2.5 #5

We say $v \in C^2(\overline{U})$ is *subharmonic* if

$$-\Delta v \le 0 \qquad \text{in } U.$$

(a) Prove for subharmonic v that

$$v(x) \le \fint_{B(x,r)} v \, dy \qquad \text{for all } B(x,r) \subset U.$$

(b) Prove that $\max_{\overline{U}} v = \max_{\partial U} v$.
(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic, $v := \phi(u)$. Prove v is subharmonic.
(d) Prove $v := |Du|^2$ is subharmonic whenever u is harmonic.

Solution

(a) Let

$$\phi(r) = \fint_{\partial B(x,r)} v \, dS(y) = \fint_{\partial B(0,1)} v(x + rz) \, dS(z),$$

where $y = x + rz$. The outer unit normal at $y \in \partial B(x,r)$ is $\nu(y) = \frac{y-x}{r}$, thus

$$\phi'(r) = \fint_{\partial B(0,1)} z \cdot Dv(x+rz) \, dS(z) = \fint_{\partial B(x,r)} \frac{y-x}{r} \cdot Dv(y) \, dS(y) = \fint_{\partial B(x,r)} \nu(y) \cdot Dv(y) \, dS(y).$$

Since $-\Delta v \le 0$ in U, we have

$$0 \le \int_{B(x,r)} \Delta v(y) dy = \int_{\partial B(x,r)} v_\nu \, dS(y),$$

thus $\phi'(r) \ge 0$ for $r > 0$. Then we get for $r > 0$

$$v(x) = \phi(0) \le \phi(r) = \fint_{\partial B(x,r)} v(y) \, dS(y).$$

Thus, using polar coordinates, we get

$$
\begin{aligned}
\fint_{B(x,r)} v \, dy &= \frac{1}{\alpha(n)r^n} \int_0^r d\rho \int_{\partial B(x,\rho)} v(y) \, dS(y) \\
&\ge \frac{1}{\alpha(n)r^n} \int_0^r n\alpha(n)\rho^{n-1} v(x) \, d\rho = \frac{v(x)}{\alpha(n)r^n} \int_0^r n\rho^{n-1} \, d\rho = v(x),
\end{aligned}
$$

thus (i) is proved.

(b) Using (a), we can repeat the proof of Theorem 4 (maximum principle) in sect. 2.2.3.

(c) We compute:

$$
\begin{aligned}
Dv &= \phi'(u)Du, \\
\Delta v &= \phi'(u)\Delta u + \phi''(u)|Du|^2 = \phi''(u)|Du|^2 \ge 0,
\end{aligned}
$$

where we used harmonicity of u and convexity of ϕ.

(d) Since u is harmonic, then u_{x_i} is harmonic, for $i = 1, \ldots, n$. Then $(u_{x_i})^2$ is subharmonic by part (c). Sum of subharmonic functions is obviously subharmonic, so $|Du|^2 = \sum_{i=1}^n (u_{x_i})^2$ is subharmonic.