INTERSECTION SPACES AND HYPERSURFACE SINGULARITIES

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Abstract. We give an elementary introduction to the first author's theory of intersection spaces associated to complex projective varieties with only isolated singularities. We also survey recent results on the deformation invariance of intersection space homology in the context of projective hypersurfaces with an isolated singularity.

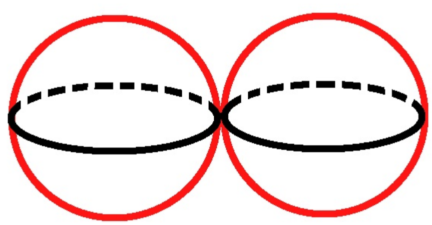
1. Introduction

Convention: By "manifold" we mean a "complex projective manifold", and by "singular space" we mean a "complex projective variety of pure complex dimension n". We are only interested in "middle-perversity" calculations, so any mentioning of other perversity functions will be ignored. Unless otherwise specified, all (intersection) (co)homology groups will be computed with rational coefficients.

Manifolds have an amazing hidden symmetry called Poincaré Duality, which ultimately is reflected in their (co)homology: ranks of (co)homology groups in complementary degrees are equal. Singular spaces, on the other hand, do not possess such symmetry. For example, consider the complex projective curve

$$X = \{(x : y : z) \in \mathbb{CP}^2 \mid xy = 0\}.$$

Then X is a union of two projective lines $\mathbb{CP}^1$ meeting at the point $(0 : 0 : 1)$. Topologically, X is just $S^2 \vee S^2$, a wedge of two 2-spheres.



As $H_0(X) = \mathbb{Q}$ and $H_2(X) = \mathbb{Q} \oplus \mathbb{Q}$, it follows that $\mathrm{rk} H_0(X) \neq \mathrm{rk} H_2(X)$.

Date: April 13, 2012.

2000 *Mathematics Subject Classification.* 32S30, 32S55, 55N33, 57P10, 32S35, 14J33.

Key words and phrases. Singularities, projective hypersurfaces, smooth deformations, Poincaré duality, intersection homology, Milnor fibration, mixed Hodge structures, mirror symmetry.

The first author was in part supported by a research grant of the Deutsche Forschungsgemeinschaft. The second author was partially supported by NSF-1005338.

Classically, much of the manifold theory, e.g., Morse theory, Lefschetz theorems, Hodge decompositions, and especially Poincaré Duality, is recovered in the singular context if, instead of the usual (co)homology, one uses Goresky-MacPherson's intersection homology groups $IH_*(X)$, see [GM80, GM83]. These are homology groups of a complex of "allowed chains", defined by imposing restrictions on how chains meet the singular strata. For example, if X is a complex projective variety of complex dimension n with only isolated singularities, then $IH_i(X)$ is the i-th homology group of the chain complex $(IC_*(X), \partial)$ defined by the following allowability conditions: if ξ is a PL i-chain on X with support $|\xi|$ (in a sufficiently fine triangulation of X compatible with the natural stratification $X \supset \mathrm{Sing}(X)$), then $\xi \in IC_i(X)$ if, and only if,

$$\dim_{\mathbb{R}} |\xi| \cap \mathrm{Sing}(X) < i - n$$

and

$$\dim_{\mathbb{R}} |\partial\xi| \cap \mathrm{Sing}(X) < i - 1 - n,$$

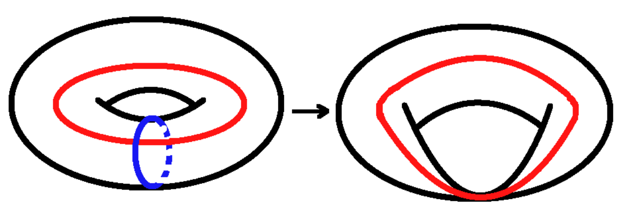
and with boundary operator induced from the usual boundary operator on chains of X. So low-dimensional allowable chains cannot meet the singularities of X.

It is known that the intersection (co)homology groups of a singular space coincide with the usual (co)homology groups of a "small" resolution (provided such a resolution exists), e.g., see [GM83]. This precludes the existence in general of a cup product (i.e., ring structure) on intersection cohomology: indeed, there are spaces having two small resolutions which have non-isomorphic cohomology rings. Intersection (co)homology is not a homotopy invariant (e.g., the intersection homology of a cone is not trivial, see [KW06, Section 4.7]), and is lacking functoriality in general (e.g., see [KW06, Section 4.8]). And, much like the usual homology theory, it is also rather unstable under deformation of singularities. We shall illustrate this last assertion by an example.

Example 1.1. Consider the equation

$$y^2 = x(x-1)(x-s)$$

or its homogeneous version $v^2w = u(u-w)(u-sw)$, defining a curve in $\mathbb{CP}^2$, where the complex parameter s is constrained to lie inside the unit disc, $|s| < 1$. For $s \neq 0$, the equation defines an elliptic curve V_s, homeomorphic to a 2-torus T^2. The curve $V := V_0$ corresponding to $s = 0$, has a nodal singularity. Thus V is homeomorphic to a pinched torus, that is, T^2 with a meridian collapsed to a point, or, equivalently, a cylinder $I \times S^1$ with coned-off boundary, where $I = [0,1]$.



The ordinary homology group $H_1(V)$ has rank one, generated by the longitudinal circle (while the meridian circle bounds the cone with vertex at the singular point of

V). The intersection homology group $IH_1(V)$ agrees with the intersection homology of the normalization S^2 of V (the longitude in V is not an "allowed" 1-cycle, while the meridian bounds an allowed 2-chain), so:

$$IH_1(V) = IH_1(S^2) = H_1(S^2) = 0.$$

Thus, as $H_1(V_s) = H_1(T^2)$ has rank 2, neither ordinary homology nor intersection homology remains invariant under the smoothing deformation $V \rightsquigarrow V_s$.

This raises the following questions: *Is there a homology theory for singular spaces, possessing Poincaré Duality, ring structure, homotopy invariance, functoriality, and which is also stable under deformations? Does this theory carry a Kähler package (i.e., Hodge decompositions, Hodge star duality, Lefschetz-type theorems; see* [H92, Section 7.3]*)?*

Parts of this question were answered positively by the first author in [Ba10]. Aspects of deformation invariance and Hodge theory were considered in [BM11]. The aim of this note is to present a quick account of these developments.

2. Intersection Spaces

The first author's approach [Ba10] consists of a homotopy-theoretic method which associates to a certain singular space X a CW complex IX, called the "intersection space of X". Moreover, as shown in [Ba10], IX is a rational Poincaré complex, i.e., its (reduced) homology groups satisfy Poincaré Duality over the rationals. One thing to notice immediately is that the "intersection space cohomology" $\widetilde{H}^*(IX)$ has an internal ring structure defined by the usual cup product in cohomology, so the cohomology of the intersection space is not generally isomorphic to the intersection cohomology of the space itself. Moreover, this theory has a DeRham interpretation (see [Ba11a]) and, as we will try to convince the reader, it is more stable under deformations of singularities.

Roughly speaking, the intersection space IX associated to a singular space X is defined by replacing links of singularities by their corresponding Moore approximations, a process the first author termed "spatial homology truncation". Let us say a few words about Moore approximations. Let L be a simply-connected CW complex, and fix an integer n. The Moore approximation construction guarantees the existence of a CW complex $L_{<n}$ together with a structural map $f : L_{<n} \to L$, so that $f_* : H_r(L_{<n}) \to H_r(L)$ is an isomorphism if $r < n$, and $H_r(L_{<n}) \cong 0$ for all $r \geq n$. (Moreover, these isomorphisms hold over the integers.) In more detail, let $C_*(L)$ be the cellular chain complex of L, and let $Z_n(L)$ denote the n-cycles. Suppose first that the following assumption holds: $Z_n(L)$ has a basis $\langle z_\alpha \rangle$ consisting of n-cells. Let $\langle y_\beta \rangle$ be the remaining n-cells, i.e., those n-cells that are not cycles. The y_β's generate a subgroup $Y \subset C_n(L)$ such that

$$C_n(L) = Z_n(L) \oplus Y,$$

and the n-skeleton of L can then be written as

$$L^{(n)} = L^{(n-1)} \cup \langle z_\alpha \rangle \cup \langle y_\beta \rangle.$$

We define:

$$L_{<n} := L^{(n-1)} \cup \langle y_\beta \rangle \overset{f=incl}{\hookrightarrow} L.$$

In the general case, any simply-connected CW complex L is homotopy equivalent relative to the $(n-1)$-skeleton to a CW complex L' which satisfies the above technical assumption. Then we set:

$$L_{<n} := (L')_{<n} \hookrightarrow L' \stackrel{h.e.}{\to} L,$$

i.e., the structural map $f: L_{<n} \to L$ is the composition of the inclusion followed by the homotopy equivalence. The process often works in the non-simply-connected case by ad-hoc considerations. For example, if $\partial_n = 0$, i.e., $C_n(L) = Z_n(L)$, we can choose $L_{<n} := L^{(n-1)}$ with structural map given by the inclusion map. In particular, if L is path-connected, then $L_{<1}$ is just a point. Note also that if $n > \dim L$, then $L_{<n} = L$.

Let us now discuss the construction of the intersection space in a very simple situation, namely when the variety X has only an isolated singularity x. Let L be the link of the singular point, and M the manifold with boundary $\partial M = L$ obtained from X by removing a small open conical neighborhood of x. Topologically,

$$X = M \cup_L cone(L).$$

Choose a CW structure on L, and let $g: L_{<n} \stackrel{f}{\to} L = \partial M \hookrightarrow M$ be the composition of inclusion of the boundary followed by the structural map of the Moore approximation of L at level $n = \dim_{\mathbb{C}} X$. The intersection space IX is then defined as the mapping cone of g, that is:

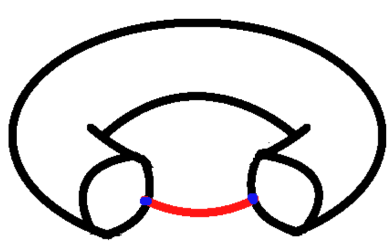
$$IX := cone(g) := M \cup_g cone(L_{<n}).$$

Therefore, in the case when the structural map of the Moore approximation of L is an inclusion, the intersection space IX is obtained from M by coning off a certain subset of the link.

If there are several isolated singularities, the intersection space is defined by performing spatial homology truncation on each of the links, simultaneously.

Note that the construction of the intersection space IX involves choices of subgroups $Y_i \subset C_n(L_i)$, where the L_i are the links of the singularities. Moreover, the chain complexes $C_*(L_i)$ depend on the CW structures on the links. However, as shown in [Ba10, Theorem 2.18], the rational homology of IX is well-defined and independent of all choices.

Example 2.1. In the case of the nodal curve V of Example 1.1, the link of the singular point is $\partial I \times S^1$, two circles. The intersection space IV of V is a cylinder $I \times S^1$ together with an interval, whose one endpoint is attached to a point in $\{0\} \times S^1$ and whose other endpoint is attached to a point in $\{1\} \times S^1$.



Thus IV is homotopy equivalent to the figure eight and

$$H_1(IV) = \mathbb{Q} \oplus \mathbb{Q},$$

which does agree with $H_1(V_s)$.

Remark 2.2. As suggested by Example 2.1, the middle homology of the intersection space IX takes into account more cycles than the corresponding intersection homology group of X. More precisely, for X of complex dimension n with only isolated singularities, $IH_n(X)$ is generally smaller than both $H_n(X - Sing(X))$ and $H_n(X)$, being a quotient of the former and a subgroup of the latter, while $H_n(IX)$, is generally bigger than both $H_n(X - Sing(X))$ and $H_n(X)$, containing the former as a subgroup and mapping to the latter surjectively, see [Ba10].

3. String Theory and Mirror Symmetry

The homology of intersection spaces addresses certain questions in type II string theory. Let us give a quick account of how this story unfolds.

In addition to the four dimensions that model our space-time classically, (super)string theory requires six dimensions for a string to vibrate. Supersymmetry considerations force these six (real) dimensions to be a Calabi-Yau space (i.e., a compact complex Kähler manifold with trivial first Chern class). However, given the multitude of known topologically distinct Calabi-Yau threefolds, the (super)string model remains undetermined. It is therefore important to have mechanisms that allow one to move from one Calabi-Yau space to another. Topologically speaking, since any two closed oriented (real) six-manifolds are bordant, and bordisms are obtained by performing a finite number of surgeries, surgery seems to be a good way to travel from one Calabi-Yau manifold to another. In Physics, a solution to this problem was first proposed by Green-Hübsch [GH1, GH2] who, motivated by Reid's fantasy [Re87], conjectured that topologically distinct Calabi-Yau's could be connected to each other by means of "conifold transitions", which should induce a phase transition between the corresponding (super)string models.

A conifold transition starts out with a nonsingular Calabi-Yau threefold, passes through a singular variety – the conifold – by a deformation of complex structure, and arrives at a topologically distinct nonsingular Calabi-Yau threefold by a small resolution of singularities. The deformation collapses embedded three-spheres (the "vanishing cycles") to isolated ordinary double points, while the resolution resolves the singular points by replacing each of them with a $\mathbb{CP}^1$. A conifold transition can be described locally by means of surgeries on the vanishing cycles (see [Cle83, Ro06]). In Physics, the topological change was interpreted by Strominger as the condensation of massive black holes to massless ones. It is then desirable to record these massless particles as classes in good (co)homology theories. In type IIA string theory, there are charged two-branes that wrap around the $\mathbb{CP}^1$ 2-cycles, and which become massless when these 2-cycles are collapsed to points by the resolution map. As intersection homology is invariant under small resolutions, the intersection homology of the conifold accounts for all of these massless two-branes, so it is the physically correct homology theory for type IIA string theory. Similarly, in type IIB string theory there are charged three-branes wrapped around the vanishing cycles, and which become massless as these vanishing cycles are collapsed by the deformation of complex structure. Neither ordinary homology nor intersection homology of the conifold account for these massless three-branes, but the homology of the intersection space of the conifold yields the correct count. So it appears that the homology of intersection spaces is the physically correct homology theory in the IIB string theory; see [Ba10, Section 3] for more details.

In relation to mirror symmetry, given a Calabi-Yau threefold X, the mirror map associates to it another Calabi-Yau threefold X° so that type IIB string theory on $\mathbb{R}^4 \times X$ corresponds to type IIA string theory on $\mathbb{R}^4 \times X^\circ$. If X and X° are nonsingular, their homology Betti numbers are related by precise algebraic identities, e.g., $\beta_3(X^\circ) = \beta_2(X) + \beta_4(X) + 2$, etc. A conjecture of Morrison [Mor99] asserts that the mirror of a conifold transition is again a conifold transition, but performed in the reverse order (so, mirror symmetry is supposed to exchange resolutions and deformations). Thus, if in the above discussion, X and X° are mirrored conifolds (in mirrored conifold transitions), the intersection space homology of one space and the intersection homology of the mirror space form a mirror-pair, in the sense that

$$\beta_3(IX^\circ) = I\beta_2(X) + I\beta_4(X) + 2, \tag{1}$$

etc., where $I\beta_i$ denotes the i-th intersection homology Betti number (see [Ba10] for details).

The above mirror symmetry considerations suggest that one could expect to be able to compute the intersection space homology $H_*(IX)$ of a variety X in terms of the topology of a smoothing family X_s, by "mirroring" known results relating the intersection homology groups $IH_*(X)$ of X to the topology of a resolution of singularities $\widetilde{X}$. This point of view will be exploited in the next section, see [BM11] for complete details. Moreover, equation (1) can serve as a beacon in constructing a mirror X° for a given singular X, as it restricts the topology of those X° that can act as a mirror of X.

4. Deformations of Singularities

All results in this section can be formulated for projective hypersurfaces with several isolated singularities. For simplicity, we restrict our attention to the case of a single isolated singularity.

Let f be a homogeneous polynomial in $n+2$ variables with complex coefficients such that the complex projective hypersurface

$$V = \{f = 0\} \subset \mathbb{CP}^{n+1}$$

has only one isolated singularity x. Let L_x, F_x and $T_x : H_n(F_x) \to H_n(F_x)$ denote the link, Milnor fiber and local monodromy operator of the isolated hypersurface singularity germ (V, x), respectively. By [Mi68], the link L_x is an $(n-2)$-connected closed oriented $(2n-1)$-dimensional manifold. Also, the Milnor fiber F_x is a parallelizable $(n-1)$-connected $2n$-dimensional manifold, which has the homotopy type of $\bigvee S^n$, a wedge of n-spheres. The number $\mu_x = \mathrm{rk} H_n(F_x)$ of these n-spheres (which are also called "vanishing cycles") is the local Milnor number at x. It is known that all eigenvalues of T_x are roots of unity. We say that the local monodromy operator T_x is trivial if all eigenvalues of T_x are equal to 1.

Theorem 4.1 ([BM11])**.** *Let $V \subset \mathbb{CP}^{n+1}$ be a complex projective hypersurface with only one isolated singular point x. Let V_s denote a nearby smoothing of V. Then:*

(a) $\widetilde{H}_i(V_s) \cong \widetilde{H}_i(IV)$*, for all* $i < 2n$, $i \neq n$*;*
(b) $\widetilde{H}_n(V_s) \cong \widetilde{H}_n(IV) \iff T_x$ *is trivial.*

Recall that intersection homology is invariant under small resolutions. Therefore, as inspired by the conifold transition picture of the previous section, we regard the local trivial monodromy condition of Theorem 4.1 as "mirroring" that of the existence of small resolutions.

In fact, the isomorphisms of Theorem 4.1 are often induced by a map. More precisely, by the construction of an intersection space, there is a canonical map $can : IV \to V$. Also, there is a specialization map $sp : V_s \to V$ which collapses the vanishing cycles to a point. In [BM11], we showed that under a mild technical assumption on the homology of the link (that is, if $n \neq 2$ and $H_{n-1}(L_x;\mathbb{Z})$ is torsion-free), one can define a map $\eta : IV \to V_s$ so that $can = sp \circ \eta$. Then the following holds:

Theorem 4.2 ([BM11]). *The isomorphisms in Theorem 4.1(a) are induced by the map η. Moreover,*

$$\eta_* : H_n(IV) \to H_n(V_s)$$

is a monomorphism, and it is an isomorphism if and only if the local monodromy operator T_x is trivial. In particular, in the latter case, the dual maps in cohomology are ring isomorphisms.

We regard the result of Theorem 4.2 as "mirroring" the fact that the intersection homology groups $IH_i(V)$ of V are vector subspaces of the corresponding homology groups $H_i(\widetilde{V})$ of any resolution $\widetilde{V}$ of V, the latter being an easy application of the Bernstein-Beilinson-Deligne-Gabber decomposition theorem, e.g., see [BBD, dCM, GM82].

Together with Remark 2.2, the result of Theorem 4.2 also yields the following bounds on the rank of the rational vector space $H_n(IV)$:

$$\text{rk}IH_n(V) \leq \text{rk}H_n(IV) \leq \text{rk}H_n(V_s).$$

In the case when the map η above can be defined, it can be used to put a mixed Hodge structure on each of the cohomology groups $\widetilde{H}^i(IV)$. Here we also use the fact that the specialization map $sp : V_s \to V$ induces mixed Hodge structure homomorphisms in cohomology, provided the cohomology groups $H^*(V_s)$ of the smoothing are considered with their "limit" mixed Hodge structures, as defined by Schmid-Steenbrink. More precisely, under the assumptions of Theorem 4.3 below, η^* can be used to transfer this limit mixed Hodge structure to the groups $\widetilde{H}^i(IV)$. We therefore get the following:

Theorem 4.3 ([BM11]). *If V is a complex projective hypersurface with only one isolated singular point x at which the local monodromy operator is trivial, then each cohomology group $\widetilde{H}^i(IV)$ carries a mixed Hodge structure so that the map $can : IV \to V$ induces a mixed Hodge structure homomorphism in (reduced) cohomology.*

The existence of mixed Hodge structures on intersection space cohomology groups (though restricted by our context and hypotheses) is already very surprising, especially after noting that the intersection space associated to a complex projective variety is not itself an algebraic variety in general. Nevertheless, one can regard the statement of Theorem 4.3 as "mirroring" the classical fact stating that for an equidimensional complex projective variety V, each intersection cohomology group $IH^i(V)$ carries a weight i pure Hodge structure, so that the natural map $H^i(V) \to IH^i(V)$ is a homomorphism of mixed Hodge structures. As already predicted by Remark 2.2, one could not expect in general to get purity on intersection space cohomology groups. However, at least when the monodromy operator of the smoothing family $\{V_s\}_s$ is trivial one does get purity. To give a concrete example, this occurs when a family of smooth genus 2 curves V_s degenerates into a union of two smooth elliptic curves meeting transversally at one double point, see [BM11, Example 5.4].

Regarding invariants of intersection spaces, it follows from [Ba10] that if $n = \dim_{\mathbb{C}} V$ is even, then the signature $\sigma(IV)$ of the intersection space (as defined via the Poincaré duality pairing on $H_n(IV)$) coincides with the Goresky-MacPherson (intersection homology) signature of V. The difference between the Euler characteristics of the two theories was computed in [BM11] as follows:

Theorem 4.4 ([BM11])**.** *For a complex projective hypersurface $V \subset \mathbb{CP}^{n+1}$ with only one isolated singular point x, we have that:*

$$\chi(\widetilde{H}_*(IV)) - \chi(IH_*(V)) = -2\chi_{<n}(L_x), \tag{2}$$

where L_x is the link of x, and $\chi_{<n} := \sum_{i<n}(-1)^i\beta_i$.

Of course, when x is actually a smooth point (so the link is a $(2n-1)$-dimensional sphere) the intersection space IV is homotopy equivalent to V with a small open ball about x excised, and the given formula is easily seen to hold. In fact, formula (2) holds more generally in the context of an even dimensional pseudomanifold X with only isolated singularities (see [Ba10, Corollary 2.14]). We indicate here a very elementary proof of this formula (as suggested by the referee) in the case when X has only an isolated singular point x. Let (M, L) be the manifold, with boundary the link L of x, obtained by excising a small open neighborhood of x. Then considering the relevant Mayer-Vietoris sequence, and using the additivity of the Euler characteristic, we have on the one hand

$$\chi(IX) = \chi(M) + \chi(cone(L_{<n})) - \chi(L_{<n}) = \chi(M) + 1 - \chi_{<n}(L),$$

and on the other

$$I\chi(X) = \chi(M) + I\chi(cone(L)) - \chi(L) = \chi(M) + \chi_{<n}(L),$$

where we have used the standard cone calculation for $IH^*(cone(L))$ and the fact that L is an odd-dimensional manifold. The result follows immediately.

5. Open Questions

We conclude this survey with a list of open problems, some of which are motivated by the "mirror symmetry" analogy discussed above.

1. Is there a sheaf-theoretic description for the intersection space (co)homology, similar to the one for intersection homology (by Deligne's intersection sheaf complex [GM83])? At least in the context of this note, the answer should be related to the complex of nearby cycles of a smoothing family.
2. Is there a version of (weak and hard) Lefschetz theorems for the (co)homology of the intersection space of a complex projective variety?
3. Is there a canonical mixed Hodge structure on the intersection space cohomology of a complex projective variety?
4. How much of the above work can be extended in the context of more general singularities? For example, consider the case of a hypersurface V with a higher-dimensional smooth singular locus Σ so that (V, Σ) is a Whitney

stratification of V. The intersection space IV was defined in [Ba10] under certain assumptions on the link bundle. In the same vein, the intersection space associated to certain stratified pseudomanifolds with depth 2 stratifications was constructed in [Ba11b]. It would be interesting to study the associated homology groups for these intersection spaces.

Acknowledgements. *The authors would like to thank Greg Friedman and the anonymous referee for carefully reading the manuscript and for giving constructive comments which helped improving the quality of the paper.*

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