6.6 #7

Let $u \in H^1(\mathbb{R}^n)$ have a compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$, and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.

Hint. Mimic the proof of interior regularity Theorem, but without the cutoff function $\zeta(\cdot)$.

Remark. *We use notation of Sect. 6.3.1, Th.1. Also, let $R > 0$ be such that*

$$\mathit{supp}(u) \subset B_R(0). \tag{1}$$

Define weak solution of the equaiton in the problem as following: $u \in H^1(\mathbb{R}^n)$ is a weak solution if

$$\int_{\mathbb{R}^n} Du \cdot Dv + c(u)v \, dx = \int_{\mathbb{R}^n} fv \, dx \tag{2}$$

for all $v \in H^1(\mathbb{R}^n)$.

Note that we need an extra assumption on $c(\cdot)$ to assure that that $c(u)v \in L^1_{loc}(\mathbb{R}^n)$ (the last inclusion would be sufficient for (2) since u has compact support and $c(0) = 0$, i.e. integration in the term $c(u)v$ can be restricted to $B := B_R(0)$, where R is from (1)). Example $c(t) = e^{t^2} - 1$ shows that some assumption might be needed.

From Sobolev inequalities $u, v \in L^{\frac{2n}{n-2}}(B)$ if $n > 2$, and $u, v \in L^p(B)$ for any $p \in [1, \infty)$ if $n = 2$. Thus we need to have $c(u) \in L^{\frac{2n}{n+2}}(B)$ if $n > 2$, and $c(u) \in L^p(B)$ for some $p \in (1, \infty)$ if $n = 2$. Then it is sufficient to assume that $|c(t)| \leq Ct^{\frac{n+2}{n-2}}$ if $n > 2$, and $|c(t)| \leq Ct^M$ for some $M \geq 0$ if $n = 2$.

Note also, that, at least for $n > 2$, the above assumptions do not imply that $c(u) \in L^2_{loc}(\mathbb{R}^n)$, i.e. the assertion in the problem does not follow from the regularity results for linear equations by considering $-\Delta u = g$ where $g = f - c(u)$.