2.5 #15

Given $g : [0,\infty) \to \mathbb{R}$ with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)ds$$

for a solution of the initial-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t=0\}, \\ u = g & \text{on } \{x=0\} \times [0,\infty). \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection $v(x,t) := -v(-x,t)$ for $x < 0$)

Solution All foregoing calculations are formal. Rigorously, after the formula is derived, we need to check if it gives a solution.

Denote by $w(x,t)$ the odd extension of $v(x,t)$. Note that $w(\cdot,t)$ is continuous at $x = 0$ for every $t > 0$, and thus, using the definition of odd reflection, $w(\cdot,t)$ is continuously differentiable across $x = 0$. Then (heuristically, formally), $w(x,t)$ satisfies:

$$\begin{cases} w_t - w_{xx} = -\text{sign}(x) g'(t) & \text{in } \mathbb{R} \times (0,\infty), \\ u = 0 & \text{on } \mathbb{R} \times \{t=0\}, \end{cases}$$

where

$$\text{sign}(z) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ -1 & \text{if } z < 0. \end{cases}$$

Thus, using formula (13) of sect. 2.3, we find a solution in the form

$$w(x,t) = -\int_0^t \frac{1}{(4\pi(t-s))^{1/2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}} \text{sign}(y) g'(s)\, dyds.$$

Note that for $x > 0$, $t > s > 0$:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}} \text{sign}(y)\, dy &= \int_0^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}}\, dy - \int_{-\infty}^0 e^{-\frac{(x-y)^2}{4(t-s)}}\, dy \\ &= \int_{-x}^{x} e^{-\frac{y^2}{4(t-s)}}\, dy \\ &= 2\int_0^x e^{-\frac{y^2}{4(t-s)}}\, dy \\ &= 2\sqrt{4(t-s)} \int_0^{\frac{x}{\sqrt{4(t-s)}}} e^{-z^2} dz, \end{aligned}$$

where we changed variables by $z = \frac{y}{\sqrt{4(t-s)}}$ in the last line. Thus we get for $x > 0,\ t > 0$:

$$w(x,t) = -\frac{2}{(\pi)^{1/2}} \int_0^t g'(s) \int_0^{\frac{x}{\sqrt{4(t-s)}}} e^{-z^2} dz ds.$$

Integrating by parts in s, and using that

$$\frac{d}{ds} \int_0^{\frac{x}{\sqrt{4(t-s)}}} e^{-z^2} dz = \frac{x}{2\sqrt{4}(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}},$$

we get:

$$w(x,t) = -\frac{2}{(\pi)^{1/2}} g(t) \int_0^{\infty} e^{-z^2} dz + \frac{1}{(\pi)^{1/2}} \int_0^t \frac{x}{\sqrt{4}(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds.$$

Using that $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$ and thus $\int_0^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2}$, we get that, for $x > 0,\ t > 0$

$$v(x,t) = w(x,t) = -g(t) + \frac{x}{(4\pi)^{1/2}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds.$$

Thus, $u(x,t) = g(t) + v(x,t)$ is

$$u(x,t) = \frac{x}{(4\pi)^{1/2}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds.$$