2.5 #9

Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \le 1$. Show Du is *not* bounded near $x = 0$.
Hint: Estimate $\frac{u(\lambda e_n)-u(0)}{\lambda}$.

Solution

We first show that it is sufficient to prove that

$$\left|\frac{u(\lambda e_n) - u(0)}{\lambda}\right| \to \infty \quad \text{as } \lambda \to 0+ . \tag{1}$$

Indeed, if $|Du(x)| \le M$ for all $x \in \mathbb{R}^n_+ \cap B_r(0)$ for some $r > 0$, then, for any $\lambda, \epsilon \in (0, r)$ with $\lambda > \epsilon$, using $u \in C^\infty(\mathbb{R}^n_+)$, we get

$$|u(\lambda e_n) - u(\epsilon e_n)| = \left|\int_\epsilon^\lambda u_{x_n}(se_n)ds\right| \le M(\lambda - \epsilon). \tag{2}$$

Since $g(x)$ is continuous for $|x| < 1$, then by properties of Poisson integral, Theorem 14(iii) of Sect. 2.2, we get $u(\epsilon e_n) \to u(0)$ as $\epsilon \to 0+$. Thus, sending $\epsilon \to 0+$ in (2) we contradict (1). Thus Du is not bounded near $x = 0$ if (1) holds.

It remains to prove (1). Using again Theorem 14(iii) of Sect. 2.2, get $u(0) = g(0) = 0$. Thus, using $|\lambda e_n - y|^2 = \lambda^2 + |y|^2$ for $y \in \mathbb{R}^{n-1}$, we get from Poisson's formula

$$\frac{u(\lambda e_n) - u(0)}{\lambda} = \frac{u(\lambda e_n)}{\lambda} = \frac{2\lambda}{n\alpha(n)\lambda}\int_{\mathbb{R}^{n-1}} \frac{g(y)\,dy}{(\lambda^2 + |y|^2)^{n/2}} = \frac{2}{n\alpha(n)}\int_{\mathbb{R}^{n-1}} \frac{g(y)\,dy}{(\lambda^2 + |y|^2)^{n/2}} = I_1 + I_2,$$

where I_1 and I_2 are the integrals over $B(0, 1)$ and $\mathbb{R}^{n-1} \setminus B(0, 1)$ respectively. We estimate I_1, I_2 for $\lambda \in (0, 1)$.

First estimate I_1. Using $g(y) = |y|$ on $B(0, 1)$, and changing variables by $y = \lambda z$, we have

$$\begin{aligned} I_1 &= \frac{2}{n\alpha(n)}\int_{B(0,1)} \frac{|y|\,dy}{(\lambda^2 + |y|^2)^{n/2}} = \frac{2}{n\alpha(n)}\int_{B(0,1/\lambda)} \frac{\lambda|z|\,\lambda^{n-1}dz}{\lambda^n(1 + |z|^2)^{n/2}} \\ &= \frac{2}{n\alpha(n)}\int_{B(0,1/\lambda)} \frac{|z|\,dz}{(1 + |z|^2)^{n/2}} \ge \frac{1}{2^{\frac{n}{2}-1}n\alpha(n)}\int_{1<|z|<\frac{1}{\lambda}} \frac{dz}{|z|^{n-1}}, \end{aligned}$$

where, to get the last inequality, we use that the integrand is positive, and first restrict to the domain $B(0, 1/\lambda) \setminus B(0, 1)$, and then use that $2|z|^2 > 1 + |z|^2$ for $|z| > 1$. Using polar coordinates to compute the last integral, obtain

$$\frac{u(\lambda e_n) - u(0)}{\lambda} \ge C(n)\int_1^{1/\lambda} \frac{r^{n-2}}{r^{n-1}}dr = -C(n)\log\lambda \to +\infty \quad \text{as } \lambda \to 0+ .$$

Now estimate I_2. Since $g(y)$ is bounded, say $g|y| \le L$ on $\mathbb{R}^{n-1}$, we have

$$|I_2| \le L\int_{\mathbb{R}^{n-1}\setminus B(0,1)} \frac{dy}{(\lambda^2 + |y|^2)^{n/2}} \le L\int_{\mathbb{R}^{n-1}\setminus B(0,1)} \frac{dy}{|y|^n} = LC(n)\int_1^\infty \frac{r^{n-2}}{r^n}dr = LC(n).$$

Thus $I_1 + I_2 \to +\infty$ as $\lambda \to 0+$, and (1) is proved.