined in Lemma 2.8. If R_1 in the definition of $\tilde{U}$ is chosen sufficiently large depending only on Ω, T and $\|\nabla P_0\|_{L^\infty(\Omega)}$, then for every $t \in [0, T]$*

$$(2.60) \quad \alpha_t = \Phi_t \# \alpha_0.$$

Moreover, for every $t \in [0, T]$

$$(2.61) \quad \alpha_t(x) = \alpha_0(\Phi_t^*(x)) \quad \text{for a.e. } x \in \mathbf{R}^3,$$

where the map Φ_t^* is defined in Lemma 2.9(ii).

Proof. We choose R_1 to be defined by (2.58). Then, in particular, $R_1 \geq R_0 + 1$ where $R_0 = S(1 + T)$ is the number in (2.14).

We use notations introduced in the proof of Theorem 2.1, and for h, k considered there define

$$(2.62) \quad \tilde{U}_h^k(X) = J \left[H(X) - \nabla Q_h^k(X) \right],$$

where Q_h^k is defined by (2.18), and $H(X)$ is defined by (2.46)–(2.47) with R_1 given by (2.58). Define functions $\tilde{U}_h$ on $[0, T] \times \Omega$ by setting them equal to $\tilde{U}_h^k$ on the time interval $t \in [kh, (k+1)h)$. Then, from (2.19) and (2.46)–(2.47), we have $\tilde{U}_h \in L^\infty([0, T] \times \mathbf{R}^3)$ with

$$(2.63) \quad \|\tilde{U}_h(t, \cdot)\|_{L^\infty(\mathbf{R}^3)} \leq S + R_1 + 1 \quad \text{for all } t \in [0, T], \quad h \in (0, 1).$$

Let $h_j \rightarrow 0$ be a sequence for which (2.27)–(2.32) hold. From (2.31), using convexity of $Q_{h_j}(t, \cdot)$ we conclude that

$$\nabla \overline{Q}_{h_j}(t, \cdot) \rightarrow \nabla P^*(t, \cdot) \quad \text{a.e. in } \mathbf{R}^3 \quad \text{for each } t \in [0, T],$$

and from this, noting that $U_{h_j}(X, t) = J[H(X) - \nabla \overline{Q}_{h_j}(X, t)]$ on $[0, T] \times \mathbf{R}^3$, we get

$$(2.64) \quad \tilde{U}_{h_j}(t, \cdot) \rightarrow \tilde{U}(t, \cdot) \quad \text{a.e. in } \mathbf{R}^3 \quad \text{for each } t \in [0, T],$$

where $\tilde{U}$ is defined by (2.46)–(2.48). By (2.63), (2.64) and the Dominated Convergence Theorem

$$(2.65) \quad \tilde{U}_{h_j} \rightarrow \tilde{U} \quad \text{in } L^r([0, T] \times B(0, R); \mathbf{R}^3) \quad \text{for all } R > 0, \quad r \in [1, \infty).$$

Our choice $R_1 \geq R_0 + 1$ implies that $\overline{U}_h(t, X) = \tilde{U}_h(t, X)$ in $[0, T] \times B(0, R_0 + 1)$. Then from (2.24) it follows that α_h satisfy the equations (2.20) with U_h^k replaced by $\tilde{U}_h^k$. Thus each α_h is a weak solution of

$$(2.66) \quad \frac{\partial \alpha_h}{\partial t} + \operatorname{div}(\alpha_h \tilde{U}_h) = 0 \quad \text{in } \mathbf{R}^3 \times (0, T),$$

$$\alpha_h(0, X) = \alpha_h^0(X).$$

For each $h \in (0, 1)$, from (2.19), (2.46)–(2.47), and (2.62)

$$(2.67) \quad \|\nabla \tilde{U}_h\|_{L^\infty((0,T) \times \mathbf{R}^3)} \leq C \left(\frac{S}{h} + 1 \right),$$

where C depends only on functions η, ζ . Also, from its definition, $\tilde{U}_h$ is a divergence-free vector field. Then there exists a unique Lagrangian flow $\Phi_h : \mathbf{R}^3 \times \mathbf{R}^1 \rightarrow \mathbf{R}^3$ induced by $\tilde{U}_h$, and for each t , the map $(\Phi_h)_t : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is $\mathcal{L}^3$ measure-preserving (see e.g. [8, Theorem III.2]). From this, for the sequence $\{h_j\}$ considered above, using also the properties (2.63), (2.67) of $\tilde{U}_{h_j}$, the properties (2.45) of $\tilde{U}$, and the convergence in (2.65), we can apply [1, Theorem 6.5] to conclude that for each $t \in [0, T]$

$$(2.68) \quad (\Phi_{h_j})_t \rightarrow \Phi_t \quad \text{in } L^1_{loc}(\mathbf{R}^3) \quad \text{as } j \rightarrow \infty.$$

Let $q > 1$ be as in (2.13). Then $\|\alpha_h^0\|_{L^q(\mathbf{R}^3)} \leq \|\alpha_0\|_{L^q(\mathbf{R}^3)}$. From Lemma 2.10, it follows that the function $v(t, \cdot) = (\Phi_h)_t \# \alpha_h^0$ satisfies $v \in L^\infty([0, T], L^q(\mathbf{R}^3))$ and v is a weak solution of (2.66). Since $\tilde{U}_h \in L^\infty((0, T), W^{1,\infty}(\mathbf{R}^3))$ by (2.63) and (2.67), it follows that (2.66) has at most one weak solution in $v \in L^\infty([0, T], L^q(\mathbf{R}^3))$, see e.g. [8, Theorem II.2]. Using (2.26), we conclude that

$$(2.69) \quad \alpha_h(t, \cdot) = (\Phi_h)_t \# \alpha_h^0.$$

From (2.69) and (2.27), we get for any $t > 0$, $j = 1, \dots$ and any $\varphi \in C_c(\mathbf{R}^3)$

$$\int_{\mathbf{R}^3} \varphi(\Phi_{h_j}(t, X)) \alpha_{h_j}^0(X) dX = \int_{\mathbf{R}^3} \varphi(y) \alpha_{h_j}(t, y) dy.$$

Passing to the limit $j \rightarrow \infty$ in the last equality, using (2.68), the fact that $\alpha_{h_j}^0 \rightarrow \alpha_0$ in $L^q(\mathbf{R}^3)$, and the Dominated Convergence theorem in the left-hand side, and using (2.29) in the right hand side, we obtain

$$(2.70) \quad \int_{\mathbf{R}^3} \varphi(\Phi(t, X)) \alpha_0(X) dX = \int_{\mathbf{R}^3} \varphi(y) \alpha(t, y) dy$$

for any $\varphi \in C_c(\mathbf{R}^3)$. This implies (2.60).

Since Φ_t is a measure-preserving map, we use Lemma 2.9 (ii) to conclude that the left-hand side of (2.70) is equal to $\int_{\mathbf{R}^3} \varphi(y) \alpha_0(\Phi_t^*(y)) dy$, and now (2.70) implies (2.61). $\square$

2.4. Lagrangian flow in the physical space. Through this section we assume that $\Omega, T, P_0, \alpha, P, U, \tilde{U}, \Phi$ are as in Proposition 2.11. Moreover, we fix R_1 in the definition of $\tilde{U}$ sufficiently large so that the conclusions of Proposition 2.11 hold.

Below we use the following notation: for a function $g(t, x)$ we denote $g_t(x) := g(t, x)$.

We intend to define a Lagrangian flow in the physical space $F : [0, T) \times \Omega \rightarrow \Omega$ by defining $F_t : \Omega \rightarrow \Omega$ for $t \in [0, T)$ by

$$(2.71) \quad F_t := \nabla P_t^* \circ \Phi_t \circ \nabla P_0,$$

where P_t^* is the convex dual of P_t , and $\Phi_t : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is the Lagrangian flow in the dual space constructed in Lemma 2.8. For that, we need to prove first

Lemma 2.12. *The right-hand side of (2.71) is defined $\mathcal{L}^4$ - a.e. in $[0, T) \times \Omega$. Moreover, for any $t \in [0, T)$ the right-hand side of (2.71) is defined $\mathcal{L}^3$ - a.e. in Ω . The map $F : [0, T) \times \Omega \rightarrow \Omega$ defined by (2.71) is Borel.*

Proof. Since P_0 is a bounded convex function on B , then ∇P_0 exists on $\Omega \setminus N_0^1$ where N_0^1 is a Borel subset of Ω with $\mathcal{L}^3(N_0^1) = 0$. Also, since P^* is a bounded Borel function on $[0, T] \times \mathbf{R}^3$ and for every t the function $P^*(t, \cdot)$ is convex in $\mathbf{R}^3$, then ∇P^* exists on $([0, T] \times \mathbf{R}^3) \setminus N^2$, where N^2 is a Borel subset of $[0, T] \times \mathbf{R}^3$ with $\mathcal{L}^4(N^2) = 0$, and moreover, denoting $N_t^2 := N^2 \cap (\{t\} \times \mathbf{R}^3)$, we have $\mathcal{L}^3(N_t^2) = 0$ for every $t \in [0, T]$.

Then the right-hand side of (2.71) is defined for all

$$(t, x) \in ([0, T] \times \Omega) \setminus (([0, T] \times N_0^1) \cup M),$$

$$\text{where } M = \{(s, y) \in [0, T] \times (\Omega \setminus N_0^1) \mid (s, \Phi(s, \nabla P_0(y))) \in N^2\}$$

From its definition, M is a Borel set.

It remains to prove that $\mathcal{L}^4(M) = 0$, and $\mathcal{L}^3(M_t) = 0$ for every $t \in [0, T]$, where $M_t := M \cap (\{t\} \times \mathbf{R}^3)$. By Fubini's theorem, it is sufficient to prove that $\mathcal{L}^3(M_t) = 0$ for every $t \in [0, T]$.

Fix $t \in [0, T]$. Then, since $\Phi_t : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is $\mathcal{L}^3$ -measure-preserving, we have $\mathcal{L}^3(\Phi_t^{-1}(N_t^2)) = 0$, and, using that $\nabla P_0 \# \chi_\Omega = \alpha_0$ thus $\nabla P_0 \# \chi_{\Omega \setminus N_0^1} = \alpha_0$, we compute:

$$|M_t| = |\{x \in \Omega \setminus N_0^1 \mid \nabla P_0(x) \in \Phi_t^{-1}(N_t^2)\}| = \int_{\Phi_t^{-1}(N_t^2)} \alpha_0(z) dz = 0$$

since $\alpha_0 \in L_{loc}^1(\mathbf{R}^3)$.

Thus we can define $F : [0, T] \times \Omega \rightarrow \Omega$ by (2.71). Then by Lemma 2.8, F is a Borel mapping. $\square$

It remains to prove that if F is defined by (2.71), then (F, P) is a weak Lagrangian solution in the sense of Definition 2.4.

We first show that the initial condition for the flow is satisfied:

Proposition 2.13. $F(0, x) = x$ for a.e. $x \in \Omega$.

Proof. By (2.71), $F_0(x) = \nabla P_0^* \circ \Phi_0 \circ \nabla P_0(x)$ for all $x \in \Omega \setminus N_0$ where N_0 is a Borel set with $\mathcal{L}^3(N_0) = 0$.

From convexity of P_0 in B and (2.4), there exist Borel sets $N_1 \subset \Omega$, $N_2 \subset \mathbf{R}^3$ with $\mathcal{L}^3(N_1) = \mathcal{L}^3(N_2) = 0$ such that P_0 (resp. P_0^*) is differentiable on $\Omega \setminus N_1$ (resp. $\mathbf{R}^3 \setminus N_2$). Moreover, if $x \in \Omega \setminus [N_1 \cup (\nabla P_0)^{-1}(N_2)]$ then $\nabla P_0^* \circ \nabla P_0(x) = x$.

By Lemma 2.8(ii), $\Phi_0(x) = x$ in $\mathbf{R}^3 \setminus N_3$, where N_3 is a Borel set with $\mathcal{L}^3(N_3) = 0$.

Thus $F_0(x) = x$ for all $x \in \Omega \setminus [N_0 \cup N_1 \cup (\nabla P_0)^{-1}(N_2 \cup N_3)]$. In order to complete the proof of the Proposition, we need to show that $\mathcal{L}^3[(\nabla P_0)^{-1}(N_2 \cup N_3) \cap \Omega] = 0$.

Denote $\mu = \alpha_0 dx$ in $\mathbf{R}^3$. Since $\alpha_0 \in L^q(\mathbf{R}^3)$, then $\mu[N_2 \cup N_3] = 0$, and since $\nabla P_0 \# \chi_\Omega = \alpha_0$, we get $\mathcal{L}^3[(\nabla P_0)^{-1}(N_2 \cup N_3) \cap \Omega] = \mu[N_2 \cup N_3] = 0$. $\square$

The next step is to prove that the property Definition 2.4(ii) is satisfied.

Proposition 2.14. For every $t > 0$ the map $F_t : \Omega \rightarrow \Omega$ is $\mathcal{L}^3$ -measure preserving.

Proof. In order to complete the proof, we need to justify the following calculation: for any $\varphi \in C(\mathbf{R}^3)$

$$(2.72) \quad \int_{\Omega} \varphi(F_t(x)) dx = \int_{\Omega} \varphi \circ \nabla P_t^* \circ \Phi_t \circ \nabla P_0(x) dx$$

$$(2.73) \quad = \int_{\mathbf{R}^3} \varphi \circ \nabla P_t^* \circ \Phi_t(y) \alpha_0(y) dy$$

$$(2.74) \quad = \int_{\mathbf{R}^3} \varphi \circ \nabla P_t^*(z) \alpha_t(z) dz$$

$$(2.75) \quad = \int_{\Omega} \varphi(x) dx.$$

Then the Proposition follows from Lemma A.2.

Now we prove these equalities.

Equality (2.72) follows from the definition of F_t .

Since $\nabla P_0 \# \chi_{\Omega} = \alpha_0$, and $\alpha_0 \in L^q(\mathbf{R}^3)$ with compact support, Corollary A.3 implies

$$\int_{\Omega} \psi(\nabla P_0(x)) dx = \int_{\mathbf{R}^3} \psi(y) \alpha_0(y) dy \quad \text{for all } \psi \in L^{\infty}(\mathbf{R}^3).$$

Choosing $\psi = \varphi \circ \nabla P_t^* \circ \Phi_t \in L^{\infty}(\mathbf{R}^3)$, we conclude that the right-hand side of (2.72) is equal to (2.73).

Equality of expressions (2.74) and (2.73) follows from (2.60) and Corollary A.3, since $\varphi \circ \nabla P_t^* \in L^{\infty}(\mathbf{R}^3)$ and $\alpha_0 \in L^q(\mathbf{R}^3)$ with compact support.

Equality of expressions (2.74) and (2.75) follows from $\nabla P_t^* \# \alpha_t = \chi_{\Omega}$. $\square$

Now we prove that (2.35) holds for all $r \in [1, \infty)$.

Proposition 2.15. *For any $t_0 \in [0, T)$, any $r \in [1, \infty)$*

$$\lim_{t \rightarrow t_0, t \in [0, T)} \int_{\Omega} |F_t(x) - F_{t_0}(x)|^r dx = 0.$$

Proof. By Lemma 2.12, for any $t \in [0, T)$ (2.71) holds $\mathcal{L}^3$ - a.e. in Ω . Also, $\nabla P_0 \# \chi_{\Omega} = \alpha_0$. Thus we get for any $t, t_0 \in [0, T)$

$$\begin{aligned} \int_{\Omega} |F_t(x) - F_{t_0}(x)|^r dx &= \int_{\Omega} |\nabla P_t^* \circ \Phi_t \circ \nabla P_0(x) - \nabla P_{t_0}^* \circ \Phi_{t_0} \circ \nabla P_0(x)|^r dx \\ &= \int_{\mathbf{R}^3} |\nabla P_t^* \circ \Phi_t(y) - \nabla P_{t_0}^* \circ \Phi_{t_0}(y)|^r \alpha_0(y) dy \\ &\leq C \int_{\mathbf{R}^3} |\nabla P_t^* \circ \Phi_t(y) - \nabla P_{t_0}^* \circ \Phi_t(y)|^r \alpha_0(y) dy \\ &\quad + C \int_{\mathbf{R}^3} |\nabla P_{t_0}^* \circ \Phi_t(y) - \nabla P_{t_0}^* \circ \Phi_{t_0}(y)|^r \alpha_0(y) dy \\ &=: C(I_1 + I_2) \end{aligned}$$

We show first that $I_1 \rightarrow 0$ as $t \rightarrow t_0$. Note that $\|\alpha_t\|_{L^q(\mathbf{R}^3)} = \|\alpha_0\|_{L^q(\mathbf{R}^3)}$ for $t \in (0, T)$, this follows from (2.61) since Φ_t^* is a $\mathcal{L}^3$ -measure preserving map. Now we use (2.60) to

estimate

$$\begin{aligned}
I_1 &= \int_{\mathbf{R}^3} |\nabla P_t^* \circ \Phi_t(y) - \nabla P_{t_0}^* \circ \Phi_t(y)|^r \alpha_0(y) dy \\
&= \int_{\mathbf{R}^3} |\nabla P_t^*(y) - \nabla P_{t_0}^*(y)|^r \alpha_t(y) dy \\
&\leq \|\nabla P_t^* - \nabla P_{t_0}^*\|_{r_{q'}}^r \|\alpha_t\|_q \\
&= \|\nabla P_t^* - \nabla P_{t_0}^*\|_{r_{q'}}^r \|\alpha_0\|_q \rightarrow 0 \quad \text{as } t \rightarrow t_0
\end{aligned}$$

by Theorem 2.1.

Now we show that $I_2 \rightarrow 0$ as $t \rightarrow t_0$. Since $\nabla P_t^*(y) \in B$ for a.e. y for each t , and $\alpha_0 \in L^1(\mathbf{R}^3)$ (since α_0 is in L^q and has compact support), then by the Dominated Convergence theorem it remains to prove that, for every t_0 ,

$$(2.76) \quad \nabla P_{t_0}^* \circ \Phi_t(y) - \nabla P_{t_0}^* \circ \Phi_{t_0}(y) \rightarrow 0 \quad \text{as } t \rightarrow t_0$$

for a.e. $y \in \mathbf{R}^3$. First we note that, since Φ_t is measure preserving, then it follows from Lemma 2.8(i) that

$$\Phi_t(y) \rightarrow \Phi_{t_0}(y) \text{ as } t \rightarrow t_0 \text{ in } [0, T], \text{ for a.e. } y \in \mathbf{R}^3.$$

If y is such a point, and if in addition $\Phi_{t_0}(y)$ is a point of continuity for $\nabla P_{t_0}^*$, then convergence in (2.76) holds at y . As $P_{t_0}^*$ is convex, $\nabla P_{t_0}^*$ is differentiable a.e. Since Φ_{t_0} is measure preserving, it follows that $\Phi_{t_0}(y)$ is a point of continuity for $\nabla P_{t_0}^*$ for a.e. y . Thus (2.76) holds for a.e. $y \in \mathbf{R}^3$. The proposition is proved. $\square$

It remains to show that the properties (iii, iv) in Definition 2.4 are satisfied. The first step is the following:

Lemma 2.16. $\nabla P_t \circ F_t(x) = \Phi_t \circ \nabla P_0(x)$ for a.e. $(t, x) \in (0, T) \times \Omega$.

Proof. To prove the lemma, we need to justify the following formal computation:

$$\nabla P_t \circ F_t = \nabla P_t \circ \nabla P_t^* \circ \Phi_t \circ \nabla P_0 = \Phi_t \circ \nabla P_0$$

since $\nabla P_t \circ \nabla P_t^*$ is the identity on the support of α_t .

Now we make this argument rigorous.

Since $P(t, x)$ is a bounded Borel function in $[0, T] \times B$ and $P(t, \cdot)$ is a convex function in B for every $t \in [0, T]$, then ∇P exists in $([0, T] \times \Omega) \setminus N^1$ where N^1 is a Borel subset of $[0, T] \times \Omega$ with $\mathcal{L}^4(N^1) = 0$, and moreover, denoting $N_t^1 := N^1 \cap (\{t\} \times \mathbf{R}^3)$, we have $\mathcal{L}^3(N_t^1) = 0$ for every $t \in [0, T]$.

Then $F^{-1}(N^1)$ is a Borel subset of $[0, T] \times \Omega$ satisfying

$$\mathcal{L}^4(F^{-1}(N^1)) = 0.$$

Indeed, it is enough to show that $\mathcal{L}^3(F_t^{-1}(N_t^1)) = 0$ for every $t \in [0, T]$, and this follows from $\mathcal{L}^3(N_t^1) = 0$ and Proposition 2.14.

Now we can conclude, using Lemma 2.12, that

$$Z_t(x) = \nabla P_t \circ \nabla P_t^* \circ \Phi_t \circ \nabla P_0(x) \quad \text{for } (t, x) \in ([0, T] \times \Omega) \setminus \tilde{N} \quad \text{where } \mathcal{L}^4(\tilde{N}) = 0.$$

Let $\tilde{M} = \{(t, x) \in [0, T] \times \mathbf{R}^3 \mid \nabla P(\nabla P^*(t, x), t) \neq x\}$, where we include (t, x) such that either $\nabla P^*(t, x)$ or $\nabla P(\nabla P^*(t, x), t)$ do not exist. Then $\tilde{M}$ is a Borel set.

The proof of the lemma will be completed if we show that

$$(2.77) \quad \mathcal{L}^4 \left[\{(t, x) \in ([0, T] \times \Omega) \setminus \tilde{N} \mid (\Phi_t \circ \nabla P_0(x), t) \in \tilde{M}\} \right] = 0.$$

Since $\nabla P_t \# \chi_\Omega = \alpha_t$ and $\nabla P_t^* \# \alpha_t = \chi_\Omega$, it follows that for any t

$$\nabla P_t \circ \nabla P_t^*(x) = x \quad \text{for } \alpha_t\text{-a.e. } x \in \mathbf{R}^3.$$

Then, denoting $\tilde{M}_t = \tilde{M} \cap (\mathbf{R}^3 \times \{t\})$, we have

$$\int_{\tilde{M}_t} \alpha_t(x) dx = 0 \quad \text{for any } t \in [0, T].$$

Thus for any $t \in [0, T]$ we get, using that $\mathcal{L}^3(\tilde{N}_t) = 0$ and thus $\nabla P_0 \# \chi_{\Omega \setminus \tilde{N}_t} = \alpha_0$, and also using Lemma 2.8(iv), Lemma 2.9(ii) and (2.61):

$$\begin{aligned} \mathcal{L}^3 \left[\{x \in \Omega \setminus \tilde{N}_t \mid \Phi_t \circ \nabla P_0(x) \in \tilde{M}_t\} \right] &= \mathcal{L}^3 \left[\{x \in \Omega \setminus \tilde{N}_t \mid \nabla P_0(x) \in \Phi_t^*(\tilde{M}_t)\} \right] \\ &= \int_{\Phi_t^*(\tilde{M}_t)} \alpha_0(x) dx = \int_{\tilde{M}_t} \alpha_0(\Phi_t^*(x)) dx \\ &= \int_{\tilde{M}_t} \alpha_t(x) dx = 0. \end{aligned}$$

Now (2.77) follows from Fubini's theorem. The lemma is proved. $\square$

Now we show existence of the map F^* from Definition 2.4(iii).

Proposition 2.17. *The map F satisfies property (iii) in Definition 2.4.*

Proof. Similar to Lemma 2.12 we can show that for every $t \in [0, T]$ the expression $\nabla P_0^* \circ \Phi_t^* \circ \nabla P_t(x)$ is defined for a.e. $x \in \Omega$, and the map $F_t^* : \Omega \rightarrow \Omega$ defined by $F_t^* = \nabla P_0^* \circ \Phi_t^* \circ \nabla P_t$ is Borel.

Since F_t is measure preserving, then $F_t^* \circ F_t(x) = \nabla P_0^* \circ \Phi_t^* \circ \nabla P_t \circ F_t(x)$ for a.e. $x \in \Omega$. Using Lemma 2.16, get $F_t^* \circ F_t(x) = \nabla P_0^* \circ \Phi_t^* \circ \Phi_t \circ \nabla P_0(x)$ a.e. in Ω . Since $\Phi_t^* \circ \Phi_t(y) = y$ for a.e. y and thus for α_0 -a.e. $y \in \mathbf{R}^3$, and $\nabla P_0 \# 1_\Omega = \alpha_0$, then $\Phi_t^* \circ \Phi_t \circ \nabla P_0(x) = \nabla P_0(x)$ for a.e. $x \in \Omega$. Thus $F_t^* \circ F_t(x) = \nabla P_0^* \circ \nabla P_0(x) = x$ for a.e. $x \in \Omega$.

By a similar argument, $F_t \circ F_t^*(x) = x$ for a.e. $x \in \Omega$. $\square$

Finally we show the property (iv) in Definition 2.4.

Proposition 2.18. *Equality (2.38) holds for any $\phi \in C_c^1((0, T) \times \Omega, \mathbf{R}^3)$. Moreover, possibly after modifying $Z(t, x)$ on a negligible subset of $(0, T) \times \Omega$, we have $Z(\cdot, x) \in W^{1, \infty}([0, T]; \mathbf{R}^3)$ for a.e. $x \in \Omega$, and (2.43) holds.*

Proof. From the definition of the Lagrangian flow Φ , i.e. properties (i)–(iii) of Lemma 2.8, for a.e. $X \in \mathbf{R}^3$ and every $t \in [0, T]$

$$\Phi(t, X) = X + \int_0^t \tilde{U}(s, \Phi_s(X)) ds.$$

Thus the above equality holds for all $X \in \mathbf{R}^3 \setminus N$ where $|N| = 0$. Since $\nabla P_0 \# \chi_\Omega = \alpha_0$ and $\alpha_0 \in L^q(\mathbf{R}^3)$, it follows that $|(\nabla P_0)^{-1}(N) \cap \Omega| = \int_N \alpha_0(z) dz = 0$. Thus for a.e.

$x \in \Omega$ and every $t \in [0, T)$

$$(2.78) \quad \Phi(t, \nabla P_0(x)) = \nabla P_0(x) + \int_0^t U(s, \Phi_s(\nabla P_0(x))) ds,$$

where we replaced $\tilde{U}(s, \Phi_s(\nabla P_0(x)))$ by $U(s, \Phi_s(\nabla P_0(x)))$ based on (2.54), (2.55). Multiplying the last equality by $\partial_t \eta(t, x)$, where $\eta \in C_c^1([0, T) \times \mathbf{R}^3)$, and integrating we get

$$\begin{aligned} \int_{[0, T) \times \Omega} \partial_t \eta(t, x) \Phi(t, \nabla P_0(x)) dt dx &= \int_{[0, T) \times \Omega} \partial_t \eta(t, x) \nabla P_0(x) dt dx \\ &\quad + \int_{[0, T) \times \Omega} \partial_t \eta(t, x) \int_0^t U(s, \Phi_s(\nabla P_0(x))) ds dt dx. \end{aligned}$$

Note that $\eta(T, x) \equiv 0$. In the right-hand side, we perform the integration with respect to t in the first integral, and integrate by parts with respect to t in the second integral, to get

$$(2.79) \quad \begin{aligned} \int_{[0, T) \times \Omega} \partial_t \eta(t, x) \Phi(t, \nabla P_0(x)) dt dx &= - \int_{\Omega} \eta(0, x) \nabla P_0(x) dx \\ &\quad - \int_{[0, T) \times \Omega} \eta(t, x) U(t, \Phi_t(\nabla P_0(x))) dt dx. \end{aligned}$$

Now we compute using (2.7), (2.71), and Lemma 2.16

$$U(t, \Phi_t(\nabla P_0(x))) = J[\Phi_t(\nabla P_0(x)) - \nabla P_t^*(\Phi_t(\nabla P_0(x)))] = J[Z(t, x) - F(t, x)]$$

for a.e. (t, x) . Substituting this into the right-hand side of (2.79) and using Lemma 2.16 to replace $\Phi(t, \nabla P_0(x))$ by $Z(t, x)$ in the left-hand side of (2.79), we obtain (2.38)

Finally, $Z(\cdot, x) \in W^{1, \infty}([0, T); \mathbf{R}^3)$ for a.e. $x \in \Omega$ (possibly after modifying $Z(t, x)$ on a negligible subset of $(0, T) \times \Omega$) follows from Lemma 2.16 and Lemma 2.8(i). Then (2.78) and Lemma 2.16 imply (2.43). $\square$

Now the properties of P in Theorem 2.1, and the properties of (F, P) proved in Propositions 2.13, 2.14, 2.15, 2.17, 2.18 imply Theorem 2.2.

3. LAGRANGIAN SOLUTIONS OF 2-DIMENSIONAL SEMI-GEOSTROPHIC SHALLOW WATER EQUATIONS IN PHYSICAL SPACE

3.1. Model formulation and background. In this section we extend the approach of the previous section to the model considered by Cullen and Gangbo in [5]. They study a shallow water approximation to the following free boundary problem for the system (2.1) in an evolving region $\mathcal{D}(t)$: the free boundary condition is

$$(3.1) \quad u \cdot \nu = \text{normal speed of the boundary, on } \partial [\cup_{t \in [0, T)} \{t\} \times \mathcal{D}(t)],$$

and the initial conditions are

$$(3.2) \quad \begin{aligned} \mathcal{D}(0) &= \mathcal{D}_0, \\ p(0, x) &= p_0(x) \quad \text{in } \Omega, \end{aligned}$$

where $\mathcal{D}_0, p_0(x)$ are a given set and function respectively. In [5], problem (2.1), (3.1), (3.2) is considered in the following shallow water approximation:

The fluid is contained within a region Ω of (x_1, x_2) -plane but the height $h(t, x_1, x_2)$ above the reference level is unknown and can evolve in time:

$$(3.3) \quad \mathcal{D}(t) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid (x_1, x_2) \in \Omega, \quad 0 \leq x_3 \leq h(t, x_1, x_2)\}.$$

The pressure on the top boundary of the fluid is a given constant p_0 , and

$$(3.4) \quad p(t, x_1, x_2, x_3) = [h(t, x_1, x_2) - x_3] + p_0.$$

The horizontal components of velocity are independent of x_3 .

Then the problem (2.1), (3.1), (3.2) in an evolving 3-dimensional domain can be rewritten as a problem in the 2-dimensional domain Ω , for the unknown height function $h(t, x)$ defined on $[0, T] \times \Omega$, and horizontal components of velocity $v = (v_1, v_2)$. As noted by ([5]), it is possible for h to become zero on part of Ω . We set $v = 0$ in such regions. Then the problem (2.1), (3.1), (3.2) in the shallow water approximation (3.3)–(3.4) can be written as the following problem for $h(t, x)$ and $v = (v_1, v_2)$ defined on $[0, T] \times \Omega$ and $v = (v_1, v_2)$ defined in Ω :

$$(3.5) \quad D_t X = J(X - x), \quad \text{in } [0, T] \times \Omega,$$

$$(3.6) \quad \partial_t h + \operatorname{div}(hv) = 0, \quad \text{in } [0, T] \times \Omega,$$

$$(3.7) \quad X = \nabla P, \quad \text{where } P(t, x_1, x_2) = h(t, x_1, x_2) + \frac{1}{2}(x_1^2 + x_2^2),$$

$$(3.8) \quad D_t = \partial_t + v \cdot \nabla, \quad \nabla = (\partial_{x_1}, \partial_{x_2}),$$

$$(3.9) \quad v \cdot \nu = 0, \quad \text{on } [0, T] \times \partial\Omega,$$

$$(3.10) \quad h(0, x) = h_0(x) \quad \text{in } \Omega.$$

Here $J \equiv J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that (3.6) corresponds to the condition (3.1) on the surface $\{(x_1, x_2, x_3) \mid h(x_1, x_2) > 0, x_3 = h(x_1, x_2)\}$, which is the top of the fluid.

The Cullen-Purser stability condition implies that the function $x \rightarrow P(t, x)$ is convex in Ω for any t , where $P(t, x)$ is defined in (3.7). Thus the initial height function $h_0(x)$ must satisfy:

$$h_0 + \frac{1}{2}|x|^2 \text{ is convex in } \Omega.$$

Cullen and Gangbo in [5] rewrite the system (3.5)–(3.10) in dual variables (t, X) :

$$(3.11) \quad \partial_t \alpha + \nabla \cdot (U\alpha) = 0 \quad \text{in } [0, T] \times \mathbf{R}^2,$$

$$(3.12) \quad \nabla P_t \# h_t = \alpha_t \quad \text{for any } t \in [0, T];$$

$$(3.13) \quad U(t, X) = J_2(X - \nabla P^*(t, X)) \quad \text{in } [0, T] \times \mathbf{R}^2,$$

$$(3.14) \quad P^*(t, X) = \sup_{x \in \Omega} (x \cdot X - P(t, x)) \quad \text{for } (t, X) \in [0, T] \times \mathbf{R}^2,$$

$$(3.15) \quad \alpha_0(X) \text{ is prescribed on } \mathbf{R}^2.$$

Cullen and Gangbo in [5] prove existence of a weak solution of system (3.11)–(3.15). The precise statement of their result is Theorem 2.1 with the following changes: $\mathbf{R}^3$ replaced by $\mathbf{R}^2$; $DP_0 \# \chi_\Omega$ replaced by $DP_0 \# h_0$ in (2.13); and (2.5) – (2.8) replaced by (3.11)–(3.15) in the assertion Theorem 2.1(v), where $h(t, x) = P(t, x) - \frac{1}{2}|x|^2$.

In this section we study the shallow water semigeostrophic model (3.5)–(3.10) in physical space. The unknown functions in this problem are the (modified) pressure P and

the velocity v . The Eulerian formulation of weak solutions is obtained from the following argument. The Eulerian form of (3.5) is

$$(3.16) \quad \partial_t X + v \cdot \nabla X = J(X - x).$$

Multiplying (3.16) by h and adding (3.5) multiplied by X , we get

$$\partial_t(hX) + \sum_{j=1}^2 \partial_{x_j}(hv_j X) = hJ(X - x).$$

Thus we have the following

Definition 3.1. *Let $v : [0, T] \times \Omega \rightarrow \mathbf{R}^2$, $P : [0, T] \times \Omega \rightarrow \mathbf{R}^1$, satisfy $v \in L^1([0, T] \times \Omega, \mathbf{R}^2)$, $\nabla P \in L^\infty([0, T] \times \Omega) \cap C([0, T]; L^1(\Omega))$, and $P(t, \cdot)$ is convex in Ω for every $t \in [0, \infty)$. Let $h(t, x) = P(t, x) - \frac{1}{2}|x|^2$. The pair (v, P) is a weak Eulerian solution of (3.5)–(3.10) if*

$$(3.17) \quad \int_{(0,T) \times \Omega} \{ \nabla P(t, x) \cdot [\partial_t \phi(t, x) + (v(t, x) \cdot \nabla) \phi(t, x)] \\ + J[\nabla P(t, x) - x] \cdot \phi(t, x) \} h(t, x) dt dx + \int_{\Omega} [\nabla h_0(x) + x] \cdot \phi(0, x) h_0(x) dx = 0.$$

for any $\phi \in C_c^1([0, T] \times \Omega; \mathbf{R}^2)$, and

$$(3.18) \quad \int_{[0,T] \times \Omega} [\partial_t \psi(t, x) + v(t, x) \cdot \nabla \psi(t, x)] h dt dx + \int_{\Omega} \psi(0, x) h_0(x) dx = 0$$

for any $\psi \in C_c^1([0, T] \times \bar{\Omega})$.

Remark 3.2. • Equality (3.17) is a weak form of the equation (3.5). In the second integral in the left-hand side of (3.17), we used the fact that $P_0(x) = h_0(x) + \frac{1}{2}|x|^2$.
• Equality (3.18) is a weak form of the equation (3.6) with the boundary condition (3.9) and initial condition (3.10).

Remark 3.3. Equality (3.17) of Definition 3.1 essentially requires the evolution equation (3.5) to hold only in the fluid region $\mathcal{D} = \{(t, x) \in [0, T] \times \Omega \mid h(t, x) > 0\}$. This is natural since, physically, the evolution is defined only in $\mathcal{D}$.

As in the case of the incompressible model (2.3), we can find P by solving the problem (3.11) – (3.15) in the dual coordinates, and v then is expressed by (2.12), but regularity of this solution is not sufficient to show that such v, P satisfy (3.5) in the Eulerian sense. Thus we define a Lagrangian solution.

3.2. Lagrangian solutions for shallow water model in physical space. Similar to Section 2.2 we rewrite (3.5)–(3.10) in terms of (F, P) where $F : [0, T] \times \Omega \rightarrow \Omega$ is a (formal) Lagrangian flow corresponding to the full wind velocity $v = (v_1, v_2)$, and then we define the corresponding weak solution F, P . The difference is now that the vector field v is not divergence-free, but instead the transport equation $\partial_t h + \operatorname{div}(hv) = 0$ holds. Since F is a Lagrangian flow of v , solutions h of this transport equation satisfy $F_t \# h_0 = h_t$ for any $t \in (0, T)$. This property in the present replaces the Lebesgue-measure preserving property of F in section 2.2.

Then we come to the following

Definition 3.4. Let $\Omega \subset \mathbf{R}^2$ be an open set, and $T > 0$. Let $P_0(x)$ be a convex bounded function in Ω such that $h_0(x) = P_0(x) - \frac{1}{2}|x|^2 \geq 0$ in Ω . Let $r \in [1, \infty)$. Let $P : [0, T) \times \Omega \rightarrow \mathbf{R}^1$ satisfy

$$(3.19) \quad P \in L^\infty([0, T); W^{1,\infty}(\Omega)) \cap C([0, T); W^{1,r}(\Omega)),$$

$$(3.20) \quad P(t, \cdot) \text{ is convex in } \Omega \text{ for each } t \in [0, T).$$

Let $h(t, x) = P(t, x) - \frac{1}{2}|x|^2$. Let $F : [0, T) \times \Omega \rightarrow \Omega$ be a Borel map, satisfying

$$(3.21) \quad F \in C([0, T); L^r(\Omega, h_0 dx)).$$

The pair (P, F) is called a weak Lagrangian solution of (3.5)–(3.10) in $[0, T) \times \Omega$ if

- (i) $F(0, x) = x$ for $h_0 \mathcal{L}^2$ -a.e. $x \in \Omega$, and $P(0, x) = P_0(x)$ for a.e. $x \in \Omega$,
- (ii) for any $t \in [0, T]$ the mapping $F_t = F(t, \cdot) : \Omega \rightarrow \Omega$ satisfies $F_t \# h_0 = h_t$;
- (iii) There exists a Borel map $F^* : [0, T) \times \Omega \rightarrow \Omega$ such that for every $t \in (0, T)$ the map $F_t^* = F^*(t, \cdot) : \Omega \rightarrow \Omega$ satisfies $F_t^* \# h_t = h_0$, and $F_t^* \circ F_t(x) = x$ for $h_0 \mathcal{L}^2$ -a.e. $x \in \Omega$, $F_t \circ F_t^*(x) = x$ for $h_t \mathcal{L}^2$ -a.e. $x \in \Omega$;
- (iv) The function

$$(3.22) \quad Z(t, x) = \nabla P(t, F(t, x))$$

is a weak solution of

$$(3.23) \quad \begin{aligned} \partial_t Z(t, x) &= J[Z(t, x) - F(t, x)] && \text{on } \text{supp } h_0 \text{ in } (0, T) \times \Omega, \\ Z(0, x) &= \nabla P_0(x) && \text{on } \text{supp } h_0 \text{ in } \Omega, \end{aligned}$$

in the following sense: for any $\phi \in C_c^1([0, T) \times \Omega; \mathbf{R}^2)$

$$(3.24) \quad \begin{aligned} \int_{(0,T) \times \Omega} [Z(t, x) \cdot \partial_t \phi(t, x) + J(Z(t, x) - F(t, x)) \phi(t, x)] h_0(x) dt dx \\ + \int_{\Omega} \nabla P_0(x) \cdot \phi(0, x) h_0(x) dx = 0. \end{aligned}$$

Remark 3.5. Remark 2.5, with obvious modifications, applies to the present case. In particular, continuity in time of P, F in (3.19), (3.21), combined with initial conditions in part (i) of Definition 3.4, imply that

$$(3.25) \quad \begin{aligned} \lim_{t \rightarrow 0+} \int_{\Omega} |F_t(x) - x|^r h_0(x) dx &= 0, \\ \lim_{t \rightarrow 0+} \|P_t - P_0\|_{W^{1,r}(\Omega)} &= 0, \quad \lim_{t \rightarrow 0+} \|h_t - h_0\|_{W^{1,r}(\Omega)} = 0, \end{aligned}$$

where $r \in [1, \infty)$ is from (3.19), (3.21)

Remark 3.6. In Definition 3.4 we essentially consider the flow mapping F_t and its inverse F_t^* only in the fluid region $\mathcal{D} = \{(t, x) \in [0, T) \times \Omega \mid h(t, x) > 0\}$. That is, for any $t \in [0, T)$, Definition 3.4 does not contain any conditions for the map F_t away from $\mathcal{D}(0) = \{x \in \Omega \mid h_0(x) > 0\}$, and for the map F_t^* away from $\mathcal{D}(t) = \{x \in \Omega \mid h_t(x) > 0\}$. This is natural since, physically, the evolution is defined only in $\mathcal{D}$. A similar feature of weak Eulerian solutions is discussed in Remark 3.3. In particular, we can define F and F^* arbitrarily outside the domains $[0, T) \times \mathcal{D}_0$ and $\mathcal{D}$ respectively. Note also that the set $\Omega \setminus \mathcal{D}(t)$ is $h_t \mathcal{L}^2$ -negligible for each t , thus Definition 3.4 determines the maps F_t and F_t^* almost everywhere with respect to the measures $h_0 \mathcal{L}^2$ and $h_t \mathcal{L}^2$ respectively.

We justify Definition 3.4 by showing that a smooth Lagrangian solution (F, P) with additional regularity property $\partial_t F \in L^\infty([0, T) \times \Omega)$ determines a weak Eulerian solution of (3.5)–(3.10):

Lemma 3.7 (consistency of weak solutions for shallow water model). *Let $\Omega \subset \mathbf{R}^2$ be an open bounded set, and $T > 0$. Let (F, P) be a weak Lagrangian solution of (3.5)–(3.10) in $[0, T) \times \Omega$ satisfying $\partial_t F \in L^\infty([0, T) \times \Omega; \mathbf{R}^2)$. Then the function*

$$(3.26) \quad v(t, x) := \frac{\partial F}{\partial t}(t, F_t^*(x)) \quad \text{for } (t, x) \in [0, T) \times \Omega,$$

satisfies $v \in L^\infty([0, T) \times \Omega; \mathbf{R}^2)$, and (v, P) is a weak Eulerian solution of (3.5)–(3.10) in the sense of Definition 3.1.

Proof. The proof is similar to the proof of Lemma 2.6(i). Thus we only sketch the argument and emphasize the structural differences between the models (2.3) and (3.5)–(3.10).

We show (3.18). Fix $t \in [0, T)$. Since $F_t \# h_0 = h_t$, then for each $\psi \in C_c^1([0, T) \times \bar{\Omega})$

$$\int_{\Omega} (\partial_t \psi)(t, F_t(x)) h_0(x) dx = \int_{\Omega} \partial_t \psi(t, x) h(t, x) dx.$$

Integrating with respect to t and using $\partial_t F \in L^\infty([0, T) \times \Omega)$, we get

$$\begin{aligned} & \int_{[0, T) \times \Omega} \{ \partial_t [\psi(t, F_t(x)) h_0(x)] - \partial_t F_t(x) \cdot (\nabla \psi)(t, F_t(x)) h_0(x) \} dt dx \\ &= \int_{[0, T) \times \Omega} \partial_t \psi(t, x) h(t, x) dt dx. \end{aligned}$$

Since $\psi(T, x) \equiv 0$ and $F(0, x) = x$ for all $x \in \Omega$ and using that $h \in C([0, T); W^{1,r}(\Omega))$ by (3.25), we get

$$\begin{aligned} & - \int_{\Omega} \partial_t \psi(0, x) h_0(x) dx - \int_{[0, T) \times \Omega} \partial_t F(t, x) \cdot (\nabla \psi)(t, F_t(x)) h_0(x) dt dx \\ &= \int_{[0, T) \times \Omega} \partial_t \psi(t, x) h(t, x) dt dx. \end{aligned}$$

Now, making change of variables $y = F_t(x)$ in the second integral in the left-hand side using $F_t \# h_0 = h_t$, we rewrite this integral as $\int_{[0, T) \times \Omega} (\partial_t F)(t, F_t^*(x)) \cdot \nabla \psi(t, x) h(t, x) dt dx$.

Recalling the definition (3.26) of v , we get (3.18).

To show (3.17), we make change of variables $y = F_t(x)$ in (3.24), and get using $F_t \# h_0 = h_t$

$$\begin{aligned} & \int_{(0, T) \times \Omega} [\nabla P(t, y) \cdot \partial_t \phi(t, F_t^*(y)) + J(\nabla P(t, y) - y) \phi(t, F_t^*(y))] h(t, y) dt dy \\ &+ \int_{\Omega} \nabla P_0(x) \cdot \phi(0, x) h_0(x) dx = 0, \end{aligned}$$

for all $\phi \in C_c^1([0, T) \times \Omega; \mathbf{R}^2)$. Now, similar to the proof of Lemma 2.6(i), we show that this equality holds for all ϕ of the form $\phi(t, x) = \eta(t, F_t(x))$, where $\eta \in C_c^1([0, T) \times \Omega)$, and this, with use of the property (iii) in Definition 3.4, implies (3.17). $\square$

The main result of this section is

Theorem 3.1. *Let $\Omega \subset \mathbf{R}^2$ be an open bounded set and $\bar{\Omega} \subset B$ where B is an open ball $B(0, S)$. Let $h_0(x) \geq 0$ be such that $P_0(x) = h_0(x) + \frac{1}{2}|x|^2$ is a convex bounded function in B , and that*

$$(3.27) \quad DP_0 \# h_0 \in L^q(\nabla P_0(\Omega)) \quad \text{for some } q > 1.$$

Then for any $T > 0$ there exists a weak Lagrangian solution (P, F) of (3.5)–(3.10) in $[0, T) \times \Omega$, where the properties (3.19) are satisfied for any $r \in [1, \infty)$. Moreover, the function $Z(t, x)$ defined by (3.22) satisfies $Z(\cdot, x) \in W^{1, \infty}([0, T]; \mathbf{R}^2)$ for $h_0 \mathcal{L}^2$ -a.e. $x \in \Omega$, and (3.23) is satisfied, in addition to the weak form (3.24), in the following sense:

$$(3.28) \quad \begin{aligned} \partial_t Z(t, x) &= J(Z(t, x) - F(t, x)) && \text{for } h_0 \mathcal{L}^2 \times \mathcal{L}^1\text{-a.e. in } (t, x) \in, \\ Z(0, x) &= \nabla P_0(x) && \text{for } h_0 \mathcal{L}^2\text{-a.e. in } x \in \Omega. \end{aligned}$$

The proof of this theorem follows closely the proof of Theorem 2.2. Indeed, since in the present case we have $U \in BV(\mathbf{R}^2)$ and $\operatorname{div} U(t, \cdot) = 0$ in the sense of distributions by (3.13), we can repeat the argument of Section 2.3 to define Lagrangian flow $\Phi_t : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ in the dual space, with same properties as Φ_t in Section 2.3. Then we define $F_t : \Omega \rightarrow \Omega$ by (2.71), and repeat the argument of Section 2.4 with some obvious changes which come from the fact that the property $\nabla P_t \# \chi_\Omega = \alpha_t$ is now replaced by $\nabla P_t \# h_t = \alpha_t$.

APPENDIX A. SOME PROPERTIES OF MEASURE-PRESERVING MAPS

Definition A.1. Let $\Psi : \mathbf{R}^m \rightarrow \mathbf{R}^n$ be a Borel map.

- (i) Let μ, ν be Radon measures on $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. The map Ψ pushes μ forward to ν , denoted $\Psi \# \mu = \nu$, is $\mu[\Psi^{-1}(A)] = \nu[A]$ for all Borel $A \subset \mathbf{R}^n$.
- (ii) Let $f \in L^1(\mathbf{R}^m)$ and $g \in L^1(\mathbf{R}^n)$. The map Ψ pushes f forward to g , denoted $\Psi \# f = g$, if Ψ pushes the measure $f \mathcal{L}^m$ forward to $g \mathcal{L}^n$.

Lemma A.2. Let $\Psi : \mathbf{R}^m \rightarrow \mathbf{R}^n$ be a Borel map. Let μ, ν be Radon measures on $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Then $\nu = \Psi \# \mu$ is and only if

$$\int_{\mathbf{R}^m} \phi(\Psi(x)) d\mu(x) = \int_{\mathbf{R}^n} \phi(y) d\nu(y)$$

for all ν -integrable functions ϕ on $\mathbf{R}^n$.

Proof. Since every ν -measurable function ϕ on $\mathbf{R}^n$ can be represented as

$$f = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}$$

for some ν -measurable sets A_k in $\mathbf{R}^n$ [9, 1.1.2, Th. 7], and since μ and ν are Radon measures, Lemma A.2 follows directly from Definition A.1. $\square$

Corollary A.3. Let Ψ, μ, ν be as in Lemma A.2. Assume also that $\nu = g \mathcal{L}^n$ where $g \in L^1(\mathbf{R}^n)$. If $\nu = \Psi \# \mu$, then

$$(A.1) \quad \int_{\mathbf{R}^m} \phi(\Psi(x)) d\mu(x) = \int_{\mathbf{R}^n} \phi(y) g(y) dy \quad \text{for all } \phi \in L^\infty(\mathbf{R}^n).$$

In particular, if $f \in L^1(\mathbf{R}^m)$, $g \in L^1(\mathbf{R}^n)$, and $\Psi \# f = g$, then

$$(A.2) \quad \int_{\mathbf{R}^m} \phi(\Psi(x)) f(x) dx = \int_{\mathbf{R}^n} \phi(y) g(y) dy \quad \text{for all } \phi \in L^\infty(\mathbf{R}^n).$$

Proof. If $\nu = g \mathcal{L}^n$ where $g \in L^1(\mathbf{R}^n)$, then every $\phi \in L^\infty(\mathbf{R}^n)$ is ν -integrable. Now we apply Lemma A.2 to get (A.1). Finally, (A.2) follows from (A.1). $\square$

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REFERENCES

- [1] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. Preprint, 2003.
- [2] J.-D. Benamou, Y. Brenier. Weak existence for the semigeostrophic equations formulated as a coupled Monge-Ampere/transport problem. *SIAM J. Appl. Math.*, 58 (1998), no. 5, 1450–1461
- [3] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* 44 (1991), no. 4, 375–417.
- [4] M. Cullen. On the accuracy of the semi-geostrophic approximation. *Quart. J. Roy. Meteorol. Soc.*, 126 (2000), 1099–1115.
- [5] M. Cullen, W. Gangbo. A variational approach for the 2-dimensional semi-geostrophic shallow water equations. *Arch. Rational Mech. Anal*, 156 (2001), 241–273.
- [6] M. Cullen, H. Maroofi. The fully compressible semi-geostrophic system from meteorology. *Arch. Ration. Mech. Anal*, 167 (2003), no. 4, 309–336.
- [7] M. Cullen, R.J. Purser. An extended Lagrangian theory of semi-geostrophic frontogenesis. *J. Atmos. Sci.*, 41 (1984), 1477–1497
- [8] R.J. DiPerna, P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98 (1989), no. 3, 511–547.
- [9] L.C. Evans, R.F. Gariepy. Measure theory and fine properties of functions. *Studies in Advanced Mathematics*. CRC Press, Boca Raton, FL, 1992.
- [10] H. Federer. *Geometric Measure Theory*. Springer-Verlag, New York, 1969.
- [11] B.J. Hoskins. The geostrophic momentum approximation and the semi-geostrophic equations. *J. Atmos. Sci.*, 32 (1975), 233–242.
- [12] G.J. Shutts, M.Cullen. Parcel stability and its relation to semi-geostrophic theory. *J. Atmos. Sci.*, 44 (1987), 1318–1330.

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