2.5 #15

Given $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)ds$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\}, \\ u = g & \text{on } \{x = 0\} \times [0,\infty). \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection $v(x,t) := -v(-x,t)$ for $x < 0$)