6.6 #7

Let $u \in H^1(\mathbb{R}^n)$ have a compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $f \in L^2(\mathbb{R}^n)$, and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.
Hint. Mimic the proof of interior regularity Theorem, but without the cutoff function $\zeta(\cdot)$.

Solution. We use notation of Sect. 6.3.1, Th.1. Also, let $R > 0$ be such that

$$\operatorname{supp}(u) \subset B_R(0). \tag{2}$$

Define weak solution as following: $u \in H^1(\mathbb{R}^n)$ is a weak solution of (1) if

$$\int_{\mathbb{R}^n} Du \cdot Dv + c(u)v\, dx = \int_{\mathbb{R}^n} fv\, dx \tag{3}$$

for all $v \in H^1(\mathbb{R}^n)$.

Remark. *Strictly speaking, for the definition of weak solution given above, we need an extra assumption on $c(\cdot)$ to assure that that $c(u)v \in L^1_{loc}(\mathbb{R}^n)$ (the last inclusion would be sufficient for (3) since u has compact support and $c(0) = 0$, i.e. integration in the term $c(u)v$ can be restricted to $B := B_R(0)$, where R is from (2)). Example $c(t) = e^{t^2} - 1$ shows that some assumption might be needed.*

From Sobolev inequalities $u, v \in L^{\frac{2n}{n-2}}(B)$ if $n > 2$, and $u, v \in L^p(B)$ for any $p \in [1, \infty)$ if $n = 2$. Thus we need to have $c(u) \in L^{\frac{2n}{n+2}}(B)$ if $n > 2$, and $c(u) \in L^p(B)$ for some $p \in (1, \infty)$ if $n = 2$. Then it is sufficient to assume that $|c(t)| \leq Ct^{\frac{n+2}{n-2}}$ if $n > 2$, and $|c(t)| \leq Ct^M$ for some $M \geq 0$ if $n = 2$.

Note also, that, at least for $n > 2$, the above assumptions do not imply that $c(u) \in L^2(\mathbb{R}^n)$, i.e. the assertion in the problem does not follow from the regularity results for linear equations by considering $-\Delta u = g$ where $g = f - c(u)$.

We choose $h \neq 0$ with $|h| \leq 1$, $k \in \{1, \dots, n\}$

$$v = -D_k^{-h} D_k^h u.$$

Then $v \in H^1(\mathbb{R}^n)$ and has compact support. Using this v in (3), and using integration by parts and the "difference quotients integration by parts" formula, and using that u has compact support, get

$$\int_{\mathbb{R}^n} \left(|D_k^h Du|^2 + D_k^h(c(u)) D_k^h u \right) dx = -\int_{\mathbb{R}^n} f \left(D_k^{-h} D_k^h u \right) dx. \tag{4}$$

We calculate:

$$(D_k^h(c(u)))(x) = \frac{c(u(x + he_k)) - c(u(x))}{h} = \eta_h(x) \frac{u(x + he_k) - u(x)}{h} = \eta_h(x) D_k^h u(x),$$

where

$$\eta_h(x) = \int_0^1 c'(ta + (1-t)b)\, dt, \quad \text{where } \ a = u(x + he_k),\ b = u(x).$$

Thus $c' \geq 0$ implies $\eta_h(x) \geq 0$, and we get

$$D_k^h(c(u))D_k^h u = \eta_h(x)(D_k^h u(x))^2 \geq 0. \tag{5}$$

Also, using that $0 < |h| \leq 1$ and applying Theorem 3 from Sect. 5.8.2 in the region $B_{R+1}(0)$

$$\left| \int_{\mathbb{R}^n} f\Big(D_k^{-h} D_k^h u)\Big) dx \right| = \left| \int_{B_{R+1}(0)} f\Big(D_k^{-h} D_k^h u)\Big) dx \right| \leq \varepsilon \int_{\mathbb{R}^n} |D\, D_k^h u|^2 dx + \frac{C}{\varepsilon} \int_{\mathbb{R}^n} f^2 dx. \tag{6}$$

From (4), (5), (6)

$$\int_{\mathbb{R}^n} |D_k^h Du|^2 \leq \varepsilon \int_{\mathbb{R}^n} |D_k^h Du|^2 dx + \frac{C}{\varepsilon} \int_{\mathbb{R}^n} f^2 dx.$$

Choosing $\varepsilon = \frac{1}{2}$, get

$$\int_{\mathbb{R}^n} |D_k^h Du|^2 \leq C \int_{\mathbb{R}^n} f^2 dx.$$

This is true for each $k \in \{1, \ldots, n\}$, $0 < |h| < 1$. Thus, applying Theorem 3 from Sect. 5.8.2 in the region $B_{R+1}(0)$, get

$$\int_{B_{R+1}(0)} |D^2 u|^2 \leq C \int_{\mathbb{R}^n} f^2 dx,$$

thus

$$\int_{\mathbb{R}^n} |D^2 u|^2 = \int_{B_{R+1}(0)} |D^2 u|^2 \leq C \int_{\mathbb{R}^n} f^2 dx.$$

Combining this with $u \in H^1(\mathbb{R}^n)$, get $u \in H^2(\mathbb{R}^n)$.