2.5 #12

Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(i) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ solves the heat equation for each $\lambda \in \mathbb{R}$.

(ii) Use (i) to show that $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

Solution

(i) We compute:

$$\partial_t u_\lambda(x,t) = \lambda^2 \partial_t u(\lambda x, \lambda^2 t), \qquad \partial_{x_i x_i} u_\lambda(x,t) = \lambda^2 \partial_{x_i x_i} u(\lambda x, \lambda^2 t),$$

thus

$$(\partial_t - \Delta) u_\lambda(x,t) = \lambda^2 (\partial_t - \Delta) u(\lambda x, \lambda^2 t) = 0.$$

(ii) Since

$$(\partial_t - \Delta) u_\lambda(x,t) = 0 \quad \text{for all} \;\; (x,t,\lambda) \in \mathbb{R}^n \times (0,\infty) \times \mathbb{R}$$

by (i), we differentiate with respect to λ, thus get

$$(\partial_t - \Delta)(\partial_\lambda u_\lambda)(x,t) = 0 \qquad \text{for all} \;\; (x,t,\lambda) \in \mathbb{R}^n \times (0,\infty) \times \mathbb{R}.$$

We compute:

$$(\partial_\lambda u_\lambda)(x,t) = \partial_\lambda [u(\lambda x, \lambda^2 t)] = x \cdot Du(\lambda x, \lambda^2 t) + 2\lambda t u_t(\lambda x, \lambda^2 t).$$

Thus, for each fixed $\lambda \in \mathbb{R}$, the function $v_\lambda(x,t) = x \cdot Du(\lambda x, \lambda^2 t) + 2\lambda t u_t(\lambda x, \lambda^2 t)$ solves the heat equation. Choosing $\lambda = 1$ we get assertion (ii).