1. Let Ω be the half plane $\{x_2 > 0\}$ in $\mathbf{R}^2$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic:

$$\Delta u = 0 \qquad \text{in } \ \Omega.$$

Under the additional assumption that $u(\cdot)$ is bounded from above in Ω, prove that

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

Note: The additional assumption is needed to exclude examples like $u(x_1, x_2) = x_2$.
Hint: Take for $\varepsilon > 0$ the harmonic in Ω function

$$v(x_1, x_2) = u(x_1, x_2) - \varepsilon \log \sqrt{x_1^2 + (x_2 + 1)^2}.$$

Apply the maximum principle in an appropriate *bounded* region. Let $\varepsilon \to 0$.

2. Let $U \subset \mathbf{R}^n$ be open and bounded, let $U_T = U \times (0, T]$ be the parabolic cylinder, and and $\Gamma_T = \overline{U_T} \setminus U_T$ be the parabolic boundary. Let nonnegative $u \in C_1^2(U_T) \cap C(\overline{U_T})$ satisfies

$$u_t = \Delta u + cu \qquad \text{in } \ U_T,$$

where $c(x, t)$ is continuous in $\overline{U_T}$.

(a) Prove that if $c(x, t) < 0$ in $\overline{U_T}$, then

$$\max_{\overline{U_T}} u = \max_{\Gamma_T} u.$$

Hint. Show that $\max_{\overline{U_T}} u$ cannot be assumed in U_T unless $\max_{\overline{U_T}} u \le 0$. In order to do that, consider the possible signs of u_t and Δu at a point of maximum $(x_0, t_0) \in U_T$. Note that it is possible that $t_0 = T$.

(b) Show that more generally

$$\max_{\overline{U_T}} u \le e^{CT} \max_{\Gamma_T} u,$$

where

$$C = \max(0, \max_{\overline{U_T}} c)$$

Hint. Substitute $u(x, t) = e^{\gamma t} v(x, t)$, where $\gamma > C$.

3.(2.5 #24 from Evans book). Let $u \in C^2(\mathbb{R} \times [0, \infty))$ solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \ \mathbb{R} \times (0, \infty), \\ u = g, \ u_t = h & \text{on } \ \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose g, h have compact support. The *kinetic energy* is $k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx$, and the *potential energy* is $p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx$. Prove:

(a) $k(t) + p(t)$ is constant in t;

(b) $k(t) = p(t)$ at all large enough times t.

4. Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = g, \ u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where g, h are smooth and have have compact support. Show there exists constant C such that

$$|u(x,t)| \leq C/t \qquad (x \in \mathbb{R}^3, \ t > 0).$$