Due Thursday December 14 (put in my mailbox).

1. Sect. 6.6 #9
Assume u is a smooth solution of $Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = f$ in U, $u = 0$ on ∂U, where f is bounded. Fix $x^0 \in \partial U$. A *barrier* at x^0 is a C^2 function w such that

$$Lw \geq 1 \text{ in } U, \quad w(x^0) = 0, \quad \text{and} \quad w \geq 0 \text{ on } \partial U.$$

Show that if w is a barrier at x^0, there exists a constant C such that

$$|Du(x^0)| \leq C \left| \frac{\partial w}{\partial \nu}(x^0) \right|.$$

Remark. You may assume that ∂U is smooth.

2. Sect. 3.5 #5.
Solve using characteristics:
(a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u, \quad u(x_1, 1) = g(x_1)$;
(c) $u u_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2} x_1$.

3. Sect. 3.5 #13.
Prove that Hopf-Lax formula reads

$$\begin{aligned} u(x,t) &= \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \\ &= \min_{y \in B(x,Rt)} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \end{aligned}$$

for $R = \sup_{\mathbb{R}^n} |DH(Dg)|$, $H = L^*$. (This proves *finite propagation speed* for Hamilton-Jacobi equation.)
Hint: use problem 3.5 #11. You can use the fact asserted in problem 3.5 #11 without proof.

4. Sect. 3.5 #14.
Let E be a closed subset of $\mathbb{R}^n$. Show that *if* the Hopf-Lax formula can be applied to the initial-value problem

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = \begin{cases} 0 & x \in E \\ +\infty & x \notin E \end{cases} & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

it would give the solution

$$u(x,t) = \frac{1}{4t} \operatorname{dist}(x, E)^2.$$