HODGE POLYNOMIALS OF SINGULAR HYPERSURFACES

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Abstract. We express the difference between the Hodge polynomials of the singular and resp. generic member of a pencil of hypersurfaces in a projective manifold by using a stratification of the singular member described in terms of the data of the pencil. In particular, if we assume trivial monodromy along all strata in the singular member of the pencil, our formula computes this difference as a weighted sum of Hodge polynomials of singular strata, the weights depending only on the Hodge-type information in the normal direction to the strata. This extends previous results (cf. [19]) which related the Euler characteristics of the generic and singular members only of *generic* pencils, and yields explicit formulas for the Hodge χ_y-polynomials of singular hypersurfaces.

1. Introduction and Statements of Results

Let X be an n-dimensional non-singular projective variety and $\mathcal{L}$ be a bundle on X. Let $\mathbf{L} \subset \mathbb{P}(H^0(X, \mathcal{L}))$ be a line in the projectivization of the space of sections of $\mathcal{L}$, i.e., a pencil of hypersurfaces in X. Assume that the generic element L_t in $\mathbf{L}$ is non-singular and that L_0 is a singular element of $\mathbf{L}$. The purpose of this note is to relate the Hodge polynomials of the singular and resp. generic member of the pencil, i.e., to understand the difference $\chi_y(L_0) - \chi_y(L_t)$ in terms of invariants of the singularities of L_0. A special case of this situation was considered by Parusiński and Pragacz ([19]), where the Euler characteristic is studied for pencils satisfying the assumptions that the generic element L_t of the pencil $\mathbf{L}$ is transversal to the strata of a Whitney stratification of L_0. This led to a calculation of Parusińki's generalized Milnor number ([18]) of a singular hypersurface. In a different vein, the Hodge theory of one-parameter degenerations was considered in [5] (compare also with [9] for the case when L_0 has only isolated singularities), by making use of Hodge-theoretical aspects of the nearby and vanishing cycles associated to the degenerating family of hypersurfaces, and extending similar Euler characteristic calculations presented in Dimca's book [11].

Let us first define the invariants to be investigated in this note. A functorial χ_y-genus is defined by the ring homomorphism

$$(1.1) \qquad \chi_y : K_0(\mathrm{MHS}) \to \mathbb{Z}[y, y^{-1}]; \; [(V, F^{\bullet}, W_{\bullet})] \mapsto \sum_p \dim_{\mathbb{C}}(gr^p_F(V \otimes_{\mathbb{Q}} \mathbb{C})) \cdot (-y)^p,$$

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where $K_0(\mathrm{MHS})$ is the Grothendieck ring of the abelian category of rational mixed Hodge structures. For $K^\cdot \in D^b(\mathrm{MHS})$ a bounded complex of rational mixed Hodge structures, we set $[K^\cdot] := \sum_i (-1)^i [K^i] \in K_0(\mathrm{MHS})$, and define

$$\chi_y([K^\cdot]) := \sum_i (-1)^i \chi_y([K^i]). \tag{1.2}$$

Then, if X is any complex algebraic variety, we let

$$\chi_y(X) := \chi_y([H^*(X;\mathbb{Q})]) = \sum_j (-1)^j \cdot \chi_y([H^j(X;\mathbb{Q})]). \tag{1.3}$$

Similarly, we define $\chi_y^c(X)$ by using instead the canonical mixed Hodge structure in the cohomology with compact support of X. The specializations of $\chi_y(X)$ and $\chi_y^c(X)$ for $y=-1$ yield the topological Euler characteristic $e(X)$. Note that χ_y^c is an additive invariant, i.e., if Z is a Zariski closed subset of X, then $\chi_y^c(X) = \chi_y^c(Z) + \chi_y^c(X \setminus Z)$.

Before formulating the main results of this note, we need a couple of definitions and notations. We begin by recalling standard facts about the incidence variety of a pencil, which plays an essential role in our approach. Let $\mathcal{I} \subset X \times \mathbf{L}$ be the variety defined by the *incidence correspondence*, i.e.,

$$\mathcal{I} = \{(x,t) |\ t \in \mathbf{L}, x \in L_t\}. \tag{1.4}$$

We shall denote the projections of $\mathcal{I}$ on each factor by p_X and $p_{\mathbf{L}}$ respectively; note that both p_X and $p_{\mathbf{L}}$ are surjective. Moreover, p_X is one-to-one outside of the base locus of $\mathbf{L}$, while its fibers over any point in the latter is a $\mathbb{P}^1$ which $p_{\mathbf{L}}$ maps isomorphically onto $\mathbf{L}$.

If the intersection of the base locus with the singular locus of any element of the pencil is empty, then $\mathcal{I}$ is a non-singular variety, but it has singularities otherwise. Indeed, under the empty intersection assumption, let f_1 and f_2 be two generic elements of the pencil written in the local coordinates $(x_1,..,x_n)$ of a base point P of the pencil. Then the differentials df_1 and df_2 are independent, since if $df_1 + t_0 df_2 = 0$ then the element of the pencil given by $f_1 + t_0 f_2 = 0$ has a singularity at P. Now using the local coordinates at P in which $f_1 = x_1, f_2 = x_2$ we can view the incidence corespondence as the hypersurface in $\mathbb{C}^n \times \mathbb{C}$ given by the equation $x_1 + t x_2 = 0$, which is non-singular.

We furthermore note that the fibers of $p_{\mathbf{L}}$ are isomorphic to the corresponding elements of the pencil (and they will be denoted by the symbols $L_t^{\mathcal{I}}$ or simply L_t if there is no danger of confusion), and $p_{\mathbf{L}}$ is a locally trivial topological fibration outside a finite set of points in $\mathbf{L}$ containing the point that corresponds to L_0. We will restrict our attention to fibers of $p_{\mathbf{L}}$ near L_0, that is, we consider the restriction map $p := p_{\mathbf{L}}|_{p_{\mathbf{L}}^{-1}(D_\epsilon(0))}$, for ϵ small enough so that this restriction is a locally trivial fibration outside the special fiber $L_0 = p_{\mathbf{L}}^{-1}(0)$. Note that p is a proper holomorphic map with algebraic fibers, the generic fiber being smooth and projective. As noted above, the incidence set $\mathcal{I}$ may be singular, but it is by definition a complete intersection of dimension equal to the dimension n of X. Therefore, $\mathbb{Q}_{\mathcal{I}}[n]$ is a perverse sheaf (or, more generally, a mixed Hodge module on $\mathcal{I}$, denoted by $\mathbb{Q}_{\mathcal{I}}^H[n]$, see [22, 24]). Let $\psi_p \mathbb{Q}_{\mathcal{I}} \in D_c^b(L_0^{\mathcal{I}})$ denote the constructible complex of nearby cycles attached to p. Then $\psi_p \mathbb{Q}_{\mathcal{I}}[n-1]$ is also a perverse sheaf, thus (by Saito's theory) underlying a mixed Hodge module. We consider the shifted complex

$$\mathbf{M}(L_0, p_{\mathbf{L}}) := \psi_p^H \mathbb{Q}_{\mathcal{I}}^H[1], \tag{1.5}$$

where $\mathbb{Q}_{\mathcal{I}}^H$ denotes the "constant" Hodge sheaf, and ψ_p^H is the corresponding nearby cycle functor on the level of Saito's mixed Hodge modules (i.e., if $rat : D^b(\mathrm{MHM}(\mathcal{I})) \to D^b_c(\mathcal{I})$ is the forgetful functor associating to a complex of mixed Hodge modules the underlying rational constructible complex of sheaves, then $rat \circ \psi_p^H = \psi_p[-1] \circ rat$). So, $\mathbf{M}(L_0, p_{\mathbf{L}})$ is a complex of mixed Hodge modules associated to the pair $(L_0, \mathbf{L})$ which, moreover, is supported only on L_0. We refer to [8] (see also [11]) for the definition of the nearby cycles complex, and [22, 24] for the extension of this construction to the category of mixed Hodge modules.

The main result of this note is the following (see also its reformulation in Theorem 2.4):

Theorem 1.1. *Let $\mathcal{S}$ be a Whitney stratification of L_0 such that the base locus of $\mathbf{L}$, that is, $B_{\mathbf{L}} = L_0 \cap L_t$, is a union of strata of $\mathcal{S}$. Then to each stratum $S \in \mathcal{S}$ one can associate a Hodge polynomial invariant $\chi_y^c(S, \mathbf{L})$ such that*

$$\chi_y(L_t) = \sum_{S \in \mathcal{S}} \chi_y^c(S, \mathbf{L}). \tag{1.6}$$

More precisely, $\chi_y(L_t)$ is the total χ_y-genus of the complex $\mathbf{M}(L_0, p_{\mathbf{L}})$ associated to the pair $(L_0, \mathbf{L})$ as in (1.5). In particular, if the monodromy of the restriction of $\mathbf{M}(L_0, p_{\mathbf{L}})$ to each stratum is trivial or, more generally, has finite order and the corresponding local system extends to the closure of the stratum, then:

$$\chi_y(L_t) = \sum_{S} \chi_y^c(S) \cdot \chi_y(M_S), \tag{1.7}$$

where M_S is the Milnor fiber in $\mathcal{I}$ corresponding to a point in the stratum S of L_0. The specialization of equation (1.7) for $y = -1$ yielding the equality for Euler characteristics is valid without any monodromy assumption.

Remark 1.2. It will follow from the proof of Theorem 1.1 (see also [[5], (73)]) that, in fact, each polynomial $\chi_y^c(S, \mathbf{L})$ is an alternating sum of Hodge polynomials of S with coefficients in admissible (at infinity) variations of mixed Hodge structures. Such "twisted Hodge polynomials" were computed in [5, 6] in terms of the Deligne extension of the underlying local system on a "good" compactification of S. Therefore, (1.6) provides a complete calculation for the Hodge polynomial of L_t.

An important consequence of the proof of Theorem 1.1 is an identity comparing the Hodge polynomials $\chi_y(L_0)$ and $\chi_y(L_t)$, respectively (see Theorem 2.4 for the precise formulation). For example, if for a stratification $\mathcal{S}$ as above all strata are assumed to be simply connected, the following identity holds:

$$\chi_y(L_0) = \chi_y(L_t) - \sum_{S} \chi_y^c(S) \cdot \chi_y([\tilde{H}^*(M_S; \mathbb{Q})]). \tag{1.8}$$

where the summation runs only over the singular strata of $\mathcal{S}$, i.e., strata S satisfying $\dim(S) < \dim(L_0)$. As explained in Section §3, such a formula imposes strong obstructions on the type of singularities the singular fiber of the pencil can have.

Let us elaborate more on computational aspects related to the statement of Theorem 1.1. Note that the stalk of the cohomology sheaf $\mathcal{H}^{\cdot}(rat(\mathbf{M}(L_0, p_{\mathbf{L}})))$ at any point $B \in L_0$ is a graded algebra, which we denote by $\mathcal{H}^{\cdot}(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$. More explicitly, it follows by construction that

$$\mathcal{H}^{\cdot}(\mathbf{M}(L_0, p_{\mathbf{L}}))_B = H^{\cdot}(M_S; \mathbb{Q}), \tag{1.9}$$

where M_S is as above the Milnor fiber in $\mathcal{I}$ corresponding to the stratum S of L_0 (or of $L_0^{\mathcal{I}}$) containing the point B. In relation to the last sentence of Theorem 1.1, we note that the Euler characteristic of M_S can be computed by using the following version of A'Campo's formula:

Proposition 1.3. *Let $\pi_{\mathcal{I}} : \tilde{\mathcal{I}} \to \mathcal{I}$ be the restriction to the proper preimage of $\mathcal{I}$ of an embedded resolution $\widetilde{X \times \mathbf{L}} \to X \times \mathbf{L}$ of singularities of the triple $(X \times \mathbf{L}, \mathcal{I}, L_0)$ (i.e., $\tilde{\mathcal{I}}$ is an embedded resolution of singularities of $\mathcal{I} \subset X \times \mathbf{L}$, and the components $E_{\tilde{\mathcal{I}},k}$ of the exceptional locus of $\pi_{\mathcal{I}}$ are the intersections of the components E_k of the exceptional locus of $\widetilde{X \times \mathbf{L}} \to X \times \mathbf{L}$ with $\tilde{\mathcal{I}}$; moreover, the proper transform $\tilde{L}_0 \subset \tilde{\mathcal{I}}$ of L_0 and the components $E_{\tilde{\mathcal{I}},k} \subset \tilde{\mathcal{I}}$ of the exceptional locus $E_{\tilde{\mathcal{I}}} = \cup E_{\tilde{\mathcal{I}},k}$ of $\pi_{\mathcal{I}}$ form a normal crossings divisor in $\tilde{\mathcal{I}}$).*

Let m_{E_k} be the multiplicity of the pullback of $p_{\mathbf{L}} : X \times \mathbf{L} \to \mathbf{L}$ along $E_k \subset \widetilde{X \times \mathbf{L}}$.

Let D_B be a ball in a germ of a smooth subspace of X which is transversal at $B \in X$ to the stratum of L_0 containing B.

Then the Euler characteristic of $\mathcal{H}^{\cdot}(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$ is given by

$$\sum e\left((E_{\mathcal{I},k} - E_{\mathcal{I},k} \cap \tilde{L}_0) \cap \pi_{\mathcal{I}}^{-1}(D_B)\right) \cdot m_{E_{\mathcal{I},k}}$$

Proof. The proof follows by standard arguments used in the proof of A'Campo formula for the Euler characteristic of the monodromy of the generic fiber of a base point free pencil. We apply such arguments to the restriction of the pullback of $p_{\mathbf{L}}$ to $\widetilde{X \times \mathbf{L}}$ on an appropriate subspace of the latter. More precisely, let H be a germ of a smooth submanifold in X containing $B \in L_0$ and D_B be a small ball about B in H. Then the proper preimage of D_B in $\tilde{\mathcal{I}}$ is a resolution of its preimage in $\mathcal{I}$ (indeed a small neighbourhood of B in X has a decomposition as $D_S \times D_B$ where D_S is neighbourhood of B in the stratum S, and the map $\tilde{\mathcal{I}} \to \mathcal{I}$ is a locally trivial fibration over D_S with fiber the total preimage of D_B in $\mathcal{I}$; hence this total preimage is smooth as well). Let $t \in D_\epsilon(0) \subset \mathbf{L}$ with ϵ sufficiently small so that the fibers L_t are transversal to D_B for $t \neq 0$, and yield a fibration of the preimage of D_B in $\mathcal{I}$ with one degenerate fiber $L_0 \cap D_B$. Now we apply A'Campo formula to the morphism of the proper preimage $\tilde{D}_B$ of D_B in $\tilde{\mathcal{I}}$ which is the restriction on the latter of the pullback of $p_{\mathbf{L}}$ on $\tilde{\mathcal{I}}$. The components of the exceptional locus of $\pi_{\mathcal{I}}|_{\tilde{D}_B}$ are the intersections of the components of the exceptional locus of $\widetilde{X \times \mathbf{L}} \to X \times \mathbf{L}$, i.e. $E_{\tilde{\mathcal{I}},k}$, and the multiplicity of $p_{\mathbf{L}}$ along E_k and along its restriction on $\tilde{D}_B$ along $E_{\mathcal{I},k}$ are the same by transversality. Hence the above formula indeed follows from A'Campo's result [1], since as pointed out earlier the Euler characteristic of $\mathcal{H}^{\cdot}(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$ coincides with the Euler characterisitc of the Milnor fiber in the transversal direction to the stratum.

□

Remark 1.4. It follows from (1.9) and Saito's theory (or see [16, 17]) that $\mathcal{H}^{\cdot}(\mathbf{M}(L_0, p_{\mathbf{L}}))_B$ also carries canonical mixed Hodge structures. This property will be needed in the proof of Theorem 1.1.

We now illustrate the identity (1.7) with a couple of examples, mainly for the case of Euler characteristics (but see also Example 1.6 for a calculation of Hodge polynomials), where Proposition 1.3 above is used to compute the contributions of strata to the Euler characteristic of the generic member of the pencil.[1]

Example 1.5. Let $X = \mathbb{P}^3$ and L_0 be the union of a non-singular hypersurface V_{n-1} of degree $n-1$ and a transversal hyperplane H. Let $\mathbf{L}$ be the pencil generated by V_n and $V_{n-1} \cup H$. The stratification of the singular locus $V_{n-1} \cap H$ of $V_{n-1} \cup H$ consists of its intersection with the base point locus of the pencil containing $V_{n-1} \cup H$ and V_n, i.e. $V_{n-1} \cap H \cap V_n$, and the complement to this intersection in $V_{n-1} \cap H$. The contribution of the stratum $V_{n-1} \cap H - V_{n-1} \cap H \cap V_n$ is zero, the contribution of each point in $V_{n-1} \cap H \cap V_n$ is 1 (provided V_n is transversal to $V_{n-1} \cap H$) and the contribution of each non singular point of $V_{n-1} \cup H$ is 1. Since the Euler characteristic of a non singular hypersurface of degree n in $\mathbb{P}^3$ is $n^3 - 4n^2 + 6n$, the Euler characteristics of strata $V_{n-1} - V_{n-1} \cap H$ and $H - V_{n-1} \cap H$ are $(n-1)^3 - 4(n-1)^2 + 6(n-1) - 3(n-1) + (n-1)^2$ and $3 - 3(n-1) + (n-1)^2$ respectively. Then the identity (1.7) in the case of Euler characteristic (i.e., for $y = -1$) verifies as:

$$n^3-4n^2+6n = (n-1)^3-4(n-1)^2+6(n-1)-3(n-1)+(n-1)^2+3-3(n-1)+(n-1)^2+n(n-1).$$

The identity for the Euler characteristic in Example 1.5 can be refined to a similar calculation for the Hodge polynomials.

Example 1.6. If, in the notations of Example 1.5 we moreover let $n = 4$, then we obtain the following: $\chi_y(V_4) = 2 - 20y + 2y^2$, $\chi_y(V_3) = 1 - 7y + y^2$, $\chi_y(V_1) = 1 - y + y^2$, $\chi_y(V_3 \cap V_1) = 0$ (cf. [13]). From the additivity of the χ_y^c-polynomial we have that: $\chi_y^c(V_3 - V_3 \cap V_1) = 1 - 7y + y^2, \chi_y^c(V_1 - V_3 \cap V_1) = 1 - y + y^2, \chi_y^c(V_3 \cap V_1 - V_3 \cap V_4 \cap V_1) = -12$. Moreover, the contributions of the fibers of the nearby cycles over $V_3 \cap V_1 - V_3 \cap V_1 \cap V_4$ is $1 + y$ (corresponding to the Hodge polynomial of the Milnor fiber of a node singularity since the monodromy is trivial [2]), and the contribution of a point in the intersection $V_3 \cap V_1 \cap V_4$ is

[1]In the case $y = -1$, we refer to $e(M_S)$ as the (multiplicity of) contribution of (a point in) the stratum S of L_0 to the Euler characteristic of L_t.

[2]In local coordinates near a base point the pencil has the form $xy + zt = 0$, with $xy = 0$ and $z = 0$ being the local equations of the reducible and respectively irreducible fibers. The germ of the stratum of the singular member of the pencil is given by $x = y = 0$, and the monodromy around the base point $x = y = z = 0$ of the pencil is given by $z = exp(2\pi i\theta)$, which is the same as the monodromy action on the cohomology of the Milnor fiber $xy = s$.

1. Hence, the identity:

$$\chi_y(V_4) = \chi_y^c(V_3 - V_3 \cap V_1) + \chi_y^c(V_1 - V_1 \cap V_3) +$$
$$\chi_y^c(V_3 \cap V_1 - V_3 \cap V_1 \cap V_4) \cdot \chi_y([(\psi_p \mathbb{Q}_{\mathcal{I}})_{x \in V_3 \cap V_1 - V_3 \cap V_1 \cap V_4}]) +$$
$$\chi_y^c(V_3 \cap V_1 \cap V_4) \cdot \chi_y([(\psi_p \mathbb{Q}_{\mathcal{I}})_{x \in V_3 \cap V_1 \cap V_4}])$$

becomes the relation:

$$2 - 20y + 2y^2 = (1 - 7y + y^2) + (1 - y + y^2) + (-12) \cdot (1 + y) + 12,$$

representing equation (1.7).

Example 1.7. Let us consider the following higher dimensional version of Example 1.5: the base locus of L_0 is given by $x_2 = ... = x_n = 0$, along which L_0 has a A_1 singularity. In appropriate coordinates the pencil has the form:

$$x_2^2 + ... + x_n^2 + x_1 t = 0$$

This is also the equation of the incidence correspondence $\mathcal{I}$ near the base point of the pencil. Blowing up of the A_1 singularity of $\mathcal{I}$ yields its resolution which has as the exceptional set E_I the non-singular quadric in $\mathbb{P}^n$. The intersection of the proper preimage of L_0 and E_I is given by $x_2^2 + ... + x_n^2 = 0$ which has a A_1 singularity at $t = x_2 = \cdots = x_n = 0, x_1 = 1$. In this case it follows from the Proposition 1.3 that this point may give additional contribution to the Euler characteristic of L_t. Since $E_I \cup L_0$ near this point is given by

$$(u_1 + u_2^2 + .. + u_n^2)u_1,$$

its contribution coming from the standard A'Campo formula is the Euler characteristic of the complement in $\mathbb{P}^{n-1}$ of a union of two hyperplanes. However, the Euler characteristic of the latter is zero, so we have no additional contribution. The contribution of the complement in the non-singular quadric to a tangent hyperplane is equal to 1, and hence so is the contribution of the base point of such pencil.

Example 1.8. Let us assume that $L_0 \subset \mathbb{P}^3$ has a A_2 singularity along a non-singular stratum to which L_1 is transversal. We can choose the local coordinates near the base point so that the non-singular stratum is given by $x_3 = x_2 = 0$ and L_1 is given by $x_1 = 0$. Then the pencil has the form:

$$x_3^3 + x_2^2 + x_1 t = 0$$

The incidence correspondence $\mathcal{I}$ in $\mathbb{C}^3 \times \mathbb{P}^1$ has a 3-dimesnional A_2-singularity. The blow-up at the singular point produces the proper preimage I^1 which is non-singular but tangent to the exceptional locus E, since $I^1 \cap E$ is the quadratic cone in $\mathbb{P}^3$. Blowing up the tangency point of I^1 and E yields as proper preimage I^2 of I^1 the $\mathbb{P}^1$-bundle over plane quadric curve. The pullback of t has multiplicity 1 along I^2, and the intersection of the proper preimage of $t = 0$ with the exceptional locus in I^2 is the double fiber of this fibration. The complement to this fiber has the Euler characteristic equal to 2. Hence the contribution of the singularity at the base point is equal to 2.

Example 1.9. Consider the case $n = 2$ of the previous example, but assume that V_2 is tangent to the singular locus $V_1 \cap H$ of $V_1 \cup H$. An example of such pencil is given by

$$t(xy) + s[x(y+u) + y(x+u) + z^2)].$$

Now the zero-dimensional stratum consists of one point ($x = y = z = 0, u = 1$) and its contribution is 2. To see this, notice that the pencil near the zero-dimensional stratum (i.e. in the chart $u = 1$) has in $\mathbb{C}^4$ the form

$$xy + s[x(y+1) + y(x+1) + z^2] = 0.$$

The proper transform of $\mathcal{I}$ after blowing up the origin is a quadric with one ordinary quadratic point, i.e., the resolution of $\mathcal{I}$ is a union of the Hirzebruch surface and the quadric intersecting along the rational curve. The proper preimage of xy is the chain of three rational curves dual to the graph A_2. Hence the Euler characteristic of the complement in the exceptional locus to the proper preimage of $xy = 0$ has the Euler characteristic $(4+4-2) - (2 \times 3 - 2) = 2$. So the contribution of this stratum in the formula (1.7) is 2.

2. Proof of Theorem 1.1

Recall our setting: L_0 is a singular hypersurface of the n-dimensional projective manifold X, and we fix a Whitney stratification of L_0; place L_0 in a pencil $\mathbf{L}$ of hypersurfaces with non-singular generic member L_t, and refine (if necessary) the stratification of L_0 so that the base locus of $\mathbf{L}$, i.e. $B_{\mathbf{L}} = L_0 \cap L_t$, is a union of strata; call the new stratification $\mathcal{S}$. The idea of the proof is to lift all the computations at the level of the incidence variety $\mathcal{I}$ of $\mathbf{L}$. More precisely, by using the incidence variety $\mathcal{I}$ of the pencil (cf. (1.4) for the definition), we construct a one-parameter family, denoted $\{L_t^{\mathcal{I}}\}$, of smooth complex projective varieties degenerating onto the singular variety $L_0^{\mathcal{I}}$. We denote by p the projection map onto a disk $\Delta \subset \mathbb{C}$ so that $L_0^{\mathcal{I}} = p^{-1}(0)$, and note that the domain of p is locally a complete intersection of pure dimension n. We furthermore note, as in the introduction, that the fibers $L_t^{\mathcal{I}}$ and resp. $L_0^{\mathcal{I}}$ of this family are in fact isomorphic to the generic member L_t and resp. singular member L_0 of the given pencil $\mathbf{L}$. In order to simplify the exposition, in what follows we drop the upper/lower-script $\mathcal{I}$ from the notation whenever we work on the incidence variety.

Consider the nearby cycles $\psi_p\mathbb{Q}_{\mathcal{I}}$ and, respectively, the vanishing cycles $\phi_p\mathbb{Q}_{\mathcal{I}}$ associated to the one-parameter family, and note that the following identifications hold:

$$H^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\psi_p\mathbb{Q}_{\mathcal{I}})_x, \quad \tilde{H}^j(M_x;\mathbb{Q}) = \mathcal{H}^j(\phi_p\mathbb{Q}_{\mathcal{I}})_x, \tag{2.1}$$

where M_x denotes the Milnor fiber of p at $x \in L_0$ in the incidence variety $\mathcal{I}$ of the pencil. In particular, these groups inherit canonical mixed Hodge structures since the nearby and vanishing cycles lift to Saito's category of mixed Hodge modules (cf. [24]), or see [16, 17].

There is a long exact sequence of mixed Hodge structures (e.g., see [16, 17], or use the fact that the nearby and vanishing cycles lift to the category of mixed Hodge modules):

$$\cdots \to H^j(L_0;\mathbb{Q}) \to \mathbb{H}^j(L_0;\psi_p\mathbb{Q}) \to \mathbb{H}^j(L_0;\phi_p\mathbb{Q}) \to \cdots, \tag{2.2}$$

where $\mathbb{H}^j(L_0;\psi_p\mathbb{Q})$ carries the *limit mixed Hodge structure* defined on the cohomology of the canonical fiber L_∞ of the one-parameter degeneration p (e.g., see [[21], §11.2]). The existence of the limit mixed Hodge structure is also a consequence of Saito's theory, since

$$\mathbb{H}^j(L_0;\psi_p\mathbb{Q}) = rat(H^j(k_*\psi_p^H\mathbb{Q}_{\mathcal{I}}^H[1])), \tag{2.3}$$

for $k:\mathcal{I}\to pt$ the constant map. Moreover, a consequence of the definition of the limit mixed Hodge structure is that (cf. [[21], Cor.11.25])

$$\dim_{\mathbb{C}} F^pH^j(L_\infty;\mathbb{C}) = \dim_{\mathbb{C}} F^pH^j(L_t;\mathbb{C}), \tag{2.4}$$

where L_t is the generic fiber of the family (and of p). Therefore,

$$\chi_y(L_\infty) := \chi_y([\mathbb{H}^\bullet(L_0;\psi_p\mathbb{Q})]) = \chi_y(L_t). \tag{2.5}$$

The rest of the proof follows from the following three lemmas:

Lemma 2.1. (Additivity of the χ_y^c-polynomial.)
Let $\mathcal{S}$ be the set of components of strata of an algebraic Whitney stratification of the complex algebraic variety Z. Then for any $\mathcal{M}^\bullet\in D^bMHM(Z)$ so that $rat(\mathcal{M}^\bullet)$ is constructible with respect to $\mathcal{S}$,

$$\chi_y([\mathbb{H}_c^\bullet(Z;\mathcal{M}^\bullet)]) = \sum_{S\in\mathcal{S}}\chi_y([\mathbb{H}_c^\bullet(S;\mathcal{M}^\bullet)]). \tag{2.6}$$

Proof. e.g., see [[5], Cor.3].

□

Lemma 2.2. (Trivial monodromy.)
In the notations of Lemma 2.1, let $\mathcal{F}^\bullet$ denote the rational constructible complex associated to $\mathcal{M}^\bullet$. Assume moreover that the local systems $\mathcal{H}^j(\mathcal{F}^\bullet)|_S$ are constant on S for each $j\in\mathbb{Z}$, e.g. $\pi_1(S)=0$. Then

$$\chi_y([\mathbb{H}_c^\bullet(S;\mathcal{F}^\bullet)]) = \chi_y^c(S)\cdot\chi_y([\mathcal{F}_x^\bullet]), \tag{2.7}$$

where $[\mathcal{F}_x^\bullet] := [i_x^*\mathcal{F}^\bullet] = [\mathcal{H}^\bullet(\mathcal{F}^\bullet)_x]\in K_0(MHS)$ is the complex of mixed Hodge structures induced by the pullback of $\mathcal{M}^\bullet$ over the point $x\in S$ under the inclusion $i_x:\{x\}\hookrightarrow S$.

Proof. e.g., see [[5], Prop.3] (compare also with [[10], Thm.6.1] for the case of coefficients in geometric variations).

□

If each local system $\mathcal{H}^j(\mathcal{F}^\bullet)|_S$ has a finite order monodromy, and is the restriction of a local system defined on a compactification of the stratum, then one has a similar multiplicative formula. This follows from the following:

Lemma 2.3. (Finite order monodromy, extending to a compactification.)
Let S be a connected complex algebraic manifold of dimension n, and $\mathcal{V}$ a local system on S underlying an admissible variation of mixed Hodge structures with quasi-unipotent monodromy at infinity. Assume the monodromy representation of $\mathcal{V}$ is of finite order and, moreover, $\mathcal{V}$ extends as a local system to some (possibly singular) compactification $\bar{S}$ of S. Then the twisted Hodge polynomial

$$\chi_y^c(S;\mathcal{V}) := \chi_y([H_c^*(S;\mathcal{V})])$$

[3] *is computed by the multiplicative formula:*

$$\chi_y^c(S;\mathcal{V}) = \chi_y^c(S)\cdot \chi_y([\mathcal{V}_x]), \tag{2.8}$$

for $[\mathcal{V}_x]\in K_0(MHS)$ *the class of the fiber of* $\mathcal{V}$ *at some point* $x\in S$.

Proof. Let W be a resolution of singularities of $\bar{S}$, which is an isomorphism over S, and so that $D := W\setminus S$ is a simple normal crossing divisor. Denote by $\bar{\mathcal{V}}$ the pullback to W of the extension of $\mathcal{V}$ on $\bar{S}$.

First note that the local system $\bar{\mathcal{V}}$ underlies an admissible variation of mixed Hodge structures on W. Indeed, since both S and W are smooth, if $j: S\hookrightarrow W$ is the inclusion map then the intermediate extension (cf. [2]) $j_{!*}(\mathcal{V}[n])$ of $\mathcal{V}$ from S to W is given by:

$$j_{!*}(\mathcal{V}[n]) = IC_W(\mathcal{V}) \simeq IC_W(\bar{\mathcal{V}}) \simeq IC_W\otimes\bar{\mathcal{V}} \simeq \mathbb{Q}[n]\otimes\bar{\mathcal{V}}\simeq\bar{\mathcal{V}}[n]. \tag{2.9}$$

This yields the identification

$$\bar{\mathcal{V}} = j_{!*}\mathcal{V}. \tag{2.10}$$

Moreover, if $\mathcal{V}^H$ denotes the smooth mixed Hodge module on S defined by $\mathcal{V}$ (cf. [24]), then the isomorphisms in (2.9) can be lifted to the level of algebraic mixed Hodge modules. Therefore, $j_{!*}\mathcal{V}^H[n]$ is a smooth mixed Hodge module on W, so $\bar{\mathcal{V}} = \mathrm{rat}(j_{!*}\mathcal{V}^H)$ underlies an admissible variation of mixed Hodge structures.

Clearly, by the additivity of the χ_y^c-polynomial, we have that

$$\chi_y^c(S;\mathcal{V}) = \chi_y(W;\bar{\mathcal{V}}) - \chi_y(D;\bar{\mathcal{V}}|_D) \tag{2.11}$$

where, by the inclusion-exclusion principle,

$$\chi_y(D;\bar{\mathcal{V}}|_D) = \sum_{i_0<\cdots<i_k}(-1)^k\chi_y(D_{i_0}\cap\cdots\cap D_{i_k};\bar{\mathcal{V}}|_{D_{i_0}\cap\cdots\cap D_{i_k}}) \tag{2.12}$$

for D_i the irreducible components of the divisor D. And identities similar to (2.11) and (2.12) hold, of course, for the usual Hodge polynomials with trivial coefficients (corresponding to the constant variation $\mathbb{Q}$). Since D is a simple normal crossing divisor on W, the intersections of its components are algebraic submanifolds, so in order to prove (2.8) it suffices to show that: *if X is a* compact *complex algebraic manifold, and $\mathcal{V}$ is an admissible variation of mixed Hodge structure on X with finite order monodromy, then:*

$$\chi_y(X;\mathcal{V}) = \chi_y(X)\cdot\chi_y([\mathcal{V}_x]). \tag{2.13}$$

And this can be proved by using the Atiyah-Meyer type results from [5]. Indeed, if

$$\chi_y(\mathcal{V}) := \sum_p [Gr^p_{\mathcal{F}}(\mathcal{V}\otimes_{\mathbb{Q}}\mathcal{O}_X)]\cdot(-y)^p \in K^0(X)[y,y^{-1}]$$

[3]The fact that the groups $H^i_c(S;\mathcal{V})$ carry canonical mixed Hodge structures is an easy consequence of Saito's theory of mixed Hodge modules [24].

is the K-theory χ_y-characteristic of $\mathcal{V}$ (with $\mathcal{F}^{\cdot}$ the corresponding filtration on the flat bundle $\mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_X$), then by [5] we have that:

$$\chi_y(X;\mathcal{V}) = \int_{[X]} ch^*(\chi_y(\mathcal{V})) \cup \tilde{T}^*_y(TX), \tag{2.14}$$

where $\tilde{T}^*_y(TX)$ is the un-normalized Hirzebruch class of (the tangent bundle of) X, appearing in the generalized Hirzebruch-Riemann-Roch theorem [13]. Recall here that

$$\chi_y(X) = \int_{[X]} \tilde{T}^*_y(TX).$$

The claim in (2.13) follows if we can show that the bundles $Gr^p_{\mathcal{F}}(\mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_X)$ ($p \in \mathbb{Z}$) are flat. Since flatness is a local property, it suffices to check this property on a finite cover. For this we make use of the finite monodromy assumption. Indeed, there is a finite cover $p : X' \to X$ on which the pullback of the local system $\mathcal{V}$ becomes constant. By rigidity, the pullback variation underlying this local system is constant, so the Hodge filtration (and its graded pieces) for the associated flat bundle $p^*\mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_{X'}$ is by trivial bundles. Since these are all pull-backs of the corresponding bundles from X, the claim follows.

□

At this point we remark that the methods involved in proving our Theorem 1.1 (e.g., the use of the limit mixed Hodge structure and of Atiyah-Meyer type formulae) forbid us from considering more general Hodge-theoretic invariants such as the Hodge-Deligne E-polynomial, which also takes into account the weight filtrations.

An important reformulation of Theorem 1.1 is the following generalization of some results of [5], which explicitly compares the Hodge polynomials of the singular and respectively generic fiber in a pencil of hypersurfaces.

Theorem 2.4. *Under the assumptions and notations of Theorem 1.1, we obtain the following relation between the Hodge polynomials of L_0 and L_t respectively:*

$$\chi_y(L_0) = \chi_y(L_t) - \sum_S \chi^c_y(S;\tilde{\mathbf{M}}(L_0,p_{\mathbf{L}})), \tag{2.15}$$

for $\tilde{\mathbf{M}}(L_0,p_{\mathbf{L}}) := \phi^H_p \mathbb{Q}_{\mathcal{I}}[1]$ the complex of mixed Hodge modules corresponding to Deligne's constructible complex of vanishing cycles on the incidence variety $\mathcal{I}$. Here the summation runs only over singular strata, i.e., over strata $S \in \mathcal{S}$ so that $\dim(S) < \dim(L_0)$.

In particular, if for any $S \in \mathcal{S}$ the variations of mixed Hodge structures $\mathcal{H}^i(\tilde{\mathbf{M}}(L_0,p_{\mathbf{L}}))|_S$ ($i \in \mathbb{Z}$) are constant (e.g., $\pi_1(S) = 0$), or have finite order monodromy representations that extend to the closure $\bar{S}$ of the stratum, then

$$\chi_y(L_0) = \chi_y(L_t) - \sum_S \chi^c_y(S) \cdot \chi_y(\tilde{H}^{\cdot}(M_S;\mathbb{Q})). \tag{2.16}$$

The corresponding Euler characteristic formula (i.e., for $y = -1$) holds without any restrictions on the monodromy along the singular strata of L_0.

Proof. The underlying rational constructible complex of sheaves for $\tilde{\mathbf{M}}(L_0, p_{\mathbf{L}})$ is the complex $\phi_p\mathbb{Q}$ of the vanishing cycles associated with the one-parameter family $\{L_t\}$ on the incidence variety. It is supported only on the singular locus of the singular fiber L_0.

The identity in (2.15) follows from the functoriality of the χ_y-genus, the fact that the long exact sequence (2.2) is a sequence of mixed Hodge structures, and the additivity of the χ_y^c-genus of Lemma 2.1.

Under the trivial (resp. finite order) monodromy assumption, Lemma 2.2 (resp. Lemma 2.3) and the identification in (2.1) yield (2.16).

□

Remark 2.5. We want to point out that our formulae (1.7) and (2.16), which were obtained in the case of very simple monodromy situations at each stratum, admit reformulations expressed entirely only in terms of the Hodge polynomials of closures of strata in the singular fiber of the pencil. (This makes it easier to identify each of these formulae as the degree-zero part of a conjectural corresponding characteristic class formula for the motivic Hirzebruch classes of [3].) Indeed, for a given stratum S of a Whitney stratification as in Theorem 1.1, let us define inductively

$$\widehat{\chi}_y(\bar{S}) := \chi_y(\bar{S}) - \sum_{P<S} \widehat{\chi}_y(\bar{P}), \tag{2.17}$$

where the summation is over (boundary) strata $P \subset \bar{S} \setminus S$. By the additivity of the χ_y^c-genus and since L_0 is compact, it is then easy to see that in fact we have

$$\widehat{\chi}_y(\bar{S}) = \chi_y^c(S) = \chi_y(\bar{S}) - \chi_y(\bar{S} \setminus S). \tag{2.18}$$

So formula (1.7) for example can be now re-written as

$$\chi_y(L_t) = \sum_S \widehat{\chi}_y(\bar{S}) \cdot \chi_y(M_S), \tag{2.19}$$

and similarly for formula (2.16).

Remark 2.6. As already noted in the introduction, our main results should be regarded as a Hodge-theoretic extension to arbitrary pencils of the Parusiński-Pragacz formula for the Euler characteristic of singular hypersurfaces ([19]). A characteristic class version of this formula was obtained in [20] (see also [25, 26, 28]) by studying the Milnor class of a complex hypersurface, that is, the difference between the Fulton-Johnson class [12] and the Chern-MacPherson class [15]. On the other hand, the difference between the Hodge polynomials of the singular and resp. generic fiber of a one-parameter family $\{X_t\}$ of projective hypersurfaces is just the degree of a certain Hodge-theoretic Milnor class, which can be defined as the difference between the motivic Hirzebruch class of X_0 (as defined in [3]) and the Hirzebruch class of its virtual tangent bundle in the ambient variety. It would be interesting to understand the higher dimensional components of this generalized Milnor class in terms of invariants of the singular locus of the special fiber in the family. This will be addressed elsewhere.

3. Applications and Examples

3.1. **Quadrics.** For a quadric Q_0 given by the equation

$$f_r(x_0, ..., x_n) = 0 \tag{3.1}$$

where f_r is a quadratic form of rank r, the singular locus $Sing(Q_0)$ is a linear space of dimension $n-r$. The strata of the stratification suitable for the application of Theorem 1.1 to the pencil generated by Q_0 and the generic quadric Q_1 consist of

$$S_1 = Q_0 - Sing(Q_0) - Q_1 \cap Q_0, \;\; S_2 = Sing(Q_0) - Sing(Q_0) \cap Q_1, \;\; S_3 = Sing(Q_0) \cap Q_1. \tag{3.2}$$

Note that S_3 is a generic quadric in $\mathbb{P}^{n-r}$. Only S_2 and S_3 will be used for the calculation of the Euler characteristic $e(Q_0)$ by using Theorem 2.4 (in the case $y=-1$). We have:

$$e(Q_1) = n + 1 - \frac{1}{2}(1 + (-1)^n), \;\; e(S_3) = (n-r+1) - \frac{1}{2}(1 + (-1)^{n-r}), \tag{3.3}$$

$$e(S_2) = e(\mathbb{P}^{n-r}) - e(S_3) = \frac{1}{2}(1 + (-1)^{n-r}).$$

The calculation of the Euler characteristic of Milnor fibers at points of S_3 is similar to that of Example 1.5. The incidence correspondence is locally given in $\mathbb{C}^{n+1}$ by the equation

$$x_0^2 + ... + x_{r-1}^2 + t x_k = 0 \quad (k \geq r) \tag{3.4}$$

i.e., it is a singular quadric of rank $r+2$. The Milnor fiber M_{S_3} is a r-dimensional manifold given in the affine space $\mathbb{C}^{r+2}$, with coordinates $x_0, ..., x_{r-1}, x_k$, by the equation (3.4) with $t = \epsilon \neq 0$. It is biregular to $\mathbb{C}^r$. The Milnor fiber of the stratum S_2 is the Milnor fiber of an A_1-singularity in $\mathbb{C}^r$. Therefore:

$$e(M_{S_2}) = 1 + (-1)^{r+1}, \qquad e(M_{S_3}) = 1. \tag{3.5}$$

Back in Theorem 2.4 and for $y=-1$, we now obtain:

$$e(Q_0) = n + 1 - (1 + (-1)^{r-1})/2 = n + (1 + (-1)^r)/2 \tag{3.6}$$

Also note that the relation of the Theorem 1.1 specializes (for $y=-1$) into:
(3.7)

$$(n+1) - \frac{1}{2}(1+(-1)^n) = e(S_1) + \frac{1}{2}(1+(-1)^{n-r}) \cdot (1+(-1)^{r+1}) + (n-r+1) - \frac{1}{2}(1+(-1)^{n-r})$$

which can be used to calculate $e(S_1) = r + (1+(-1)^{r-1})/2)$. (Alternatively one can compute $e(S_1)$ by using the fibration of S_1 over a non-singular quadric in $\mathbb{P}^{r-1}$ with the fiber $\mathbb{C}^{n-r}$).

Perhaps the easiest way to calculate the χ_y-genus of Q_0 is to use the fibration of its resolution $\tilde{Q}_0$ which is the proper preimage of Q_0 in the blow-up of $\mathbb{P}^n$ at $Sing(Q_0) \subset \mathbb{P}^n$. One has the following fibration with the fiber $\mathbb{P}^{n-r+1}$, base of which is a non-singular quadric Q_{ns}^{r-2} of dimension $r-2$:

$$\tilde{Q}_0 \stackrel{\mathbb{P}^{n-r+1}}{\longrightarrow} Q_{ns}^{r-2} \tag{3.8}$$

The exceptional locus of the resolution (3.8) is a Q_{ns}^{r-2}-fibration over $\mathbb{P}^{n-r}$. Hence:

$$\chi_y(Q_0) = \chi_y(Q_{ns}^{r-2})(-1)^{n-r+1} y^{n-r+1} + \chi_y(\mathbb{P}^{n-r}) = \tag{3.9}$$

$$(\sum_{i=0}^{i=r-2}(-1)^i y^i+\frac{(1+(-1)^r)}{2}(-y)^{r-2})(-1)^{n-r+1}y^{n-r+1}+\sum_{i=0}^{i=n-r}(-1)^i y^i$$

This calculation can also be obtained by using the χ_y-version of Theorem 1.1 or Theorem 2.4, instead of the Euler characteristic version used above.

3.2. **Miscellanea.** Similar calculations can be done for other singular hypersurfaces of low degree. For example, using Theorem 2.4 and the classification of cubic surfaces with one-dimensional singular locus is given in [4] one can obtain χ_y-polynomials of all singular cubic surfaces in $\mathbb{P}^3$ with non-isolated singularities. The possibilities are irreducible surfaces which are cones over nodal and cuspidal plane cubics, and surfaces given by the equations:

$$F : x_0^2x_2 + x_1^2x_3, \quad G : x_0^2x_2 + x_0x_1x_3 + x_1^3 \tag{3.10}$$

For a cubic in any dimension the singular locus of codimension 1 is a linear space, since a transversal plane section is an irreducible cubic hence has only one singularity. It is not difficult to work explicit formulas for the χ_y-polynomials in this case as well.

Also, images of generic projections $X^n \to \mathbb{P}^{n+1}$ provide an interesting class of hypersurfaces with codimension one singular locus. The numerology of singularities is given in [14]. Our Theorem 1.1 can be used to compute Hodge-theoretical invariants of strata.

Finally note that the relation between the Euler characteristic of a singular curve in $\mathbb{P}^2$ and its smoothing provides an important restriction on the number of singular points of a curve. Similarly, Theorem 1.1 yields restrictions on data of singular strata in higher dimensions.

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