2.5 #19

(a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using the change of variables $\xi = x + t$, $\eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a) and (b) to rederive d'Alembert's formula.

Solution.

(a) First, any function $u(x, y)$ of the form $u(x, y) = F(x) + G(y)$ clearly satisfies $u_{xy} = 0$.

Now suppose $u_{xy} = 0$. Denote $v = u_x$. Then $u_{xy} = 0$ implies $v_y = 0$, thus

$$u_x(x, y) = v(x, y) = f(x)$$

for some function f. Let F be such that $F'(x) = f(x)$. Then integrating with respect to x we get

$$u(x, y) = F(x) + G(y)$$

for some function G.

(b) From the definition of ξ, η, we get $u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = u_\xi + u_\eta$, and similarly $u_t = u_\xi \frac{\partial \xi}{\partial t} + u_\eta \frac{\partial \eta}{\partial t} = u_\xi - u_\eta$. Differentiating again, get

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \quad u_{tt} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.$$

Thus

$$u_{tt} - u_{xx} = -4u_{\xi\eta},$$

and (b) follows.

(c) If u satisfies $u_{tt} - u_{xx} = 0$, then from part (b), get $u_{\xi\eta} = 0$. Thus by part (a),

$$u = F(\xi) + G(\eta) = F(x + t) + G(x - t).$$

This implies

$$u_t = F'(x + t) - G'(x - t).$$

If

$$u = g, \; u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\},$$

get

$$g(x) = u(x, 0) = F(x) + G(x), \quad h(x) = u_t(x, 0) = F'(x) - G'(x).$$

Thus, if

$$H(x) := F(x) - G(x),$$

then

$$H'(x) = h(x).$$

From the above equalities, we find:

$$F(x) = \frac{1}{2}(g(x) + H(x)), \qquad G(x) = \frac{1}{2}(g(x) - H(x))$$

Thus

$$\begin{aligned}
u(x,t) &= F(x+t) + G(x-t) \\
&= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}(H(x+t) - H(x-t)) \\
&= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.
\end{aligned}$$